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THE INTEGRABILITY OF RIEMANN SUMMABLE TRIGONOMETRIC SERIES

BY

P. S. BULLEN AND S. N. MUKHOPADHYAY

ABSTRACT. It is shown that if a trigonometric series is $(R, 3)$, respectively $(R, 4)$, summable then its $(R, 3)$ sum, respectively $(R, 4)$ sum, is James P^3-, respectively P^4-, integrable and that such series are Fourier series with respect to these integrals.

1. **Introduction.** The problem of constructing an integral with respect to which the coefficients of a convergent trigonometric series can be represented in Fourier form has been solved in various ways. If convergence is replaced by (C, k) summability or by Abel summability solutions to this problem have also been given; see [5, 7, 10] and for a general survey see [1]. However, the problem remains open in the case of Riemann summable trigonometric series. Verblunsky, [11, 12], has shown that $(R, 3)$, $(R, 4)$, summable trigonometric series are Fourier series if their $(R, 3)$, $(R, 4)$, sums are equivalent to Denjoy-Perron integrable functions. The present paper improves these results of Verblunsky in the sense that it is shown that $(R, 3)$, respectively $(R, 4)$ summable trigonometric series are Fourier series with respect to the P^3-, respectively P^4-, integral of James [2, 3, 4, 7]; in particular the $(R, 3)$ and $(R, 4)$ sums of such series need not be Denjoy-Perron integrable.

Suppose f is a real valued function defined in some neighbourhood of x. If there are real numbers α_i, $1 \leqq i \leqq n$, depending on x but not h such that

$$f(x+h) - f(x) = \sum_{i=1}^{n} \alpha_i \frac{h^i}{i!} + o(h^n), \ (h \longrightarrow 0),$$

then α_n is called Peano derivative of f at x of order n and we write $\alpha_n = f_{(n)}(x)$. If there are real numbers β_{2r}, $1 \leqq r \leqq m$, depending on x but not on h such that

$$\frac{f(x+h) + f(x-h) - 2f(x)}{2} = \sum_{r=1}^{m} \beta_{2r} \frac{h^{2r}}{(2r)!} + o(h^{2m}), \ (h \longrightarrow 0),$$

then β_{2m} is called the symmetric de la Vallée Poussin Derivative of f at x of order $2m$ and we write $\beta_{2m} = D^{2m} f(x)$. Similarly if there are real numbers β_{2r+1}, $0 \leqq r \leqq m$, such that

$$\frac{f(x+h) - f(x-h)}{2} = \sum_{r=1}^{m} \beta_{2r+1} \frac{h^{2r+1}}{(2r+1)!} + o(h^{2m+1}), \ (h \longrightarrow 0),$$

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then β_{2m+1} is called the symmetric de la Vallée Poussin derivative of f at x of order $2m+1$, $\beta_{2m+1} = D^{2m+1}f(x)$.

If $D^{2m}f(x)$ exists and if we define

$$\theta_{2m+2}(f,x,h) = \frac{(2m+2)!}{h^{2m+2}} \left\{ \frac{f(x+h)+f(x-h)-2f(x)}{2} - \sum_{r=1}^{m} \frac{h^{2r}}{(2r)!} D^{2r}f(x) \right\}$$

then the upper and lower symmetric de la Vallée Poussin derivatives of f at x of order $2m+2$ are defined by

$$\overline{D}^{2m+2}f(x) = \limsup_{h\to 0} \theta_{2m+2}(f,x,h),$$
$$\underline{D}^{2m+2}f(x) = \liminf_{h\to 0} \theta_{2m+2}(f,x,h),$$

respectively. If $\lim_{h\to 0} h\theta_{2m+2}(f,x,h) = 0$, then f is said to be (de la Vallée Poussin) smooth at x of order $2m+2$; see [8]. The quantities $\theta_{2m+3}(f,x,h)$, $\overline{D}^{2m+3}f(x)$, $\underline{D}^{2m+3}f(x)$ and smoothness of order $2m+3$ are defined analogously.

For a given positive integer r let

$$\Delta^r(f;x,h) = \sum_{i=0}^{r} (-1)^i \binom{r}{i} f\left(x + \frac{1}{2}rh - ih\right);$$

the upper and lower symmetric Riemann derivatives of f at x of order r are defined to be

$$\overline{RD}^r f(x) = \limsup_{h\to 0} \frac{\Delta^r(f,x,h)}{h^r},$$
$$\underline{RD}^r f(x) = \liminf_{h\to 0} \frac{\Delta^r(f,x,h)}{h^r},$$

respectively. If $\overline{RD}^r f(x) = \underline{RD}^r f(x)$ the common value is called the symmetric Riemann derivative of f at n of order r, written $RD^r f(x)$.

It can be shown that if $D^r f(x)$ exists finitely then $RD^r f(x)$ exists with the same value; see [6].

A function is said to be Riemann smooth, or just R-smooth, at x of order r if $\lim_{h\to 0} h^{-r+1}\Delta^r(f,x,h) = 0$. It can be shown if f is smooth at x of order r then it is R-smooth at x of order r; see [6].

Consider the trigonometric series

$$(1) \qquad \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) = \sum_{n=0}^{\infty} c_n(x), \quad \text{say.}$$

If for a given positive integer k the series

$$(2) \qquad \frac{1}{2}a_o + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\left(\frac{\sin nh}{nh}\right)^k$$

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is convergent for small $|h|$, $(h \neq 0)$, then the upper and lower limits of (2) as $h \longrightarrow 0$ are called the upper and lower (R, k) sums of (1). If these are equal then (1) is said to be (R, k) summable and the common value is the (R, k) sum of (1). It can be verified that if the series (1) is integrated term by term k times, and if the integrated series converges in some neighbourhood of x to the function ϕ then $\Delta^k(\phi, x; h)/h^k$ is just (2). Hence $\overline{RD}^k\phi(x)$, and $\underline{RD}^k\phi(x)$ are the upper and lower (R, k) sums of (1).

A function g on $[a, b]$ is said to be continuous in the generalized sense, or simply (CG), on $[a, b]$, written $g \in (CG)\ (a, b)$ if $[a, b]$ can be expressed as a countable union of closed sets on each of which g is continuous; equivalently every $P \subset [a, b]$, $P \neq \phi$, closed, contains a portion $Q =]c, d[\cap P \neq \phi$ and $g \mid Q$ is continuous. This property, also referred to as property R or Baire* $-$ 1, has been used by many authors and is useful in the theory of trigonometric series; see [9, 11, 13].

The theory of the P^n-integral is rather technical and reference should be made to the basic papers [4, 5], as well as to [7]. It is sufficient to say this integral is defined in a Perron manner using major and minor functions in which the lower derivative of the major function is the lower symmetric de la Vallée Poussin derivative of order n, and the upper derivative of the minor function is the upper symmetric de la Vallée Poussin derivative of order n. The resulting P^n-integral will then integrate finite symmetric de la Vallée Poussin derivatives of order n; more general functions can also be integrated see for instance [7, Theorem 5.1]. Further, certain classes of summable trigonometric series are P^n-Fourier series and a modified form of the classical Fourier formulae for the coefficients have been given; see [5, (4.5) and (4.6)] or [7, (8.4) and (8.5)].

2. **Some Preliminary Lemmas.**

We shall need a few auxiliary lemmas.

LEMMA 1. *If $a_n = o(n)$, $b_n = o(n)$ and the upper and lower $(R, 3)$, or $(R, 4)$, sums of (1) are finite at x then*

$$\sum_{n=1}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{n^2} \tag{3}$$

converges.

This is due to Verblunsky [11. Lemma 7; also 12].

LEMMA 2. *Let $a_n = o(n)$, $b_n = o(n)$ and put*

$$\phi(x) = \sum_{n=1}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{n^4} \tag{4}$$

If $\overline{RD}^4\phi(x_0)$ and $\underline{RD}^4\phi(x_0)$ are both finite then $D^2\phi(x_0)$ exists and $\overline{D}^4\phi(x_0)$ and $\underline{D}^4\phi(x_0)$ are finite.

PROOF. By Lemma 1 series (3) converges at x_0 and its coefficients are $o(1/n)$; hence this series is $(R, 2)$ summable at x_0 [14, vol. I, p. 319 Theorem 2.4] and so

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$D^2\phi(x_0)$ exists. We may suppose that $x_0 = 0 = \phi(x_0)$, $\phi(x) = \phi(-x)$, and (by adding $-\frac{1}{3}D^2\phi(0)\{\cos x - \cos 2x\}$ if necessary) that $D^2\phi(0) = 0$. Then the finiteness of $\overline{RD}^4\phi(0)$ and $\underline{RD}^4\phi(0)$ imply that as $x \longrightarrow 0$, $\phi(x) = o(x^2)$ and $\phi(2x) - 4\phi(x) = O(x^4)$. Hence

$$\phi\left(\frac{t}{2^n}\right) - 4\phi\left(\frac{t}{2^{n+1}}\right) = O\left(\frac{t^4}{4^{2n+2}}\right)$$

or equivalently

$$4^n\phi\left(\frac{t}{2^n}\right) - 4^{n+1}\phi\left(\frac{t}{2^{n+1}}\right) = O\left(\frac{t^4}{4^{n+2}}\right). \tag{5}$$

Substituting $n = 0, 1, 2, \ldots$ in (5) and adding gives

$$\phi(t) - 4^{n+1}\phi\left(\frac{t}{2^{n+1}}\right) = \sum_{i=0}^{n} O\left(\frac{t^4}{4^{i+2}}\right).$$

Since $\phi(x) = o(x^2)$ as $x \longrightarrow 0$, we get by letting $n \longrightarrow \infty$ that $\phi(t) = O(t^4)$ and so $\overline{D}^4\phi(0)$ and $\underline{D}^4\phi(0)$ are both finite. □

LEMMA 3. *Let $a_n = o(n)$, $b_n = o(n)$ and put*

$$\psi(x) = \sum_{n=1}^{\infty} \frac{b_n \cos nx - a_n \sin nx}{n^3} \tag{6}$$

If $\overline{RD}^3\psi(x_0)$ and $\underline{RD}^3\psi(x_0)$ are both finite then $D^1\psi(x_0)$ exists and $\overline{D}^3\psi(x_u)$ and $\underline{D}^3\psi(x_0)$ and $\underline{D}^3\psi(x_0)$ are finite.

PROOF. By Lemma 1 series (3) converges at x_0 and so $D^1\psi(x_0)$ exists [14 vol. I, p. 322 Theorem 2.18]. We may suppose that $x_0 = 0 = D^1\psi(x_0)$, $\psi(x) = -\psi(-x)$ and so by the finiteness of $\overline{RD}^3\psi(0)$ we have, as $x \longrightarrow 0$, that $\psi(x) = o(x)$ and $\psi(3x) - 3\psi(x) = O(x^3)$. Hence

$$\psi\left(\frac{t}{3^n}\right) - 3\psi\left(\frac{t}{3^{n+1}}\right) = O\left(\frac{t^3}{3^{3n+3}}\right)$$

or equivalently

$$3^n\psi\left(\frac{t}{3^n}\right) - 3^{n+1}\psi\left(\frac{t}{3^{n+1}}\right) = O\left(\frac{t^3}{3^{2n+3}}\right) \tag{7}$$

Substituting $n = 0, 1, 2, \ldots$ in (7) and adding gives

$$\psi(t) - 3^{n+1}\psi\left(\frac{t}{3^{n+1}}\right) = \sum_{i=0}^{n} O\left(\frac{t^3}{3^{2n+3}}\right).$$

Since $\psi(x) = o(x)$ as $x \longrightarrow 0$, we get by letting $n \longrightarrow \infty$ that

$$\psi(t) = O(t^3)$$

and so $\overline{D}^3\psi(0)$ and $\underline{D}^3\psi(0)$ are finite. □

LEMMA 4. *If*

$$\lim_{r\to 1-}(1-r)\sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)\,r^n = 0 \tag{8}$$

then

$$\lim_{r\to 1-}\sum_{n=1}^{\infty}\frac{a_n\cos nx + b_n\sin nx}{n^2}\,r^n \tag{9}$$

exists. If further (8) *holds for all x then the function defined by* (9) *is* (CG).

This is a result of Verblunsky [13, Lemma 21] in the present notation.

LEMMA 5. *If $a_n = o(n)$, $b_n = o(n)$ and* (8) *holds and if ϕ is defined by* (4) *then $D^2\phi(x)$ exists. If further ϕ is R-smooth at x of order* 4 *then ϕ is smooth at x of order* 4.

PROOF. By Lemma 4 series (3) is A-summable at x and so since its coefficients are $o(1/n)$ it converges [14 vol. I, p. 81 Theorem 1.38]. Hence (3) is $(R,2)$ summable at x and so $D^2\phi(x)$ exists.

As in Lemma 2 we may suppose that $x = 0 = \phi(x) = D^2\phi(x)$, and $\phi(x) = \phi(-x)$; since ϕ is R-smooth at 0 of order 4 $\phi(x) = o(x^2)$ and $\phi(2x) - 4\phi(x) = o(x^3)$, as $x \longrightarrow 0$. Proceeding as in Lemma 2 this gives

$$\phi(t) - 4^{n+1}\phi\left(\frac{t}{2^{n+1}}\right) = \sum_{i=0}^{n} o\left(\frac{t^3}{2^{i+3}}\right);$$

and so as in Lemma 2 letting $n \longrightarrow \infty$ we get

$$\phi(t) = o(t^3),$$

Which shows ϕ to be smooth at 0 of order 4. □

LEMMA 6. *Under the hypotheses of Lemma* 5, *if ψ is given by* (6) *then $D^1\psi(x)$ exists. If further ψ is R-smooth at x of order* 3 *then ψ is smooth at x of order* 3.

PROOF. The proof is similar to that of Lemma 5. In fact since series (3) converges at x with coefficients $o(1/n)$, $D^1\psi(x)$ exists. The rest follows the proof of Lemma

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5, replacing Lemma 2 by Lemma 3, ϕ by ψ and with simple modifications in the arguments. □

3. **The Main Results.** We now state and prove the main results.

THEOREM 1. *Hypotheses:* $a_n = o(n)$, $b_n = o(n)$, *the upper and lower* $(R, 4)$ *sums of* (1) *are finite except on a countable set* E, *at all points of* E (8) *holds, and at all points of* E, ϕ, *given by* (4), *is R-smooth of order* 4.

Conclusion: (1) *is a* P^4*-Fourier series.*

PROOF. Since (1) has finite upper and lower $(R, 4)$ sums except on E, $\overline{RD}^4\phi$ and $\underline{RD}^4\phi$ are finite except on E. Hence by [6, Theorem 1] $\phi_{(4)}$ exists almost everywhere. If then

$$f(x) = \phi_{(4)}(x) + \frac{1}{2}a_0, \quad \text{if } \phi_{(4)}(x) \text{ exists,}$$
$$= 0, \text{ elsewhere,}$$

then (1) is $(R, 4)$ summable to f almost everywhere.

If $x \notin E$ then by Lemma 2 $\overline{D}^4\phi(x)$ and $\underline{D}^4\phi(x)$ are finite and so by [8. Corollary 1] (8) holds. Hence, by hypothesis, (8) holds everywhere and so by Lemma 4, (3) is A-summable for all x. However the coefficients in (3) are $o(1/n)$ and so the series converges. Hence, by Lemma 4, the sum of this series is (CG).

Define $G(x)$ to be the sum of (3) and $\psi(x)$ as in (6). Since, by hypotheses the coefficients in (3) are $o(1/n)$ we have that $D^1\psi = G$, [14 vol. I, p. 322 Theorem 2.18]. Also, by [14 vol. I, p. 320 Theorem 2.8] ψ is smooth and so $D^1\psi = \psi'$; thus $\psi' = G$. By a similar argument ϕ'' exists everywhere and, being G, is (CG).

Further if $x \in E$, ϕ is R-smooth at x of order 4 and (8) holds. Hence, by Lemma 5, ϕ is smooth at x of order 4 if $x \in E$. Thus ϕ has the following properties:

(i) ϕ is continuous,

(ii) ϕ'' is (CG),

(iii) $\phi_{(4)}$ exists almost everywhere,

(iv) $\overline{D}^4\phi$ and $\underline{D}^4\phi$ are finite except on E,

(v) ϕ is smooth of order 4 at every point of E;

The same properties are satisfied by $F + \phi$, where $F(x) = (1/2)a_0x^4/4!$ and so by [7, Theorem 5.1] $D^4(F + \phi)$ is P^4-integrable over $(\alpha_i; x)$, $1 \leqq i \leqq 4$, where $\alpha \leqq \alpha_1 \leqq \alpha_2 \leqq \alpha_3 \leqq \alpha_4 \leqq \beta$, $[\alpha, \beta]$ being any closed interval. Moreover if $\alpha_r \leqq x < \alpha_{r+1}$ then

$$(-1)^r \int_{(\alpha_i)}^{x} D^4(F + \phi)d_4t = (F + \phi)(x) - \sum_{i=1}^{4} \lambda(x; \alpha_i)(F + \phi)(\alpha_i),$$

where

$$\lambda(x; \alpha_i) = \prod_{\substack{j=1 \\ j \neq i}}^{4} \left(\frac{x - \alpha_j}{\alpha_i - \alpha_j}\right)$$

Taking $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (-4\pi, -2\pi, 2\pi, 4\pi)$ and observing that $D^4(F + \phi) = (1/2)a_0 + \phi_{(4)} = f$, almost everywhere, we have that

$$\int_{(\alpha_i)}^{0} f(t)d_4t = F(0) + \phi(0) - \sum_{i=1}^{4} \lambda(0; \alpha_i)F(\alpha_i) - \sum_{i=1}^{4} \lambda(0; \alpha_i)\phi(\alpha_i)$$

$$= \sum_{i=1}^{4} (-\alpha_i)\{V_4(F; \alpha_1, \alpha_2, 0, \alpha_3, \alpha_4) - V_4(\phi(0); \alpha_1, \alpha_2, 0, \alpha_3, \alpha_4)\}$$

where $V_4(g; x_0, x_1, x_2, x_3, x_4)$ is the fourth divided difference of g at the points x_i, $0 \leqq i \leqq 4$; if g is a constant the quantity is zero, while if $g(x) = x^4$ it is 1. Hence

$$(10) \qquad \int_{(\alpha_i)}^{0} f(t)d_4t = \frac{a_0}{2} \cdot \frac{1}{4!}\alpha_1\alpha_2\alpha_3\alpha_4, \quad \text{or } a_0 = \frac{4!}{(2!)^2 2^3 \pi^4} \int_{(\alpha_i)}^{0} f(t)d_4t$$

We now obtain the expressions similar to (10) for a_n, b_n, $n \geqq 1$ and for this we consider the formal product of (1) with $\cos mx$, m a fixed positive integer. To do this multiply (1) by $\cos mx$ and replace in each of the terms obtained the expressions $\sin nx \cos mx$, $\cos nx \cos mx$ by sums of cosines and sines; on rearranging the terms we get a series

$$(11) \qquad \frac{1}{2}u_0 + \sum_{n=1}^{\infty} (u_n \cos nx + v_n \sin nx) = \sum_{n=0}^{\infty} w_n(x), \quad \text{say},$$

where $u_0 = a_m$, $u_n = 1/2(a_{n-m} + a_{n+m})$, $v_n = 1/2(b_{n-m} + b_{n+m})$, $n \geqq 1$, if we agree that $a_{-r} = a_r$, $b_{-r} = -b_r$.

Since $a_n = o(n)$ and $b_n = o(n)$ we also have that $u_n = o(n)$ and $v_n = o(n)$. Denote by $s_n(x)$, respectively $\sigma_n(x)$, the partial sums, respectively the $(C, 1)$ means, of (1) then $s_n(x) \cos mx$, respectively $\sigma_n(x) \cos mx$, are their partial sums, respectively $(C, 1)$ means, of the series obtained by multiplying (1) by $\cos mx$. Now let $t_n(x)$ respectively $\tau_n(x)$ be the partial sums, respectively $(C, 1)$ means, of (11) then if $n > m$

$$|s_n(x) \cos mx - t_n(x)| \leqq 1/2 \sum_{i=n-m+1}^{n+m} (|a_i| + |b_i|) = o(n)$$

Hence

$$|\dot{\sigma}_n(x) \cos mx - \tau_n(x)| = \frac{1}{n+1} \left| \sum_{r=0}^{n} \binom{n-r}{n-r} s_r(x) \cos mx - t_r(x) \right|$$

$$n$$

$$= \frac{1}{n+1} \sum_{r=0}^{n} o(r) = o(n).$$

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So the series

$$\sum_{n=0}^{\infty} \big(w_n(x) - \cos mx\, c_n(x)\big) \tag{12}$$

is $(C,1)$ summable to 0 for all x. Hence by [14 Vol. II p. 64 Theorem 2.20] the series is also $(R,4)$ summable to 0. Since (1) has finite upper and lower $(R,4)$ sums except on E and since

$$u_n \cos nx + v_n \sin nx = [w_n(x) - c_n(x)\cos mx] + [a_n \cos nx + b_n \sin nx]\cos mx \tag{13}$$

it follows that (11) also has finite upper and lower $(R,4)$ sums except in E. By the same argument the sum $\phi(x)$ of the series $\sum_{n=1}^{\infty}(u_n \cos nx + v_n \sin nx)/n^4$ is R-smooth of order 4 at all points of E. Further, since (12) is $(C,1)$ summable to 0 for all x by [14 Vol. I, p. 80 Theorem 1.34] the series (12) is A-summable to zero for all x and so

$$\lim_{r\to 1-}(1-r)\sum_{n=0}^{\infty}\big(w_n(x) - c_n(x)\cos mx\big)r^n = 0$$

for all x. Hence from (8) and (13)

$$\lim_{r\to 1-}(1-r)\sum_{n=1}^{\infty}\big(u_n \cos nx + v_n \sin nx\big)r^n = 0, \quad x \in E.$$

Hence the hypotheses of Theorem 1 are satisfied by (11) and so from the first part of the proof and in particular (10),

$$u_0 = \frac{4!}{(2!)^3 2^3 \pi^4}\int_{(\alpha_i)}^{0} g(t)d_4 t \tag{14}$$

where g is almost everywhere the $(R,4)$ sum of (11). Since the $(R,4)$ sum of (12) is everywhere 0 and the $(R,4)$ sum of (1) is almost everywhere $f(x)$ we see that $g(x) = f(x)\cos mx$ almost everywhere. Since $u_0 = a_m$, (14) gives

$$a_m = \frac{4!}{(2!)^3 2^3 \pi^4}\int_{(\alpha_i)}^{0} f(t)\cos mt\, d_4 t.$$

In a similar manner

$$b_m = \frac{4!}{(2!)^3 2^3 \pi^4}\int_{(\alpha_i)}^{0} f(t)\sin mt\, d_4 t. \qquad \square$$

THEOREM 2. *Hypotheses:* $a_n = o(1)$, $b_n = o(1)$, *upper and lower* $(R,3)$ *sums of* (1) *are finite except on a countable set* E, *at all points of* E (8) *holds and at all points of* E, ψ, *given by* (6), *is* R*-smooth of order* 3.

Conclusion. (1) *is a* P^3*-Fourier series.*

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The proof is similar to that of Theorem 1.

The results of this paper together with those in James [5] and Mukhopadhyay [7] solve the coefficient problem for (R, k)-summable trigonometric series $1 \leqq k \leqq 4$. The question of (R, k)-summable series with $k > 4$ seems to be a difficult one; the basic results of Verblunsky [11, 12] depend critically on the particular numerical coefficients in $\Delta^k(f; x, h)$, $k = 3, 4$, as do certain of our arguments. When $k > 4$ these coefficients no longer possess these properties and a completely new approach will be required.

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Department of Mathematics
University of British Columbia
Vancouver, Canada

Department of Mathematics
University of Burdwan
Burdwan, India

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