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A SIMPLE PROOF OF JACOBI'S FOUR-SQUARE THEOREM

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Abstract

A celebrated result, due to Jacobi, says that the number of representations of the positive integer n as a sum of four squares is equal to eight times the sum of the divisors of n which are not divisible by 4. We give a new and simple proof of this result which depends only on Jacobi's triple product identity.

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1.

The following theorem, due to Jacobi, is well known (see, for example Hardy and Wright (1960) Theorem 386).

Let $r(n)$ denote the number of representations of the positive integer n as a sum of four squares of integers (positive, negative or zero), with order taken into account. Thus, for example, $r(1) = 8$ since

$$\begin{aligned} 1 &= (\pm 1)^2 + 0^2 + 0^2 + 0^2 = 0^2 + (\pm 1)^2 + 0^2 + 0^2 \\ &= 0^2 + 0^2 + (\pm 1)^2 + 0^2 = 0^2 + 0^2 + 0^2 + (\pm 1)^2. \end{aligned}$$

Then

$$r(n) = 8 \sum_{d \mid n, 4 \nmid d} d. \tag{1.1}$$

It is easily verified that (1.1) is equivalent to the q-series identity

$$\left(\sum_{-\infty}^{\infty} q^{n^2}\right)^4 = 1 + 8\sum_{4\nmid n} \frac{nq^n}{1-q^n}. \tag{1.2}$$

My object is to give a proof of (1.2) which requires only Jacobi's triple product identity

$$\prod_{n\geq 1}(1+a^{-1}q^{2n-1})(1+aq^{2n-1})(1-q^{2n}) = \sum_{-\infty}^{\infty} a^n q^{n^2}, \tag{1.3}$$

a simple proof of which can be found in Hirschhorn (1976).

2.

We start by proving the identity

$$\begin{aligned}
&\prod_{n\geq 1}(1+aq^{2n-1})(1+a^{-1}q^{2n-1})(1+bq^{2n-1})(1+b^{-1}q^{2n-1})(1-q^{2n})^2\\
&\quad = \Bigg\{\prod_{n\geq 1}(1+abq^{4n-2})(1+a^{-1}b^{-1}q^{4n-2})\\
&\qquad\qquad \times(1+ab^{-1}q^{4n-2})(1+a^{-1}bq^{4n-2})\\
&\qquad\qquad +(a+b+a^{-1}+b^{-1})\,q\prod_{n\geq 1}(1+abq^{4n})\\
&\qquad\qquad \times(1+a^{-1}b^{-1}q^{4n})(1+ab^{-1}q^{4n})(1+a^{-1}bq^{4n})\Bigg\}\prod_{n\geq 1}(1-q^{4n})^2.
\end{aligned} \tag{2.1}$$

Thus, (1.3) gives

$$\begin{aligned}
&\prod_{n\geq 1}(1+aq^{2n-1})(1+a^{-1}q^{2n-1})(1+bq^{2n-1})(1+b^{-1}q^{2n-1})(1-q^{2n})^2\\
&\quad = \sum_{r=-\infty}^{\infty} a^r q^{r^2}\cdot\sum_{s=-\infty}^{\infty} b^s q^{s^2}\\
&\quad = \sum_{r,s=-\infty}^{\infty} a^r b^s q^{r^2+s^2}\\
&\quad = \sum_{n=-\infty}^{\infty}\sum_{r+s=n} a^r b^s q^{r^2+s^2}.
\end{aligned}$$

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We now consider the two cases $n = 2m$, $n = 2m + 1$. In the first, set $r = m + t$, $s = m - t$, and in the second, $r = m + t + 1$, $s = m - t$, and the sum becomes

$$= \sum_{m,t=-\infty}^{\infty} a^{m+t} b^{m-t} q^{(m+t)^2+(m-t)^2} + \sum_{m,t=-\infty}^{\infty} a^{m+t+1} b^{m-t} q^{(m+t+1)^2+(m-t)^2}$$

$$= \sum_{m=-\infty}^{\infty} a^m b^m q^{2m^2} \sum_{t=-\infty}^{\infty} a^t b^{-t} q^{2t^2} + aq \sum_{m=-\infty}^{\infty} a^m b^m q^{2m^2+2m} \sum_{t=-\infty}^{\infty} a^t b^{-t} q^{2t^2+2t}$$

which by (1.3) again,

$$= \Big\{ \prod_{n\geq 1} (1 + abq^{4n-2})(1 + a^{-1}b^{-1}q^{4n-2})(1 + ab^{-1}q^{4n-2})(1 + a^{-1}bq^{4n-2})$$
$$+ aq \prod_{n\geq 1} (1 + abq^{4n})(1 + a^{-1}b^{-1}q^{4n-4})(1 + ab^{-1}q^{4n})(1 + a^{-1}bq^{4n-4}) \Big\}$$
$$\times \prod_{n\geq 1} (1 - q^{4n})^2$$
$$= \Big\{ \prod_{n\geq 1} (1 + abq^{4n-2})(1 + a^{-1}b^{-1}q^{4n-2})(1 + ab^{-1}q^{4n-2})(1 + a^{-1}bq^{4n-2})$$
$$+ (a + b + a^{-1} + b^{-1})q \prod_{n\geq 1} (1 + abq^{4n})$$
$$\times (1 + a^{-1}b^{-1}q^{4n})(1 + ab^{-1}q^{4n})(1 + a^{-1}bq^{4n}) \Big\}$$
$$\times \prod_{n\geq 1} (1 - q^{4n})^2,$$

as required.

3.

We now apply a straightforward but somewhat tedious limiting process to (2.1) to derive the identity

$$\prod_{n\geq 1} (1 - q^n)^6 = \prod_{n\geq 1} (1 + q^{2n-1})^2(1 - q^{2n}) \Big\{ 1 + \sum_{n\geq 1} (2n + 1)^2 q^{n^2+n} \Big\}$$
$$\times -2 \prod_{n\geq 1} (1 + q^{2n})^2(1 - q^{2n}) \Big\{ \sum_{n\geq 1} 4n^2 q^{n^2} \Big\}. \tag{3.1}$$

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In (2.1), put $-aq$ for a, $-aq$ for b, then q for q^2, and we obtain

$$\begin{aligned}
&\prod_{n\geq 1}(1-aq^n)^2(1-a^{-1}q^{n-1})^2(1-q^n)^2\\
(3.2)\quad &=\Big\{\prod_{n\geq 1}(1+a^2q^{2n})(1+a^{-2}q^{2n-2})(1+q^{2n-1})^2\\
&\quad -2a^{-1}\prod_{n\geq 1}(1+a^2q^{2n-1})(1+a^{-2}q^{2n-1})(1+q^{2n})^2\Big\}\times\prod_{n\geq 1}(1-q^{2n})^2
\end{aligned}$$

which, by (1.3), equals

$$\prod_{n\geq 1}(1+q^{2n-1})^2(1-q^{2n})\sum_{-\infty}^{\infty}a^{2n}q^{n^2+n}
-2a^{-1}\prod_{n\geq 1}(1+q^{2n})^2(1-q^{2n})\sum_{-\infty}^{\infty}a^{2n}q^{n^2}.$$

That is,

$$\begin{aligned}
&(1-a^{-1})^2\prod_{n\geq 1}(1-aq^n)^2(1-a^{-1}q^n)^2(1-q^n)^2\\
(3.3)\quad &=\prod_{n\geq 1}(1+q^{2n-1})^2(1-q^{2n})\Big\{(1+a^{-2})+\sum_{n\geq 1}q^{n^2+n}(a^{2n}+a^{-2n-2})\Big\}\\
&\quad -2\prod_{n\geq 1}(1+q^{2n})^2(1-q^{2n})\Big\{a^{-1}+\sum_{n\geq 1}q^{n^2}(a^{2n-1}+a^{-2n-1})\Big\}.
\end{aligned}$$

If in (3.3) we set $a=1$, and subtract the resulting identity from (3.3), we obtain

$$\begin{aligned}
&(1-a^{-1})^2\prod_{n\geq 1}(1-aq^n)^2(1-a^{-1}q^n)^2(1-q^n)^2\\
&=\prod_{n\geq 1}(1+q^{2n-1})^2(1-q^{2n})\\
(3.4)\quad &\quad\times\Big\{-(1-a^{-2})+\sum_{n\geq 1}q^{n^2+n}\big[(a^{2n}-1)-(1-a^{-2n-2})\big]\Big\}\\
&\quad -2\prod_{n\geq 1}(1+q^{2n})^2(1-q^{2n})\\
&\quad\times\Big\{-(1-a^{-1})+\sum_{n\geq 1}q^{n^2}\big[(a^{2n-1}-1)-(1-a^{-2n-1})\big]\Big\}.
\end{aligned}$$

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If $a \neq 1$ and we divide by $(1 - a^{-1})$, we obtain

$$(3.5)\quad \begin{aligned}(1 - a^{-1}) \prod_{n \geq 1} (1 - aq^n)^2(1 - a^{-1}q^n)^2(1 - q^n)^2 \\ = \prod_{n \geq 1} (1 + q^{2n-1})^2(1 - q^{2n}) \\ \times \Big\{-(1 + a^{-1}) + \sum_{n \geq 1} q^{n^2+n}\big[(a^{2n} + a^{2n-1} + \cdots + a) \\ - (1 + a^{-1} + \cdots + a^{-2n-1})\big]\Big\} \\ -2 \prod_{n \geq 1} (1 + q^{2n})^2(1 - q^{2n}) \\ \times \Big\{-1 + \sum_{n \geq 1} q^{n^2}\big[(a^{2n-1} + a^{2n-2} + \cdots + a) \\ - (1 + a^{-1} + \cdots + a^{-2n})\big]\Big\}.\end{aligned}$$

If in (3.5) we let $a \to 1$, subtract the resulting identity from (3.5), and divide by $(1 - a^{-1})$, we obtain the identity, invariant under $a \leftrightarrow a^{-1}$,

$$(3.6)\quad \begin{aligned}\prod_{n \geq 1} (1 - aq^n)^2(1 - a^{-1}q^n)^2(1 - q^n)^2 \\ = \prod_{n \geq 1} (1 + q^{2n-1})^2(1 - q^{2n}) \\ \times \Big\{1 + \sum_{n \geq 1} q^{n^2+n}\big[a^{2n} + 2a^{2n-1} + \cdots + 2na \\ + (2n + 1) + 2na^{-1} + \cdots + a^{-2n}\big]\Big\} \\ -2 \prod_{n \geq 1} (1 + q^{2n})^2(1 - q^{2n}) \\ \times \Big\{\sum_{n \geq 1} q^{n^2}\big[a^{2n-1} + 2a^{2n-2} + \cdots + (2n - 1)a \\ + 2n + (2n - 1)a^{-1} + \cdots + a^{-2n+1}\big]\Big\}.\end{aligned}$$

If in (3.6) we let $a \to 1$, we obtain (3.1), as required.

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4.

It is now an easy matter to complete the proof of (1.2). We can write (3.1)

$$
\begin{aligned}
\prod_{n \geq 1}(1-q^n)^6 &= \tfrac{1}{2}\prod_{n\geq 1}(1+q^{2n-1})^2(1-q^{2n})\sum_{-\infty}^{\infty}(2n+1)^2 q^{n^2+n} \\
&\quad - \prod_{n\geq 1}(1+q^{2n})^2(1-q^{2n})\sum_{-\infty}^{\infty}4n^2 q^{n^2} \\
&= \tfrac{1}{2}\prod_{n\geq 1}(1+q^{2n-1})^2(1-q^{2n})\left\{\left(1+4q\frac{d}{dq}\right)\sum_{-\infty}^{\infty}q^{n^2+n}\right\} \\
&\quad - \prod_{n\geq 1}(1+q^{2n})^2(1-q^{2n})\left\{4q\frac{d}{dq}\sum_{-\infty}^{\infty}q^{n^2}\right\} \\
&= \tfrac{1}{2}\prod_{n\geq 1}(1+q^{2n-1})^2(1-q^{2n}) \\
&\quad \times\left\{\left(1+4q\frac{d}{dq}\right)2\prod_{n\geq 1}(1+q^{2n})^2(1-q^{2n})\right\} \\
&\quad - \prod_{n\geq 1}(1+q^{2n})^2(1-q^{2n})\left\{4q\frac{d}{dq}\prod_{n\geq 1}(1+q^{2n-1})^2(1-q^{2n})\right\} \\
&= \prod_{n\geq 1}(1+q^{2n-1})^2(1+q^{2n})^2(1-q^{2n})^2 \\
&\quad \times\left\{1+8\sum_{n\geq 1}\frac{2nq^{2n}}{1+q^{2n}}-4\sum_{n\geq 1}\frac{2nq^{2n}}{1-q^{2n}}\right\} \\
&\quad - \prod_{n\geq 1}(1+q^{2n-1})^2(1+q^{2n})^2(1-q^{2n})^2 \\
&\quad \times\left\{8\sum_{n\geq 1}\frac{(2n-1)q^{2n-1}}{1+q^{2n-1}}-4\sum_{n\geq 1}\frac{2nq^{2n}}{1-q^{2n}}\right\} \\
&= \prod_{n\geq 1}(1+q^n)^2(1-q^{2n})^2\left\{1-8\sum_{n\geq 1}\frac{(2n-1)q^{2n-1}}{1+q^{2n-1}}+8\sum_{n\geq 1}\frac{2nq^{2n}}{1+q^{2n}}\right\} \\
&= \prod_{n\geq 1}(1+q^n)^4(1-q^n)^2\left\{1+8\sum_{n\geq 1}\frac{(-1)^n nq^n}{1+q^n}\right\}.
\end{aligned}
\tag{4.1}
$$

It follows that

$$
\prod_{n\geq 1}\left(\frac{1-q^n}{1+q^n}\right)^4 = 1+8\sum_{n\geq 1}\frac{(-1)^n nq^n}{1+q^n}. \tag{4.2}
$$

Now,

$$\prod_{n\geq 1}\left(\frac{1-q^n}{1+q^n}\right)=\prod_{n\geq 1}\frac{(1-q^n)^2}{(1-q^{2n})}=\prod_{n\geq 1}\frac{(1-q^{2n-1})^2(1-q^{2n})^2}{(1-q^{2n})}$$
$$=\prod_{n\geq 1}(1-q^{2n-1})^2(1-q^{2n})$$
$$=\sum_{-\infty}^{\infty}(-1)^n q^{n^2},$$

so (4.2) is

$$\left(\sum_{-\infty}^{\infty}(-1)^n q^{n^2}\right)^4=1+8\sum_{n\geq 1}\frac{(-1)^n n q^n}{1+q^n}. \tag{4.3}$$

Putting $-q$ for q, we obtain

$$\begin{aligned}\left(\sum_{-\infty}^{\infty} q^{n^2}\right)^4&=1+8\sum_{n\text{ odd}}\frac{nq^n}{1-q^n}+8\sum_{n\text{ even}}\frac{nq^n}{1+q^n}\\&=1+8\sum_{n\geq 1}\frac{nq^n}{1-q^n}-8\sum_{n\text{ even}}\left\{\frac{nq^n}{1-q^n}-\frac{nq^n}{1+q^n}\right\}\\&=1+8\sum_{n\geq 1}\frac{nq^n}{1-q^n}-8\sum_{n\geq 1}\frac{4nq^{4n}}{1-q^{4n}},\\&=1+8\sum_{4\nmid n}\frac{nq^n}{1-q^n}\end{aligned}\tag{4.4}$$

which is (1.2), as required.

References

G. H. Hardy and E. M. Wright (1960), *An Introduction to the Theory of Numbers* (Fourth Edition, Clarendon Press).

M. D. Hirschhorn (1976), 'Simple proofs of identities of MacMahon and Jacobi,' *Discrete Math.* **16**, 161–162.

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