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# METANILPOTENT VARIETIES OF GROUPS

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**Abstract**

For each positive integer $n$ let $\mathbf{N}_{2,n}$ denote the variety of all groups which are nilpotent of class at most 2 and which have exponent dividing $n$. For positive integers $m$ and $n$, let $\mathbf{N}_{2,m}\mathbf{N}_{2,n}$ denote the variety of all groups which have a normal subgroup in $\mathbf{N}_{2,m}$ with factor group in $\mathbf{N}_{2,n}$. It is shown that if $G \in \mathbf{N}_{2,m}\mathbf{N}_{2,n}$, where $m$ and $n$ are coprime, then $G$ has a finite basis for its identities.

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## 1. Introduction

The finite basis question for a group $G$ asks whether the set of all identities of $G$ is equivalent to some finite set of identities. (We refer to [13] for terminology and basic results concerned with varieties of groups, but we use the term 'identity' rather than 'law'.) Between 1970 and 1973 a number of examples were published of groups for which the answer is negative: see [9] for references covering this period and see [5] for an account of more recent results. In the majority of these examples, the groups are metanilpotent (that is, nilpotent-by-nilpotent) and have finite exponent. In the simplest cases the groups belong to the variety $\mathbf{N}_{2,4}\mathbf{N}_{2,4}$: here, for any positive integer $n$, $\mathbf{N}_{2,n}$ denotes the variety of all groups which are nilpotent of class at most 2 and have finite exponent dividing $n$, and, for varieties $\mathbf{U}$ and $\mathbf{V}$, $\mathbf{VU}$ denotes the product variety, consisting of all groups which have a normal subgroup in $\mathbf{V}$ with factor group in $\mathbf{U}$. However, there are also many positive results. In particular, Lyndon [11] showed that every nilpotent group has a finite basis for its identities and Krasil'nikov [10] showed, much more generally, that the same is true for every nilpotent-by-abelian group.

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In the negative examples mentioned above in which $G$ is metanilpotent of finite exponent there is no bound on the class of the nilpotent subgroups of $G$. It seems still to be an open question whether a soluble group of finite exponent, in which the nilpotent subgroups have bounded class, has a finite basis for its identities. Our main result gives a positive answer in many simple cases.

THEOREM A. *Let $G \in \mathbf{N}_{2,m}\mathbf{N}_{2,n}$ where $m$ and $n$ are coprime positive integers. Then $G$ has a finite basis for its identities.*

A special case of this result was proved by Brady, Bryce and Cossey [2]: they showed that $G$ has a finite basis for its identities if $G$ belongs to $\mathbf{A}_m\mathbf{N}_{2,n}$, where $m$ and $n$ are coprime positive integers and $\mathbf{A}_m$ denotes the variety of all abelian groups of exponent dividing $m$. Theorem A solves a problem posed by Kovács and Newman [9]. The method adopted in [2] depends upon an analysis of the irreducible linear groups in $\mathbf{N}_{2,n}$, in prime characteristic not dividing $n$, and develops ideas of Higman [8]. However, at about the same time, Cohen [4] introduced a quite different method for tackling the finite basis question, dependent on the combinatorics of ordered sets. Cohen used this method to prove that every metabelian group has a finite basis for its identities, and the method was developed by others in later work such as [3, 10] and [12]. We apply similar methods here, for which we need the idea of a well-quasi-ordered set, defined as follows.

A *quasi-order* on a set $W$ is a binary relation $\preccurlyeq$ on $W$ which is reflexive and transitive. (We do not assume that $x \preccurlyeq y$ and $y \preccurlyeq x$ imply $x = y$, as in a partial order. Furthermore, we give no meaning to $\prec$, only to $\preccurlyeq$.) As shown in [6], the following two properties of a quasi-ordered set $(W, \preccurlyeq)$ are equivalent:

(i) for every infinite sequence $w_1, w_2, \ldots$ of elements of $W$ there exist $i$ and $j$ with $i < j$ such that $w_i \preccurlyeq w_j$;

(ii) for every subset $X$ of $W$ there exists a finite subset $Y$ of $X$ such that for every element $x$ of $X$ there exists $y \in Y$ such that $y \preccurlyeq x$.

If (either of) these conditions hold then $(W, \preccurlyeq)$ is said to be *well-quasi-ordered*. If the relation $\preccurlyeq$ is a total (or linear) order then we obtain the more familiar idea of a well-ordered set.

We need to apply this idea to bilinear forms. Let $K$ be a non-zero, finite, commutative and associative ring, with identity element, and let $S$ be a finitely generated $K$-module. By an *S-form* we mean a pair $(V, \theta)$ consisting of a finitely generated, non-zero, free $K$-module $V$ and a $K$-bilinear mapping $\theta : V \times V \to S$. If $(V, \theta)$ and $(V', \theta')$ are $S$-forms we write $(V, \theta) \preccurlyeq (V', \theta')$ if there is a $K$-module monomorphism $\xi : V \to V'$ such that $\theta(v_1, v_2) = \theta'(v_1\xi, v_2\xi)$ for all $v_1, v_2 \in V$. The first step in the proof of Theorem A is the following result (or, to be precise, a more technical version of this result stated in Section 3).

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THEOREM B. *The set of all S-forms is well-quasi-ordered under the relation* $\preccurlyeq$.

Strictly speaking, the class of all $S$-forms is not a set. However, Theorem B can be rephrased to say that every set of $S$-forms is well-quasi-ordered under $\preccurlyeq$.

A result like this for trilinear alternating forms over a finite field was obtained by Atkinson [1] in order to prove a different finite basis result.

The finite basis question for a group $G$ is equivalent to the finite basis question for the variety $\mathbf{V}$ generated by $G$ (see [13]). Furthermore, if $F$ is a free group of countably infinite rank and $\mathbf{V}(F)$ denotes the verbal subgroup of $F$ corresponding to $\mathbf{V}$ then every subvariety of $\mathbf{V}$ is finitely based if and only if $\mathbf{V}$ is finitely based and the maximal condition holds for fully invariant subgroups of the relatively free group $F/\mathbf{V}(F)$. Much of the proof of Theorem A is concerned with establishing that the maximal condition holds in some closely related situations, typically for certain ideals in group algebras.

Let $n$ be a positive integer and let $A$ be a free group of countably infinite rank in the variety $\mathbf{N}_{2,n}$. Let $\mathbb{F}$ be a field of characteristic which does not divide $n$. Let $\Psi$ be the set of all endomorphisms of $A$ and, for each positive integer $r$, let $A^{\times r}$ denote the $r$-th direct power of $A$. Each element $\psi$ of $\Psi$ acts 'diagonally' on $A^{\times r}$ by $(a_1, \dots, a_r)\psi = (a_1\psi, \dots, a_r\psi)$ for all $a_1, \dots, a_r \in A$, and this action can be extended to the group algebra $\mathbb{F}(A^{\times r})$ in the obvious way. Using the version of Theorem B mentioned above we shall prove the following result.

THEOREM C. *For each positive integer r, the maximal condition holds for* $\Psi$*-closed left ideals of* $\mathbb{F}(A^{\times r})$.

If $U$ is a left $C$-module, for some algebra $C$, and if there is also an action of $\Psi$ on $U$, we call $U$ a $(C, \Psi)$-module. The concepts of $(C, \Psi)$-submodule and homomorphism of $(C, \Psi)$-modules are defined in the obvious way.

The algebra $\mathbb{F}(A \times A)$ is isomorphic to $\mathbb{F}A \otimes \mathbb{F}A$ (where the tensor product is taken over $\mathbb{F}$) under the linear map which sends $(a, a')$ to $a \otimes a'$ for all $a, a' \in A$. We shall identify these two algebras and write $(a, a')$ or $a \otimes a'$ interchangeably. Let $R$ be the subspace of $\mathbb{F}(A \times A)$ spanned by all elements of the form $a \otimes a$ and $a \otimes a' + a' \otimes a$ for $a, a' \in A$. It is easily verified that $R$ is a subalgebra of $\mathbb{F}(A \times A)$. Thus we may regard $\mathbb{F}(A \times A)$ as a left $R$-module and, indeed, as an $(R, \Psi)$-module. Clearly $R$ is an $(R, \Psi)$-submodule of $\mathbb{F}(A \times A)$. The last main step in the proof of Theorem A is the following result.

THEOREM D. *The maximal condition holds for* $(R, \Psi)$*-submodules of* $\mathbb{F}(A \times A)$ *which contain* $R$.

The vector space $\mathbb{F}(A \times A)/R$ is isomorphic to the exterior square $\mathbb{F}A \wedge \mathbb{F}A$, which can therefore be given the structure of an $(R, \Psi)$-module. Thus Theorem D gives the

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following result.

COROLLARY. *The maximal condition holds for* $(R, \Psi)$*-submodules of* $\mathbb{F}A \wedge \mathbb{F}A$.

Theorems B, C and D will be proved in Sections 3, 5 and 6, respectively. In Section 2 we show how Theorem A can be derived from Theorems C and D.

## 2. The derivation of Theorem A

In this section we assume Theorems C and D, and we obtain Theorem A from these results.

One step in the proof of Theorem A is the special case proved in [2]. We could, of course, assume this result, but in order to illustrate our method in a comparatively simple case we first prove this special case.

Let $\mathbf{U}$ and $\mathbf{V}$ be varieties of groups. Let $A$ be a free group in $\mathbf{U}$ on a free generating set $\{x_i : i \in \mathbb{N}\}$ and let $B$ be a free group in $\mathbf{V}$ on a free generating set $\{y_i^a : i \in \mathbb{N},\ a \in A\}$. For each $i$, the element $y_i^1$ is also written as $y_i$. Each element $a'$ of $A$ induces an automorphism of $B$ in which $y_i^a \mapsto y_i^{aa'}$ for all $i \in \mathbb{N}$, $a \in A$. Accordingly we can form the semidirect product $BA$, a split extension of $B$ by $A$ in which the original action of $A$ on $B$ becomes conjugation. We denote this group $BA$ by $F_{\text{split}}(\mathbf{V}, \mathbf{U})$. The group has the following universal property implicit in [14] and straightforward to prove directly.

LEMMA 2.1. *Let $G$ be a split extension of a group $B_1$ in $\mathbf{V}$ by a group $A_1$ in $\mathbf{U}$. Then every pair of mappings $\{x_i : i \in \mathbb{N}\} \to A_1$, $\{y_i : i \in \mathbb{N}\} \to B_1$ extends (uniquely) to a homomorphism $F_{\text{split}}(\mathbf{V}, \mathbf{U}) \to G$.*

LEMMA 2.2. *Let $\mathbf{U}$ and $\mathbf{V}$ be locally finite varieties of groups of coprime exponents and write $W = F_{\text{split}}(\mathbf{V}, \mathbf{U})$. Let $\mathbf{S}$ be a subvariety of $\mathbf{VU}$. Then $\mathbf{S}$ is generated by the group $W/\mathbf{S}(W)$, where $\mathbf{S}(W)$ is the verbal subgroup of $W$ corresponding to $\mathbf{S}$.*

PROOF. Since $\mathbf{S}$ is locally finite it is generated by the finite groups it contains. By the Schur-Zassenhaus Theorem, each such finite group $G$ is a split extension of a group in $\mathbf{V}$ by a group in $\mathbf{U}$. It follows, by Lemma 2.1, that $G$ is a homomorphic image of $W/\mathbf{S}(W)$. Therefore $W/\mathbf{S}(W)$ generates $\mathbf{S}$. □

LEMMA 2.3. *Let $F$ be a relatively free group and let $U$ be an abelian fully invariant subgroup of $F$ of exponent dividing a positive integer $m$. Suppose that $U$ contains an infinite strictly ascending chain of fully invariant subgroups of $F$. Then there exists a prime $p$ dividing $m$ such that $U/U^p$ contains an infinite strictly ascending chain of fully invariant subgroups of $F/U^p$.*

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PROOF. Let $\Omega$ be the set of all endomorphisms of $F$, with $\Omega$ regarded as a set of operators. If $V$ is any fully invariant subgroup of $F$ then, since the endomorphisms of $F/V$ are precisely those induced by elements of $\Omega$, $F/V$ may be regarded as an $\Omega$-group and the $\Omega$-subgroups of $F/V$ are precisely the fully invariant subgroups of $F/V$, each being of the form $W/V$ for some $\Omega$-subgroup $W$ of $F$ containing $V$. Observe that if $N$ is an $\Omega$-subgroup of $U$ then, since $U$ contains an infinite strictly ascending chain of $\Omega$-subgroups, either $N$ or $U/N$ contains such a chain.

Since $U$ is abelian of exponent dividing $m$, we may write $U$ as a finite direct product $U = U_1 \times \cdots \times U_k$ where each $U_i$ is a non-trivial $\Omega$-subgroup of prime-power exponent dividing $m$. By repeated use of the previous observation and isomorphisms of $\Omega$-groups, we find that there exists $i \in \{1, \dots, k\}$ such that $U/\prod_{j\neq i} U_j$ contains an infinite strictly ascending chain of $\Omega$-subgroups. Thus it suffices to prove the lemma in the case where $U$ has exponent $p^s$ for some prime $p$ and positive integer $s$. By the same observation applied to the chain $U \geq U^p \geq \cdots \geq U^{p^s} = \{1\}$, there exists $r \in \{0, 1, \dots, s-1\}$ such that $U^{p^r}/U^{p^{r+1}}$ contains an infinite strictly ascending chain of $\Omega$-subgroups. Thus there are $\Omega$-subgroups $W_1, W_2, \dots$ of $U$ satisfying

$$U^{p^{r+1}} \leq W_1 < W_2 < \cdots < U^{p^r}.$$

Let $\chi : U \to U^{p^r}$ be the homomorphism defined by $u\chi = u^{p^r}$ for all $u \in U$. Note that $\chi$ is surjective. Thus $U^p \leq W_1\chi^{-1} < W_2\chi^{-1} < \cdots < U$. It is easily verified that $\chi$ is a homomorphism of $\Omega$-groups. Thus each $W_i\chi^{-1}$ is an $\Omega$-group and $W_1\chi^{-1}/U^p < W_2\chi^{-1}/U^p < \cdots$ is an infinite strictly ascending chain of fully invariant subgroups of $F/U^p$ contained in $U/U^p$. □

We shall now obtain the finite basis result of [2]. For any variety $\mathbf{V}$, $F(\mathbf{V})$ denotes the free group of $\mathbf{V}$ of countably infinite rank.

THEOREM 2.4 ([2]). *Let m and n be coprime positive integers. Then the subvarieties of* $\mathbf{A}_m\mathbf{N}_{2,n}$ *are finitely based.*

PROOF. Since $\mathbf{A}_m\mathbf{N}_{2,n}$ is finitely based by [7], it suffices to show that $F(\mathbf{A}_m\mathbf{N}_{2,n})$ satisfies the maximal condition on fully invariant subgroups. Write $H = F(\mathbf{A}_m\mathbf{N}_{2,n})$ and $U = \mathbf{N}_{2,n}(H)$. Thus $H/U \cong F(\mathbf{N}_{2,n})$. By [11], $H/U$ satisfies the maximal condition on fully invariant subgroups. Thus it suffices to show that the maximal condition holds for fully invariant subgroups of $H$ contained in $U$. By Lemma 2.3, it suffices to show that for each prime $p$ dividing $m$ the maximal condition holds for fully invariant subgroups of $H/U^p$ contained in $U/U^p$. But $H/U^p \cong F(\mathbf{A}_p\mathbf{N}_{2,n})$, so it suffices to show that the minimal condition holds for subvarieties of $\mathbf{A}_p\mathbf{N}_{2,n}$ which contain $\mathbf{N}_{2,n}$.

Let $W = F_{\text{split}}(\mathbf{A}_p, \mathbf{N}_{2,n})$ and write $W = BA$ where $A = \langle x_i : i \in \mathbb{N}\rangle \cong F(\mathbf{N}_{2,n})$ and $B = \langle y_i^a : i \in \mathbb{N},\ a \in A\rangle$. Thus $B$ is free in $\mathbf{A}_p$. By Lemma 2.2, the subvarieties

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of $\mathbf{A}_p\mathbf{N}_{2,n}$ which contain $\mathbf{N}_{2,n}$ are in one-one correspondence with the corresponding verbal subgroups of $W$, and these verbal subgroups are contained in $B$. Thus it suffices to prove that the maximal condition holds for fully invariant subgroups of $W$ contained in $B$.

We can write $B$ additively as a vector space over $\mathbb{F}_p$, the field with $p$ elements, and $B$ has basis $\{y_i^a : i \in \mathbb{N},\ a \in A\}$. Let $T$ be the subspace with basis $\{y_1^a : a \in A\}$. There is an $\mathbb{F}_p$-space isomorphism $\mu : \mathbb{F}_pA \to T$ satisfying $a\mu = y_1^a$ for all $a \in A$. Hence we can give $T$ the structure of a left $\mathbb{F}_pA$-module in such a way that $\mu$ is a module isomorphism. Let $\Psi$ be the set of all endomorphisms of $A$. By Lemma 2.1, each element $\psi$ of $\Psi$ can be extended to an endomorphism of $W$ by taking $y_i\psi = y_i$ for each $i$. Thus $\Psi$ acts on $W$. Clearly $T$ is $\Psi$-closed and the map $\mu : \mathbb{F}_pA \to T$ is an isomorphism of $(\mathbb{F}_pA, \Psi)$-modules.

For each $a \in A$, let $\xi_a$ be the endomorphism of $W$ satisfying $x_i\xi_a = x_i$ for all $i$, $y_1\xi_a = y_1^a$ and $y_i\xi_a = y_i$ for all $i > 1$. Clearly $T$ is invariant under each $\xi_a$, and $\xi_a$ acts on $T$ in the same way as $a$ acts (when $T$ is regarded as a left $\mathbb{F}_pA$-module). It follows that if $V$ is a fully invariant subgroup of $W$ then $V \cap T$ is an $(\mathbb{F}_pA, \Psi)$-submodule of $T$.

For each $i, j \in \mathbb{N}$, let $\delta_{ij}$ be the endomorphism of $W$ determined by $x_k\delta_{ij} = x_k$ for all $k$, $y_i\delta_{ij} = y_j$ and $y_k\delta_{ij} = 1$ for all $k \in \mathbb{N} \setminus \{i\}$. Let $V$ be a fully invariant subgroup of $W$ contained in $B$ and let $v \in V$. Then there exists $r \in \mathbb{N}$ such that $v$ belongs to the span of $\{y_i^a : 1 \leq i \leq r, a \in A\}$. We have $v = v\delta_{11}\delta_{11} + v\delta_{21}\delta_{12} + \cdots + v\delta_{r1}\delta_{1r}$, where $v\delta_{11}, v\delta_{21}, \ldots, v\delta_{r1} \in V \cap T$. Thus $V$ is generated as a fully invariant subgroup by $V \cap T$.

Suppose that $V_1 \leq V_2 \leq \ldots$ is an ascending chain of fully invariant subgroups of $W$ contained in $B$. Then $V_1 \cap T \leq V_2 \cap T \leq \cdots$ is an ascending chain of $(\mathbb{F}_pA, \Psi)$-modules. Hence $(V_1 \cap T)\mu^{-1} \leq (V_2 \cap T)\mu^{-1} \leq \cdots$ is an ascending chain of $\Psi$-closed left ideals of $\mathbb{F}_pA$. By Theorem C, this chain becomes stationary. Therefore, so does $V_1 \cap T \leq V_2 \cap T \leq \cdots$, and so does $V_1 \leq V_2 \leq \cdots$, which completes the proof of Theorem 2.4. □

PROOF OF THEOREM A. Let $m$ and $n$ be coprime positive integers, and write $F = F(\mathbf{N}_{2,m}\mathbf{N}_{2,n})$. By [7], $\mathbf{N}_{2,m}\mathbf{N}_{2,n}$ is finitely based. Thus it suffices to show that $F$ satisfies the maximal condition on fully invariant subgroups. Let $U$ be the verbal subgroup of $F$ corresponding to $\mathbf{A}_m\mathbf{N}_{2,n}$. Thus $F/U \cong F(\mathbf{A}_m\mathbf{N}_{2,n})$ and, by Theorem 2.4, it suffices to show that the maximal condition holds for fully invariant subgroups of $F$ contained in $U$. By Lemma 2.3 it suffices to show that, for each prime $p$ dividing $m$, the maximal condition holds for fully invariant subgroups of $F/U^p$ contained in $U/U^p$. Let $\mathbf{V}$ be the variety of all groups $G$ such that $G$ is nilpotent of class at most two, $G$ has exponent dividing $m$ and $G'$ has exponent dividing $p$. Thus $F/U^p \cong F(\mathbf{V}\mathbf{N}_{2,n})$. It suffices to show that the minimal condition holds for subvarieties of $\mathbf{V}\mathbf{N}_{2,n}$ which contain $\mathbf{A}_m\mathbf{N}_{2,n}$.

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Let $W = F_{\text{split}}(\mathbf{V}, \mathbf{N}_{2,n})$ and write $W = BA$ where $A = \langle x_i : i \in \mathbb{N} \rangle \cong F(\mathbf{N}_{2,n})$ and $B = \langle y_i^a : i \in \mathbb{N},\ a \in A \rangle$. Thus $B$ is free in $\mathbf{V}$. By Lemma 2.2, the subvarieties of $\mathbf{VN}_{2,n}$ which contain $\mathbf{A}_m\mathbf{N}_{2,n}$ are in one-one correspondence with the corresponding verbal subgroups of $W$, and these verbal subgroups are contained in $B'$. Thus it suffices to prove that the maximal condition holds for fully invariant subgroups of $W$ contained in $B'$. If $B' = \{1\}$ (as occurs when $p = 2$ and $m$ is not divisible by 4) then the result is trivial. Thus we may assume that $B' \neq \{1\}$.

We can write $B'$ additively as a vector space over $\mathbb{F}_p$ spanned by $\{[y_i^a, y_j^{a'}] : i, j \in \mathbb{N},\ a, a' \in A\}$. Let $T_1$ be the subspace spanned by $\{[y_1^a, y_1^{a'}] : a, a' \in A\}$ and let $T_2$ be the subspace spanned by $\{[y_1^a, y_2^{a'}] : a, a' \in A\}$. Thus $T_1$ has basis $\{[y_1^a, y_1^{a'}] : a, a' \in A,\ a > a'\}$, where $>$ is an arbitrary total order on $A$, and $T_2$ has basis $\{[y_1^a, y_2^{a'}] : a, a' \in A\}$. Thus there are $\mathbb{F}_p$-space isomorphisms $\mu_1 : \mathbb{F}_pA \wedge \mathbb{F}_pA \to T_1$ and $\mu_2 : \mathbb{F}_p(A \times A) \to T_2$ satisfying $(a \wedge a')\mu_1 = [y_1^a, y_1^{a'}]$ and $(a \otimes a')\mu_2 = [y_1^a, y_2^{a'}]$ for all $a, a' \in A$. Hence, with $R$ defined as in Section 1, we can give $T_1$ the structure of a left $R$-module and $T_2$ the structure of a left $\mathbb{F}_p(A \times A)$-module in such a way that $\mu_1$ and $\mu_2$ are module isomorphisms. Let $\Psi$ be the set of all endomorphisms of $A$. As in the proof of Theorem 2.4, $\Psi$ acts on $W$. Clearly $T_1$ and $T_2$ are $\Psi$-closed, $\mu_1$ is an isomorphism of $(R, \Psi)$-modules, and $\mu_2$ is an isomorphism of $(\mathbb{F}_p(A \times A), \Psi)$-modules.

For $a \in A$, let $\xi_a$ be the endomorphism of $W$ satisfying $x_i\xi_a = x_i$ for all $i$, $y_1\xi_a = y_1^a$ and $y_i\xi_a = y_i$ for all $i > 1$. For $a, a' \in A$, let $\xi_{a+a'}$ be the endomorphism of $W$ satisfying $x_i\xi_{a+a'} = x_i$ for all $i$, $y_1\xi_{a+a'} = y_1^a y_1^{a'}$ and $y_i\xi_{a+a'} = y_i$ for all $i > 1$. Thus $T_1$ is invariant under each $\xi_a$ and under each $\xi_{a+a'}$. Furthermore, $\xi_a$ acts on $T_1$ in the same way as $a \otimes a$ acts, while $\xi_{a+a'}$ acts on $T_1$ in the same way as $(a + a') \otimes (a + a')$ acts. It is easily verified that $R$ is spanned by the elements $a \otimes a$ and $(a+a') \otimes (a+a')$ for $a, a' \in A$. It follows that if $V$ is a fully invariant subgroup of $W$ then $V \cap T_1$ is an $(R, \Psi)$-submodule of $T_1$.

For $a, a' \in A$, let $\xi_{a,a'}$ be the endomorphism of $W$ determined by $x_i\xi_{a,a'} = x_i$ for all $i$, $y_1\xi_{a,a'} = y_1^a$, $y_2\xi_{a,a'} = y_2^{a'}$ and $y_i\xi_{a,a'} = y_i$ for all $i > 2$. Clearly $T_2$ is invariant under each $\xi_{a,a'}$. Furthermore, $\xi_{a,a'}$ acts on $T_2$ in the same way as $a \otimes a'$ acts. It follows that if $V$ is a fully invariant subgroup of $W$ then $V \cap T_2$ is an $(\mathbb{F}_p(A \times A), \Psi)$-submodule of $T_2$.

For each $i, j \in \mathbb{N}$, let $\delta_{ij}$ be the endomorphism of $W$ determined by $x_k\delta_{ij} = x_k$ for all $k$, $y_i\delta_{ij} = y_j$ and $y_k\delta_{ij} = 1$ for all $k \in \mathbb{N} \setminus \{i\}$. For each $i, j, i', j' \in \mathbb{N}$ with $i \neq j$, let $\varepsilon_{ii',jj'}$ be the endomorphism of $W$ determined by $x_k\varepsilon_{ii',jj'} = x_k$ for all $k$, $y_i\varepsilon_{ii',jj'} = y_{i'}$, $y_j\varepsilon_{ii',jj'} = y_{j'}$ and $y_k\varepsilon_{ii',jj'} = 1$ for all $k \in \mathbb{N} \setminus \{i, j\}$.

Let $V$ be a fully invariant subgroup of $W$ contained in $B'$ and let $v \in V$. Then, for some $r \in \mathbb{N}$, we can write $v = v_1 + v_2$ where $v_1$ is in the span of $\{[y_i^a, y_i^{a'}] : 1 \leq i \leq r,\ a, a' \in A\}$ and $v_2$ is in the span of $\{[y_i^a, y_j^{a'}] : 1 \leq i < j \leq r,\ a, a' \in A\}$. Then it is easily verified that $v_1 = \sum_i v\delta_{i1}\delta_{1i}$ and

$$v - v_1 = v_2 \doteq \sum_{\substack{i,j \\ 1\le i<j\le r}} v_2\varepsilon_{i1,j2}\varepsilon_{1i,2j}.$$

Here $v\delta_{i1} \in V \cap T_1$ for all $i$ and $v_2\varepsilon_{i1,j2} \in V \cap T_2$ for all $i, j$. It follows that $V$ is generated as a fully invariant subgroup by $(V \cap T_1) \cup (V \cap T_2)$.

Suppose that $V_1 \le V_2 \le \cdots$ is an ascending chain of fully invariant subgroups of $W$ contained in $B'$. Then $V_1 \cap T_1 \le V_2 \cap T_1 \le \cdots$ is an ascending chain of $(R, \Psi)$-submodules of $T_1$ while $V_1 \cap T_2 \le V_2 \cap T_2 \le \cdots$ is an ascending chain of $(\mathbb{F}_p(A \times A), \Psi)$-submodules of $T_2$. Hence $(V_1 \cap T_1)\mu_1^{-1} \le (V_2 \cap T_1)\mu_1^{-1} \le \cdots$ is an ascending chain of $(R, \Psi)$-submodules of $\mathbb{F}_pA \wedge \mathbb{F}_pA$ and

$$(V_1 \cap T_2)\mu_2^{-1} \le (V_2 \cap T_2)\mu_2^{-1} \le \cdots$$

is an ascending chain of $\Psi$-closed left ideals of $\mathbb{F}_p(A \times A)$. By Theorem C and the Corollary to Theorem D, both of the last two chains become stationary. Hence $(V_1 \cap T_1) \cup (V_1 \cap T_2) \le (V_2 \cap T_1) \cup (V_2 \cap T_2) \le \cdots$ becomes stationary. Therefore $V_1 \le V_2 \le \cdots$ becomes stationary, which proves Theorem A. □

## 3. Bilinear forms

Let $K$ be a non-zero, finite, commutative and associative ring, with identity element 1. Unless otherwise stated all $K$-modules are finitely generated (therefore finite). Let $S$ be a $K$-module. An *$S$-form* is a pair $(V, \theta)$ consisting of a non-zero free $K$-module $V$ and a $K$-bilinear map $\theta : V \times V \to S$. A $K$-linear map $\xi : V \to V'$, where $(V, \theta)$ and $(V', \theta')$ are $S$-forms, is said to be a *homomorphism of $S$-forms* if $\theta(v_1, v_2) = \theta'(v_1\xi, v_2\xi)$ for all $v_1, v_2 \in V$. We write $\xi : (V, \theta) \to (V', \theta')$. The terms *isomorphism* and *monomorphism* are defined in the obvious way. We define a quasi-order $\preccurlyeq$ on the set of all $S$-forms by defining $(V, \theta) \preccurlyeq (V', \theta')$ if there is a monomorphism $\xi : (V, \theta) \to (V', \theta')$. The main result of this section is the following.

THEOREM B. *The set of all $S$-forms is well-quasi-ordered under the relation $\preccurlyeq$.*

Let $(V, \theta)$ be an $S$-form. For any subset $U$ of $V$ we define $P(U)$ to be the subset of $S \oplus S$ given by $P(U) = \{(\theta(v_1, v_2), \theta(v_2, v_1)) : v_1, v_2 \in U\}$, and we define $Q(U) \subseteq S$ by $Q(U) = \{\theta(v, v) : v \in U\}$. Also, for $U, U' \subseteq V$ we define $\theta(U, U') \subseteq S$ by $\theta(U, U') = \{\theta(u, u') : u \in U, u' \in U'\}$. Subsets $U$ and $U'$ are said to be *orthogonal* if $\theta(U, U') = \theta(U', U) = \{0\}$.

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LEMMA 3.1. *Let $V$ be a free $K$-module and let $v_1, \dots, v_l \in V$. Then there are free $K$-submodules $U_1$, $U_2$ of $V$ such that $V = U_1 \oplus U_2$,* $\operatorname{rank}(U_1) \le |K|l$, *and* $v_1, \dots, v_l \in U_1$.

PROOF. Take elements $x_1, \dots, x_m$ of $V$ where $m$ is minimal such that $\{x_1, \dots, x_m\}$ is contained in a $K$-basis of $V$ and $v_1$ belongs to the submodule $\langle x_1, \dots, x_m \rangle$. Write $v_1 = \sum_{i=1}^{m} \alpha_i x_i$ where each $\alpha_i$ is an element of $K$. If $m > |K|$ then there exist distinct $j, k \in \{1, \dots, m\}$ such that $\alpha_j = \alpha_k$ and we may replace $x_j$ and $x_k$ by $x_j + x_k$, contrary to the minimality of $m$. Thus $m \le |K|$. Let $W$ be a free $K$-submodule of $V$ such that $V = \langle x_1, \dots, x_m \rangle \oplus W$ and, for $i = 2, \dots, l$, write $v_i = v_i' + w_i$ where $v_i' \in \langle x_1, \dots, x_m \rangle$ and $w_i \in W$. The result follows by applying an inductive argument to $w_2, \dots, w_l$ in $W$. □

LEMMA 3.2. *Let $(V, \theta)$ be an $S$-form. Suppose that $W$ is a free $K$-submodule of $V$ and let $v_1, \dots, v_l \in V$. Then there are free $K$-submodules $W_1$, $W_2$ of $W$ such that $W = W_1 \oplus W_2$,* $\operatorname{rank}(W_1) \le 2|S|l$ *and $W_2$ is orthogonal to $\{v_1, \dots, v_l\}$.*

PROOF. We assume that $l = 1$ since the general case follows easily. We shall find free submodules $U_1$, $U_2$ of $W$ such that $W = U_1 \oplus U_2$, $\operatorname{rank}(U_1) \le |S|$ and $\theta(\{v_1\}, U_2) = \{0\}$. A similar argument gives $U_2 = U' \oplus U''$ with $\operatorname{rank}(U') \le |S|$ and $\theta(U'', \{v_1\}) = \{0\}$. The result follows with $W_1 = U_1 \oplus U'$ and $W_2 = U''$.

Take basis elements $x_1, \dots, x_m$ of $W$ where $m$ is maximal subject to $\theta(v_1, x_i) = 0$ for $i = 1, \dots, m$. Let $\{x_1, \dots, x_d\}$ be a basis of $W$ containing $\{x_1, \dots, x_m\}$. If $d - m > |S|$ then there exist distinct $j, k \in \{m + 1, \dots, d\}$ such that $\theta(v_1, x_j) = \theta(v_1, x_k)$ and we may extend $\{x_1, \dots, x_m\}$ to $\{x_1, \dots, x_m, x_j - x_k\}$, contrary to the maximality of $m$. Thus $d - m \le |S|$ and we may take $U_1 = \langle x_{m+1}, \dots, x_d \rangle$, $U_2 = \langle x_1, \dots, x_m \rangle$. □

Let $N$ be a positive integer and define $N^{[i]}$, for each non-negative integer $i$, by $N^{[0]} = 0$ and $N^{[i]} = N + N^2 + \cdots + N^i$ for $i > 0$. Let $(V, \theta)$ be an $S$-form and let $\{x_1, \dots, x_d\}$ be a $K$-basis of $V$. We shall assume, in such notation, that the elements $x_i$ are distinct (that is, $d = \operatorname{rank}(V)$) and that the basis is ordered as shown, corresponding to the ordered $d$-tuple $(x_1, \dots, x_d)$. Let $m$ be the non-negative integer which satisfies $N^{[m]} < d \le N^{[m+1]}$ and write $V_1 = \langle x_1, \dots, x_{N^{[1]}} \rangle, \dots$, $V_m = \langle x_{N^{[m-1]}+1}, \dots, x_{N^{[m]}} \rangle$, $V_{m+1} = \langle x_{N^{[m]}+1}, \dots, x_d \rangle$. Thus $\operatorname{rank}(V_i) = N^i$ for $i = 1, \dots, m$ and $0 < \operatorname{rank}(V_{m+1}) \le N^{m+1}$. For $i = 1, \dots, m + 1$, write $V_i^+ = V_i \oplus \cdots \oplus V_{m+1}$. We say that $(V, \theta)$ is $N$-*regular* with respect to the ordered basis $\{x_1, \dots, x_d\}$ if $P(V_i) = P(V_i^+)$ for $i = 1, \dots, m + 1$, $Q(V_i) = Q(V_i^+)$ for $i = 1, \dots, m + 1$, and $V_{i-1}$ and $V_{i+1}^+$ are orthogonal for $i = 2, \dots, m$. A decomposition $V = V_1 \oplus \cdots \oplus V_{m+1}$ with these properties, which is obtained from some ordered basis in the way described, is called an $N$-*regular decomposition* of $V$. Note that $V_i$

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and $V_j$ are orthogonal whenever $|i-j| \geq 2$. Also $P(V_1) \supseteq P(V_2) \supseteq \cdots \supseteq P(V_{m+1})$ and $Q(V_1) \supseteq Q(V_2) \supseteq \cdots \supseteq Q(V_{m+1})$.

LEMMA 3.3. *Let $N \geq |K|(2|S|^2 + |S|)$. Then every $S$-form is $N$-regular with respect to some basis.*

PROOF. Write $s = |S|$. Let $(V, \theta)$ be an $S$-form. Let $d = \text{rank}(V)$ and define $m$ by $N^{[m]} < d \leq N^{[m+1]}$. Suppose we can find free modules $V_1^+$, $V_1$, $V_2^+$, $V_2, \ldots, V_m^+$, $V_m$, $V_{m+1}^+$ with the following properties: $V_1^+ = V$; for $i = 1, \ldots, m$, $V_i^+ = V_i \oplus V_{i+1}^+$, $\text{rank}(V_i) = N^i$, $P(V_i) = P(V_i^+)$ and $Q(V_i) = Q(V_i^+)$; and, for $i = 2, \ldots, m$, $V_{i-1}$ and $V_{i+1}^+$ are orthogonal. Then, taking $V_{m+1} = V_{m+1}^+$, we see that $V = V_1 \oplus \cdots \oplus V_{m+1}$ and $(V, \theta)$ is $N$-regular with respect to a basis of $V$ composed of bases of $V_1, \ldots, V_{m+1}$. We construct the required free modules inductively.

First define $V_1^+ = V$. If $\text{rank}(V_1^+) \leq N$ then $m = 0$ and we have finished. So suppose that $\text{rank}(V_1^+) > N$. Since $|P(V_1^+)| \leq s^2$ and $|Q(V_1^+)| \leq s$ we can choose elements $v_1, \ldots, v_{2s^2+s}$ of $V_1^+$ (not necessarily distinct) such that

$$\{(\theta(v_{2i-1}, v_{2i}), \theta(v_{2i}, v_{2i-1})) : i = 1, \ldots, s^2\} = P(V_1^+),$$
$$\{\theta(v_i, v_i) : i = 2s^2 + 1, \ldots, 2s^2 + s\} = Q(V_1^+).$$

By Lemma 3.1, we can find free submodules $U_1$ and $U_2$ of $V_1^+$ such that $V_1^+ = U_1 \oplus U_2$, $v_1, \ldots, v_{2s^2+s} \in U_1$ and $\text{rank}(U_1) \leq |K|(2s^2 + s) \leq N$. Choose free modules $V_1$ and $V_2^+$ such that $V_1^+ = V_1 \oplus V_2^+$, $\text{rank}(V_1) = N$ and $V_1 \supseteq U_1$. By the choice of $v_1, \ldots, v_{2s^2+s}$, we have $P(V_1) = P(V_1^+)$ and $Q(V_1) = Q(V_1^+)$.

Suppose that for some $k$ with $1 \leq k \leq m$ we have found free modules $V_1^+$, $V_1$, $V_2^+$, $\ldots$, $V_k$, $V_{k+1}^+$ with the required properties for these modules. If $\text{rank}(V_{k+1}^+) \leq N^{k+1}$ then $m = k$ and we have finished. So suppose that $\text{rank}(V_{k+1}^+) > N^{k+1}$. By the method used in the first part of the proof we may find free submodules $U$ and $W$ of $V_{k+1}^+$ such that $V_{k+1}^+ = U \oplus W$, $P(U) = P(V_{k+1}^+)$, $Q(U) = Q(V_{k+1}^+)$ and $\text{rank}(U) = N$. By Lemma 3.2, there are free submodules $W_1$ and $W_2$ of $W$ such that $W = W_1 \oplus W_2$, $W_2$ and $V_k$ are orthogonal and $\text{rank}(W_1) \leq 2sN^k$. Then

$$\text{rank}(U \oplus W_1) \leq N + 2sN^k \leq (1 + 2s)N^k \leq N^{k+1}.$$

Choose free modules $V_{k+1}$ and $V_{k+2}^+$ such that $V_{k+1}^+ = V_{k+1} \oplus V_{k+2}^+$, $\text{rank}(V_{k+1}) = N^{k+1}$, $V_{k+1} \supseteq U \oplus W_1$ and $V_{k+2}^+ \subseteq W_2$. Then $V_{k+1}$ and $V_{k+2}^+$ have the required properties. □

LEMMA 3.4. *Let $(V, \theta)$ be an $S$-form which has an $N$-regular decomposition $V = V_1 \oplus \cdots \oplus V_{m+1}$.*

(i) *Let $k \in \{1, \ldots, m-1\}$. Suppose that $P(V_k) = P(V_{k+2})$ and $Q(V_k) = Q(V_{k+2})$. Then $P(V_k)$ is an additive subgroup of $S \oplus S$ and $Q(V_k)$ is an additive subgroup of $S$.*

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(ii) *Let $c$ be a positive integer and let $r(1)$ and $r(2)$ be integers such that $1 \le r(1) < r(2) \le m+1$. Suppose that*

$$P(V_{r(1)}) = P(V_{r(1)+1}) = \cdots = P(V_{r(2)}) = P \subseteq S \oplus S,$$
$$Q(V_{r(1)}) = Q(V_{r(1)+1}) = \cdots = Q(V_{r(2)}) = Q \subseteq S,$$

*and $r(2) - r(1) \ge c(c+1) + 2$. Write $W = V_{r(1)+2} \oplus \cdots \oplus V_{r(2)-2}$. For all $i, j \in \{1, \dots, c\}$ with $i < j$ let $p_{ij} \in P$ and for all $i \in \{1, \dots, c\}$ let $q_i \in Q$. Then there exist $w_1, \dots, w_c \in W$ such that $(\theta(w_i, w_j), \theta(w_j, w_i)) = p_{ij}$, for all $i, j \in \{1, \dots, c\}$ with $i < j$, and $\theta(w_i, w_i) = q_i$, for all $i \in \{1, \dots, c\}$.*

PROOF. (i) Let $p, p' \in P(V_k)$. Then there exist $v, w \in V_k$ and $v', w' \in V_{k+2}$ such that $(\theta(v, w), \theta(w, v)) = p$ and $(\theta(v', w'), \theta(w', v')) = p'$. Write $V_k^+ = V_k \oplus \cdots \oplus V_{m+1}$. Since $V_k$ and $V_{k+2}$ are orthogonal,

$$p + p' = (\theta(v+v', w+w'), \theta(w+w', v+v')) \in P(V_k^+) = P(V_k).$$

Hence, since $P(V_k)$ is finite, it is a group. Similarly $Q(V_k)$ is a group.

(ii) By (i), $P$ and $Q$ are additive groups. There are $c(c+1)/2$ modules in the set $\{V_{r(1)+2}, V_{r(1)+4}, \dots, V_{r(1)+c(c+1)}\}$ and so these modules can be relabelled as $U_i$ for $1 \le i \le c$ and $U_{ij}$ for $1 \le i < j \le c$. These modules are pairwise orthogonal submodules of $W$ such that $P(U_i) = P(U_{ij}) = P$ and $Q(U_i) = Q(U_{ij}) = Q$ for all $i, j$. For $i, j \in \{1, \dots, c\}$ with $i < j$ choose $u_{ij}, v_{ij} \in U_{ij}$ such that

$$(\theta(u_{ij}, v_{ij}), \theta(v_{ij}, u_{ij})) = p_{ij}.$$

Then for each $i \in \{1, \dots, c\}$ choose $u_i \in U_i$ such that

$$\theta(u_i, u_i) = q_i - \sum_{j:j>i} \theta(u_{ij}, u_{ij}) - \sum_{j:j<i} \theta(v_{ji}, v_{ji}).$$

Finally, for $i = 1, \dots, c$, define $w_i = u_i + \sum_{j:j>i} u_{ij} + \sum_{j:j<i} v_{ji}$. It is easy to check that these elements have the required properties. □

For each $S$-form $(V, \theta)$ we need to fix an ordered basis of $V$. Thus we define an *$S$-triple* to be a triple $(V, \theta, X)$ where $(V, \theta)$ is an $S$-form and $X$ is an ordered basis of $V$.

Let $(V, \theta, X)$ and $(V', \theta', X')$ be $S$-triples, where $\operatorname{rank}(V) = d$, $\operatorname{rank}(V') = d'$, $X = \{x_1, \dots, x_d\}$ and $X' = \{x'_1, \dots, x'_{d'}\}$. We say that $(V, \theta, X)$ and $(V', \theta', X')$ are *isomorphic* if $d = d'$ and there is an $S$-form isomorphism $\xi : (V, \theta) \to (V', \theta')$ such that $x_i \xi = x'_i$ for $i = 1, \dots, d$. We write $(V, \theta, X) \preccurlyeq (V', \theta', X')$ if there is a

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one-one order-preserving map $\phi : \{1, \dots, d\} \to \{1, \dots, d'\}$ together with an $S$-form homomorphism $\xi : (V, \theta) \to (V', \theta')$ such that, for $i = 1, \dots, d$,

$$x_i\xi = x'_{i\phi} + z_i, \quad \text{for some } z_i \in \langle x'_1, x'_2, \dots, x'_{i\phi-1} \rangle. \tag{3.1}$$

Clearly $\preccurlyeq$ is a quasi-order on the set of all $S$-triples. Also, if $\xi$ satisfies (3.1) then $\xi$ is a monomorphism. Hence $(V, \theta, X) \preccurlyeq (V', \theta', X')$ implies $(V, \theta) \preccurlyeq (V', \theta')$. An $S$-triple $(V, \theta, X)$ is said to be $N$*-regular* if $(V, \theta)$ is $N$-regular with respect to $X$.

PROPOSITION 3.5. *The set of all $N$-regular $S$-triples is well-quasi-ordered under the relation $\preccurlyeq$.*

PROOF. Let $Y^{(1)}, Y^{(2)}, Y^{(3)}, \dots$ be an infinite sequence of $N$-regular $S$-triples. It suffices to show that there exist integers $i$ and $j$ with $i < j$ such that $Y^{(i)} \preccurlyeq Y^{(j)}$. For each $i$, let $Y^{(i)} = (V^{(i)}, \theta^{(i)}, X^{(i)})$ where $V^{(i)}$ has $N$-regular decomposition $V_1^{(i)} \oplus \dots \oplus V_{m(i)+1}^{(i)}$, $d(i) = \operatorname{rank}(V^{(i)})$ and $X^{(i)} = \{x_1^{(i)}, \dots, x_{d(i)}^{(i)}\}$. If $\{m(1), m(2), \dots\}$ is bounded then there are only finitely many isomorphism types in the sequence $Y^{(1)}, Y^{(2)}, Y^{(3)}, \dots$ and the result is clear. Thus we assume that $\{m(1), m(2), \dots\}$ is unbounded. By passing to an infinite subsequence we may assume that $m(i) \geq 1$ for all $i \geq 1$. There are only finitely many possibilities for the values $\theta^{(i)}(x_j^{(i)}, x_k^{(i)})$ for $j, k \in \{1, \dots, N^{[1]}\}$. Thus, by passing to an infinite subsequence, we may assume that, for all $j, k \in \{1, \dots, N^{[1]}\}$, the value $\theta^{(i)}(x_j^{(i)}, x_k^{(i)})$ is independent of $i$. Then, by passing to an infinite subsequence, we may assume that $m(i) \geq 2$ for all $i \geq 2$ and that, for all $j, k \in \{1, \dots, N^{[2]}\}$, the value $\theta^{(i)}(x_j^{(i)}, x_k^{(i)})$ is independent of $i$ for all $i \geq 2$. Continuing in this way we may pass to an infinite subsequence with the following property for all $n \in \mathbb{N}$:

$$\begin{gathered} m(i) \geq n \text{ for all } i \geq n \text{ and,} \\ \text{for all } j, k \in \{1, \dots, N^{[n]}\}, \theta^{(i)}(x_j^{(i)}, x_k^{(i)}) \text{ is independent of } i \text{ for all } i \geq n. \end{gathered} \tag{3.2}$$

Let $\overline{V}$ be a free $K$-module with countably infinite basis $\overline{X} = \{\overline{x}_1, \overline{x}_2, \dots\}$. Define a $K$-bilinear map $\overline{\theta} : \overline{V} \times \overline{V} \to S$ by taking $\overline{\theta}(\overline{x}_j, \overline{x}_k)$ to be the limiting value of $\theta^{(i)}(x_j^{(i)}, x_k^{(i)})$. Furthermore, for each positive integer $n$, let $P_n$ and $Q_n$ be the limiting values of $P(V_n^{(i)})$ and $Q(V_n^{(i)})$, respectively. Since $P_1 \supseteq P_2 \supseteq \cdots$ and $Q_1 \supseteq Q_2 \supseteq \cdots$, there exist $P \subseteq S \oplus S$, $Q \subseteq S$, and a positive integer $r$, such that $P_r = P_{r+1} = \cdots = P$ and $Q_r = Q_{r+1} = \cdots = Q$. By Lemma 3.4, $P$ and $Q$ are additive groups.

For each $i$, let $r(i)$ be the largest integer belonging to $\{1, \dots, m(i)\}$ such that $\theta^{(i)}(x_j^{(i)}, x_k^{(i)}) = \overline{\theta}(\overline{x}_j, \overline{x}_k)$ for all $j, k \in \{1, \dots, N^{[r(i)]}\}$. By construction, the set $\{r(1), r(2), \dots\}$ is unbounded. Hence, by passing to an infinite subsequence, we may

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assume that $r \leq r(1) < r(2) < \cdots$. Let

$$a(i) = N^{[r(i)-1]} = \operatorname{rank}\left(V_1^{(i)} \oplus \cdots \oplus V_{r(i)-1}^{(i)}\right),$$
$$b(i) = N^{[r(i)]} = \operatorname{rank}\left(V_1^{(i)} \oplus \cdots \oplus V_{r(i)}^{(i)}\right) = a(i) + N^{r(i)}.$$

We may pass to an infinite subsequence so that, for each $i$, we have

$$\text{(3.3)} \qquad \begin{aligned} d(i) - a(i) &\leq d(i+1) - a(i+1) \quad \text{and} \\ r(i+1) - r(i) &\geq (d(i) - a(i))(d(i) - a(i) + 1) + 2. \end{aligned}$$

We now focus on $Y^{(1)}$ and $Y^{(2)}$ and show that $Y^{(1)} \preccurlyeq Y^{(2)}$. By the choice of $r(1)$ and $r(2)$, we have

$$P\left(V_{r(1)}^{(1)}\right) = P\left(V_{r(1)}^{(2)}\right) = P\left(V_{r(1)+1}^{(2)}\right) = \cdots = P\left(V_{r(2)}^{(2)}\right) = P,$$
$$Q\left(V_{r(1)}^{(1)}\right) = Q\left(V_{r(1)}^{(2)}\right) = Q\left(V_{r(1)+1}^{(2)}\right) = \cdots = Q\left(V_{r(2)}^{(2)}\right) = Q,$$

and

$$\theta^{(1)}\left(x_i^{(1)}, x_j^{(1)}\right) = \theta^{(2)}\left(x_i^{(2)}, x_j^{(2)}\right) \quad \text{for all } i, j \in \{1, \ldots, b(1)\}.$$

Since $a(1) < a(2)$ and $d(1) - a(1) \leq d(2) - a(2)$ there exists a one-one order-preserving map $\phi : \{1, \ldots, d(1)\} \to \{1, \ldots, d(2)\}$ such that $i\phi = i$ for $i=1, \ldots, a(1)$ and $\{a(1) + 1, \ldots, d(1)\}\phi \subseteq \{a(2) + 1, \ldots, d(2)\}$.

Write $W = V_{r(1)+2}^{(2)} \oplus \cdots \oplus V_{r(2)-2}^{(2)}$ as in Lemma 3.4. Note that, for $i \in \{a(1) + 1, \ldots, d(1)\}$,

$$\theta^{(1)}\left(x_i^{(1)}, x_i^{(1)}\right) \in Q\left(V_{r(1)}^{(1)} \oplus \cdots \oplus V_{m(1)+1}^{(1)}\right) = Q\left(V_{r(1)}^{(1)}\right) = Q,$$

and

$$\theta^{(2)}\left(x_{i\phi}^{(2)}, x_{i\phi}^{(2)}\right) \in Q\left(V_{r(2)}^{(2)} \oplus \cdots \oplus V_{m(2)+1}^{(2)}\right) = Q\left(V_{r(2)}^{(2)}\right) = Q.$$

Similarly,

$$\left(\theta^{(1)}\left(x_i^{(1)}, x_j^{(1)}\right), \theta^{(1)}\left(x_j^{(1)}, x_i^{(1)}\right)\right) \in P, \quad \left(\theta^{(2)}\left(x_{i\phi}^{(2)}, x_{j\phi}^{(2)}\right), \theta^{(2)}\left(x_{j\phi}^{(2)}, x_{i\phi}^{(2)}\right)\right) \in P,$$

for all $i, j \in \{a(1) + 1, \ldots, d(1)\}$ with $i < j$. Hence, by Lemma 3.4, we can choose elements $w_{a(1)+1}, \ldots, w_{d(1)}$ of $W$ satisfying

$$\theta^{(2)}(w_i, w_i) = \begin{cases} -\theta^{(2)}\left(x_{i\phi}^{(2)}, x_{i\phi}^{(2)}\right) & \text{for } i \in \{a(1) + 1, \ldots, b(1)\}; \\ \theta^{(1)}\left(x_i^{(1)}, x_i^{(1)}\right) - \theta^{(2)}\left(x_{i\phi}^{(2)}, x_{i\phi}^{(2)}\right) & \text{for } i \in \{b(1) + 1, \ldots, d(1)\}; \end{cases}$$

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and

$$\begin{aligned}&\big(\theta^{(2)}(w_i, w_j), \theta^{(2)}(w_j, w_i)\big)\\ &= \begin{cases} \big(\theta^{(1)}(x_i^{(1)}, x_j^{(1)}), \theta^{(1)}(x_j^{(1)}, x_i^{(1)})\big) - \big(\theta^{(2)}(x_{i\phi}^{(2)}, x_{j\phi}^{(2)}), \theta^{(2)}(x_{j\phi}^{(2)}, x_{i\phi}^{(2)})\big), \\ \qquad \text{for } i < j \text{ with } i \in \{a(1)+1, \dots, b(1)\}, j \in \{b(1)+1, \dots, d(1)\}; \\ -\big(\theta^{(2)}(x_{i\phi}^{(2)}, x_{j\phi}^{(2)}), \theta^{(2)}(x_{j\phi}^{(2)}, x_{i\phi}^{(2)})\big), \\ \qquad \text{for } i < j \text{ with } i, j \in \{a(1)+1, \dots, b(1)\}; \\ \big(\theta^{(1)}(x_i^{(1)}, x_j^{(1)}), \theta^{(1)}(x_j^{(1)}, x_i^{(1)})\big) - \big(\theta^{(2)}(x_{i\phi}^{(2)}, x_{j\phi}^{(2)}), \theta^{(2)}(x_{j\phi}^{(2)}, x_{i\phi}^{(2)})\big) \\ \qquad \text{for } i < j \text{ with } i, j \in \{b(1)+1, \dots, d(1)\}. \end{cases}\end{aligned}$$

Then we define a $K$-linear map $\xi : V^{(1)} \to V^{(2)}$ by

$$x_i^{(1)}\xi = \begin{cases} x_i^{(2)} & \text{for } i \in \{1, \dots, a(1)\}; \\ x_i^{(2)} + w_i + x_{i\phi}^{(2)} & \text{for } i \in \{a(1)+1, \dots, b(1)\}; \\ w_i + x_{i\phi}^{(2)} & \text{for } i \in \{b(1)+1, \dots, d(1)\}. \end{cases}$$

Note that, in these equations, $x_i^{(2)} \in V_1^{(2)} \oplus \cdots \oplus V_{r(1)}^{(2)}$, while $w_i \in W$ and $x_{i\phi}^{(2)} \in V_{r(2)}^{(2)} \oplus \cdots \oplus V_{m(2)+1}^{(2)}$, where $V_1^{(2)} \oplus \cdots \oplus V_{r(1)}^{(2)}$, $W$ and $V_{r(2)}^{(2)} \oplus \cdots \oplus V_{m(2)+1}^{(2)}$ are pairwise orthogonal. It is straightforward to check that $\theta^{(2)}(x_i^{(1)}\xi, x_j^{(1)}\xi) = \theta^{(1)}(x_i^{(1)}, x_j^{(1)})$ in all the various cases for $i$ and $j$. Hence $\xi$ is a homomorphism of $S$-forms. Clearly $\xi$ has the form required in (3.1). Thus we have $Y^{(1)} \preccurlyeq Y^{(2)}$, as required. □

PROOF OF THEOREM B. Take any positive integer $N$ such that $N \geq |K|(2|S|^2+|S|)$. Then, by Lemma 3.3, for each $S$-form $(V, \theta)$ there exists an ordered basis $X_{(V,\theta)}$ of $V$ such that $(V, \theta, X_{(V,\theta)})$ is an $N$-regular $S$-triple. If $(V, \theta)$ and $(V', \theta')$ are $S$-forms such that $(V, \theta, X_{(V,\theta)}) \preccurlyeq (V', \theta', X_{(V',\theta')})$ then $(V, \theta) \preccurlyeq (V', \theta')$. Hence the result follows by Proposition 3.5. □

To prove our result about varieties of groups we need, in fact, not Theorem B itself but the assertion stated below as Proposition 3.7.

Let $T$ be any non-empty finite set. We consider finite sequences $(t_1, \dots, t_n)$ of elements of $T$ and write $(t_1, \dots, t_n) \preccurlyeq (t'_1, \dots, t'_{n'})$ if $(t_1, \dots, t_n)$ is a subsequence of $(t'_1, \dots, t'_{n'})$, that is, if there is a one-one order-preserving map $\phi : \{1, \dots, n\} \to \{1, \dots, n'\}$ such that $t_i = t'_{i\phi}$ for $i = 1, \dots, n$. Clearly $\preccurlyeq$ is a quasi-order (in fact a partial-order). The following result is a special case of [6, Theorem 4.3].

LEMMA 3.6. *The set of all finite sequences of elements of $T$ is well-quasi-ordered under the relation $\preccurlyeq$.*

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We define an $(S, T)$-*form* to be a quadruple $(V, \theta, X, \mathbf{t})$ where $(V, \theta, X)$ is an $S$-triple and $\mathbf{t}$ is an ordered $d$-tuple $(t_1, \dots, t_d)$ of elements of $T$, with $d = \operatorname{rank}(V)$. We say that $(S, T)$-forms $(V, \theta, X, \mathbf{t})$ and $(V', \theta', X', \mathbf{t}')$ are *isomorphic* if the $S$-triples $(V, \theta, X)$ and $(V', \theta', X')$ are isomorphic and $\mathbf{t} = \mathbf{t}'$. Let $\operatorname{rank}(V) = d$, $\operatorname{rank}(V') = d'$, $X = \{x_1, \dots, x_d\}$ and $X' = \{x'_1, \dots, x'_{d'}\}$. Write $(V, \theta, X, \mathbf{t}) \preccurlyeq (V', \theta', X', \mathbf{t}')$ if there is a one-one order-preserving map $\phi : \{1, \dots, d\} \to \{1, \dots, d'\}$ together with an $S$-form homomorphism $\xi : (V, \theta) \to (V', \theta')$ such that, for $i = 1, \dots, d$, $t_i = t'_{i\phi}$ and

$$(3.4) \qquad x_i\xi = x'_{i\phi} + z_i, \quad \text{for some } z_i \in \langle x'_1, x'_2, \dots, x'_{i\phi-1} \rangle.$$

Clearly $\preccurlyeq$ is a quasi-order on the set of all $(S, T)$-forms, and we observe that $(V, \theta, X, \mathbf{t}) \preccurlyeq (V', \theta', X', \mathbf{t}')$ implies $(V, \theta, X) \preccurlyeq (V', \theta', X')$.

An $(S, T)$-form $(V, \theta, X, \mathbf{t})$ is said to be $N$-*regular* if the $S$-triple $(V, \theta, X)$ is $N$-regular. For given $S$, $T$ and $N$ we write $\mathscr{Z}$ for the set of all $N$-regular $(S, T)$-forms.

PROPOSITION 3.7. *The set* $(\mathscr{Z}, \preccurlyeq)$ *is well-quasi-ordered.*

PROOF. Let $Z^{(1)}, Z^{(2)}, Z^{(3)}, \dots$ be an infinite sequence of $N$-regular $(S, T)$-forms. It suffices to show that there exist integers $i$ and $j$ with $i < j$ such that $Z^{(i)} \preccurlyeq Z^{(j)}$. For each $i$, let $Z^{(i)} = (V^{(i)}, \theta^{(i)}, X^{(i)}, \mathbf{t}^{(i)})$ and use further notation for $(V^{(i)}, \theta^{(i)}, X^{(i)})$ exactly as in the proof of Proposition 3.5. Also, write $\mathbf{t}^{(i)} = (t_1^{(i)}, \dots, t_{d(i)}^{(i)})$.

As in the proof of Proposition 3.5, we may assume that $\{m(1), m(2), \dots\}$ is unbounded and we may pass to a subsequence with the property (3.2) for all $n \in \mathbb{N}$. But, for each $n$ and each $k \in \{1, \dots, N^{[n]}\}$, there are only finitely many possibilities for $t_k^{(i)}$; thus we may also assume that, for all $k \in \{1, \dots, N^{[n]}\}$, $t_k^{(i)}$ is independent of $i$ for all $i \geq n$.

Define $\overline{V}$, $\overline{X}$, $\overline{\theta}$, $P$, $Q$ and $r$ as before. Also, for each $k \in \mathbb{N}$, define $\overline{t}_k$ to be the limiting value of $t_k^{(i)}$. Then define $r(i)$ as before, but with the additional requirement that $t_k^{(i)} = \overline{t}_k$ for all $k \in \{1, \dots, N^{[r(i)]}\}$.

Define $a(i)$ and $b(i)$ as before and pass to an infinite subsequence with property (3.3) for each $i$. Also, define $\mathbf{t}_i = \left(t_{a(i)+1}^{(i)}, t_{a(i)+2}^{(i)}, \dots, t_{d(i)}^{(i)}\right)$ for each $i$. By Lemma 3.6, there exist $i$ and $j$ with $i < j$ such that $\mathbf{t}_i$ is a subsequence of $\mathbf{t}_j$. Hence, by passing to an infinite subsequence of $Z^{(1)}, Z^{(2)}, \dots$, we may assume that $\mathbf{t}_1$ is a subsequence of $\mathbf{t}_2$. Thus there is a one-one order-preserving map $\phi : \{a(1) + 1, \dots, d(1)\} \to \{a(2) + 1, \dots, d(2)\}$ such that $t_i^{(1)} = t_{i\phi}^{(2)}$ for $i = a(1) + 1, \dots, d(1)$. We may extend $\phi$ to a one-one order-preserving map $\phi : \{1, \dots, d(1)\} \to \{1, \dots, d(2)\}$ by defining $i\phi = i$ for $i = 1, \dots, a(1)$.

As in the proof of Proposition 3.5, there is a homomorphism of $S$-forms $\xi : (V^{(1)}, \theta^{(1)}) \to (V^{(2)}, \theta^{(2)})$ such that $\xi$ has the form required in (3.4). For $i = 1, \dots, a(1)$, we have $t_i^{(1)} = t_i^{(2)} = \overline{t}_i$, since $a(1) \leq N^{[r(1)]} \leq N^{[r(2)]}$, and so $t_i^{(1)} = t_{i\phi}^{(2)}$,

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since $i = i\phi$. Also, for $i = a(1)+1, \ldots, d(1)$, we have $t_i^{(1)} = t_{i\phi}^{(2)}$ by the choice of $\phi$. Thus $Z^{(1)} \preccurlyeq Z^{(2)}$. □

An *alternating* $S$-form is an $S$-form $(V, \theta)$ such that $\theta(v, v) = 0$ for all $v \in V$. Consider now the case where $S = K$. An alternating $K$-form $(V, \theta)$ is called *standard* with respect to the ordered basis $\{x_1, \ldots, x_d\}$ of $V$ if $\theta(x_i, x_j) = 0$ for all $i, j$ such that $1 \leq i < j \leq d$ and $(i, j) \notin \{(1,2), (3,4), \ldots, (2[d/2]-1, 2[d/2])\}$.

LEMMA 3.8 (compare [2]). *Let $n_0$ be an integer, with $n_0 \geq 2$, and let $K = \mathbb{Z}/n_0\mathbb{Z}$. Let $(V, \theta)$ be an alternating $K$-form. Then there is a $K$-basis $\{x_1, \ldots, x_d\}$ of $V$ such that $(V, \theta)$ is standard with respect to $\{x_1, \ldots, x_d\}$.*

PROOF. Choose $u_1, u_2 \in V$ such that the additive cyclic subgroup $\langle \theta(u_1, u_2) \rangle$ of $K$ has largest possible order. Let $x_1$ be an element of $V$ of order $n_0$ such that $u_1 \in \langle x_1 \rangle$. Note that $x_1$ belongs to some basis of $V$. By maximality, $\langle \theta(u_1, u_2) \rangle = \langle \theta(x_1, u_2) \rangle$. Hence we may replace $u_1$ by $x_1$. Let $U$ be a submodule of $V$ such that $V = \langle x_1 \rangle \oplus U$. If $U = \{0\}$ then $\{x_1\}$ is the required basis, so suppose $U \neq \{0\}$. Write $u_2 = u_2' + u$ where $u_2' \in \langle x_1 \rangle$ and $u \in U$. Clearly we may replace $u_2$ by $u$. Then, as before, we may replace $u$ by an element $x_2$ which belongs to a basis of $U$. Thus $\{x_1, x_2\}$ is contained in a basis of $V$. Set $W = \{w \in V : \theta(x_1, w) = \theta(x_2, w) = 0\}$. Let $v \in V$. The choice of $x_1$ and $x_2$ shows that $\theta(x_1, x_2)$ is a generator of the cyclic group $\{\theta(x_1, u) : u \in V\}$. Hence there exists $\lambda \in K$ such that $\theta(x_1, v) = \lambda\theta(x_1, x_2)$. Similarly there exists $\mu \in K$ such that $\theta(v, x_2) = \mu\theta(x_1, x_2)$. It follows that $v - \mu x_1 - \lambda x_2 \in W$ and so $v \in \langle x_1, x_2 \rangle + W$. Therefore $V = \langle x_1, x_2 \rangle + W$. Thus we may find a basis $\{x_1, x_2, w_1, \ldots, w_{d-2}\}$ of $V$ with $w_1, \ldots, w_{d-2} \in W$. The lemma follows by an inductive argument applied to $\langle w_1, \ldots, w_{d-2} \rangle$. □

## 4. Direct powers of finite groups

In this section we shall obtain some results which will be useful for both Theorem C and Theorem D.

Let $G$ be a finite group and let $D$ be the (restricted) direct product $D = \prod_{i \in \mathbb{N}} G_i$ where $G_i = G$ for all $i$. Thus the elements of $D$ may be regarded as sequences of the form $(g_1, g_2, \ldots)$ where $g_i \in G$ for all $i$ and where $\{i : g_i \neq 1\}$ is finite.

Let $\phi : \mathbb{N} \to \mathbb{N}$ be a one-one order-preserving function. Let $X$ be a finite subset of $\mathbb{N} \setminus \mathbb{N}\phi$ and let $\sigma : X \to \mathbb{N}\phi$ be a function such that $j < j\sigma$ for all $j \in X$. Given such $\phi$, $X$ and $\sigma$, let $\xi$ be the endomorphism of $D$ defined by

$$(g_1, g_2, \ldots)\xi = (g_1', g_2', \ldots),$$

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where $g'_j = g_i$ if $j = i\phi$, $g'_j = 1$ if $j \notin \mathbb{N}\phi \cup X$, and $g'_j = g'_{j\sigma}$ if $j \in X$. Let $\Xi$ be the set of all such endomorphisms of $D$ (for all possible choices of $\phi$, $X$ and $\sigma$).

Let $\leq$ be a total order on $G$ which is arbitrary except that $1 \leq g$ for all $g \in G$. Then the set $D$ may be ordered lexicographically from the right: if $d, d' \in D$ where $d = (g_1, g_2, \dots)$ and $d' = (g'_1, g'_2, \dots)$, we set $d < d'$ if there exists $l \in \mathbb{N}$ such that $g_l < g'_l$ but $g_i = g'_i$ for all $i > l$. Clearly $(D, \leq)$ is well-ordered, and it is easy to prove the following result.

LEMMA 4.1. *Let $d, d' \in D$ and let $\xi \in \Xi$. If $d < d'$ then $d\xi < d'\xi$.*

For $d \in D$, where $d = (g_1, g_2, \dots)$, write

$$\operatorname{span}(d) = \{g \in G \setminus \{1\} : g = g_i \text{ for some } i\},$$

and, for $g \in \operatorname{span}(d)$, let $i_g(d)$ denote the largest $i$ such that $g_i = g$.

Let $d$ and $d'$ be elements of $D$, where $d = (g_1, g_2, \dots)$ and $d' = (g'_1, g'_2, \dots)$. Write $d \preccurlyeq d'$ if $\operatorname{span}(d) = \operatorname{span}(d')$ and there is a one-one order-preserving function $\phi : \mathbb{N} \to \mathbb{N}$ such that $g_i = g'_{i\phi}$ for all $i$ and $i_g(d)\phi = i_g(d')$ for all $g \in \operatorname{span}(d)$. Clearly $(D, \preccurlyeq)$ is quasi-ordered (in fact, partially-ordered).

LEMMA 4.2. *The set $(D, \preccurlyeq)$ is well-quasi-ordered.*

PROOF. Let $m = |G \setminus \{1\}|$ and assume $m \geq 1$ (the result is trivial for $m = 0$). Write $G \setminus \{1\} = \{a_1, \dots, a_m\}$. For $d \in D$ and $k = 1, \dots, m$, define $p_k(d) = i_{a_k}(d)$ if $a_k \in \operatorname{span}(d)$ and $p_k(d) = 1$ otherwise, so that we obtain an $m+1$-tuple $s(d) = (p_1(d), \dots, p_m(d), d)$. Let $d, d' \in D$, where $d = (g_1, g_2, \dots)$ and $d' = (g'_1, g'_2, \dots)$. Following the notation of [3], we write $s(d) \preccurlyeq_\Phi s(d')$ if there exists a one-one order-preserving map $\phi : \mathbb{N} \to \mathbb{N}$ such that $g_i = g'_{i\phi}$ for all $i$ and $p_i(d)\phi = p_i(d')$ for $i = 1, \dots, m$. By [3, Lemma 3.2], the set of $m+1$-tuples $s(d)$ is well-quasi-ordered under $\preccurlyeq_\Phi$. But $s(d) \preccurlyeq_\Phi s(d')$ implies $d \preccurlyeq d'$. The result follows. □

Let $\mathbb{F}$ be any field. Then each non-zero element $u$ of the group algebra $\mathbb{F}D$ can be written (uniquely) in the form $u = \lambda_1 d_1 + \cdots + \lambda_r d_r$ where $d_1, \dots, d_r \in D$, $d_1 > \cdots > d_r$ and $\lambda_1, \dots, \lambda_r \in \mathbb{F} \setminus \{0\}$. The largest group element $d_1$ is called the *leading group element* of $u$ and we write $d_1 = \operatorname{lead}(u)$. Since every endomorphism of $D$ extends to $\mathbb{F}D$, each element of $\Xi$ acts on $\mathbb{F}D$. For $S \subseteq \mathbb{F}D$ we write $\langle S \rangle_\Xi$ for the $\Xi$-closed subspace of $\mathbb{F}D$ generated by $S$.

LEMMA 4.3. *Let $u$ and $v$ be non-zero elements of $\mathbb{F}D$ with* $\operatorname{lead}(u) \preccurlyeq \operatorname{lead}(v)$. *Then there exists $v^* \in \mathbb{F}D$ such that $\langle u, v \rangle_\Xi = \langle u, v^* \rangle_\Xi$ and either $v^* = 0$ or* $\operatorname{lead}(v^*) < \operatorname{lead}(v)$.

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PROOF. Write $u = \lambda_1 d_1 + \cdots + \lambda_r d_r$ and $v = \lambda'_1 d'_1 + \cdots + \lambda'_s d'_s$ where the $d_i$ and $d'_i$ are elements of $D$, $d_1 > \cdots > d_r$, $d'_1 > \cdots > d'_s$, and the $\lambda_i$ and $\lambda'_i$ are elements of $\mathbb{F} \setminus \{0\}$. Write $d = d_1 = \text{lead}(u)$ and $d' = d'_1 = \text{lead}(v)$. Thus $d \preccurlyeq d'$. Let $d = (g_1, g_2, \ldots)$ and $d' = (g'_1, g'_2, \ldots)$, and let $\phi : \mathbb{N} \to \mathbb{N}$ be as in the definition of $d \preccurlyeq d'$. Let $X = \{j : j \notin \mathbb{N}\phi$ and $g'_j \neq 1\}$. By the definition of $d \preccurlyeq d'$ we have $i_g(d') \in \mathbb{N}\phi$ for all $g \in \text{span}(d')$. For each $j \in X$ let $j\sigma = i_g(d')$ where $g = g'_j$. Let $\xi$ be the element of $\Xi$ corresponding to $\phi$, $X$ and $\sigma$. Then it is easy to check that $d\xi = d'$. Hence, by Lemma 4.1, $\text{lead}(u\xi) = d' = \text{lead}(v)$. Let $v^* = v - \lambda'_1 \lambda_1^{-1}(u\xi)$. Then the result follows. □

PROPOSITION 4.4. *The maximal condition holds for $\Xi$-closed subspaces of $\mathbb{F}D$.*

PROOF. Let $U$ be a $\Xi$-closed subspace of $\mathbb{F}D$. It suffices to prove that $U$ is finitely generated as a $\Xi$-closed subspace. By Lemma 4.2, there exists a finite subset $S$ of $U \setminus \{0\}$ such that for all $v \in U \setminus \{0\}$ there exists $u \in S$ such that $\text{lead}(u) \preccurlyeq \text{lead}(v)$. We claim that $U = \langle S \rangle_\Xi$. Suppose, in order to get a contradiction, that there exists $v \in U$ such that $v \notin \langle S \rangle_\Xi$, and choose such $v$ so that $\text{lead}(v)$ is as small as possible in the well-ordered set $(D, \leq)$. There exists $u \in S$ such that $\text{lead}(u) \preccurlyeq \text{lead}(v)$. By Lemma 4.3, there exists $v^* \in \mathbb{F}D$ such that $\langle u, v \rangle_\Xi = \langle u, v^* \rangle_\Xi$ and either $v^* = 0$ or $\text{lead}(v^*) < \text{lead}(v)$. Since $v \notin \langle u \rangle_\Xi$, we have $v^* \neq 0$. Since $v^* \in \langle u, v \rangle_\Xi \subseteq U$, the choice of $v$ gives $v^* \in \langle S \rangle_\Xi$. Hence $v \in \langle u, v^* \rangle_\Xi \subseteq \langle S \rangle_\Xi$, and we have the required contradiction. □

Let $n$ be a positive integer and let $E$ be a free group of countably infinite rank in the variety $\mathbf{A}_n$. Let $\Gamma$ be the set of all endomorphisms of $E$.

PROPOSITION 4.5. *For each positive integer $r$, the maximal condition holds for $\Gamma$-closed subspaces of $\mathbb{F}(E^{\times r})$.*

PROOF. Clearly we may assume $n > 1$. Let $\{x_1, x_2, \ldots\}$ be a free generating set for $E$. For each $i \in \mathbb{N}$, let $G_i$ be the subgroup of $E^{\times r}$ generated by the elements $(x_i, 1, \ldots, 1), (1, x_i, 1, \ldots, 1), \ldots, (1, \ldots, 1, x_i)$. Write $G = G_1$. Thus $G$ is a finite group. Clearly $E^{\times r}$ is the direct product of the groups $G_i$ and, for each $i$, there is an obvious isomorphism from $G$ to $G_i$. Thus we may identify $E^{\times r}$ with the direct power $D$ of $G$ considered above. The result will follow from Proposition 4.4 if we can show that every element of $\Xi$ is induced by some element of $\Gamma$. Let $\xi \in \Xi$ and suppose that $\xi$ is associated with $\phi$, $X$ and $\sigma$, in the notation used before. Define a homomorphism $\gamma : E \to E$ by $x_i\gamma = x_{i\phi} \prod_{j \in X,\, j\sigma = i\phi} x_j$, for each $i$, where the product is taken over all those values of $j$, if any, which lie in $X$ and satisfy $j\sigma = i\phi$. It is straightforward to verify that $\gamma$ induces $\xi$. □

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## 5. Proof of Theorem C

We use the notation of Section 1. In particular, $n$ is a positive integer, $A$ is a free group of $\mathbf{N}_{2,n}$ of countably infinite rank, $\Psi$ is the set of endomorphisms of $A$ and $\mathbb{F}$ is a field of characteristic not dividing $n$. We shall describe the proof of Theorem C only in the case $r = 2$. The proof for general $r$ is essentially the same, but greater notational complexity is required for $r > 2$.

Let $\overline{\mathbb{F}}$ be the algebraic closure of $\mathbb{F}$. If $I$ is a $\Psi$-closed left ideal of $\mathbb{F}(A \times A)$ then $\overline{\mathbb{F}} \otimes_{\mathbb{F}} I$ is a $\Psi$-closed left ideal of $\overline{\mathbb{F}}(A \times A)$, and $I = \mathbb{F}(A \times A) \cap \overline{\mathbb{F}} \otimes_{\mathbb{F}} I$. Therefore we may assume that $\mathbb{F} = \overline{\mathbb{F}}$. We write $\mathbb{F}^{\times}$ for the multiplicative group $\mathbb{F} \setminus \{0\}$.

Let $\{x_i : i \in \mathbb{N}\}$ be a free generating set of $A$ and, for each positive integer $k$, let $A_k$ be the subgroup $\langle x_1, \ldots, x_k \rangle$. Define $n_0$ by $n_0 = n$ if $n$ is odd and $n_0 = n/2$ if $n$ is even. For all $a, b \in A$ we have $(ab)^n = 1$ and hence $[a, b]^{n_0} = [a^{n_0}, b] = 1$. Thus $(A')^{n_0} = \{1\}$ and $A^{n_0}$ is central in $A$. It is easily verified that the relations $x_i^n = 1$ and $[x_i, x_j]^{n_0} = 1$, for all $i, j \in \{1, \ldots, k\}$, imposed on the free nilpotent group of class 2 on free generators $x_1, \ldots, x_k$, give a group of exponent $n$, which is therefore isomorphic to $A_k$. It follows that $A_k'$ is a free abelian group of exponent $n_0$ with basis $\{[x_i, x_j] : 1 \leq i < j \leq k\}$. If $n \leq 2$, then $A$ is the free group of countably infinite rank in the variety $\mathbf{A}_n$, and, in this case, Theorem C follows from Proposition 4.5. Thus we assume that $n > 2$, so that $n_0 > 1$.

Let $K = \mathbb{Z}/n_0\mathbb{Z}$ and let $\omega$ be a primitive $n_0$-th root of unity in $\mathbb{F}$. Thus $\omega^{\lambda}$ is well-defined for all $\lambda \in K$, and $\{\omega^{\lambda} : \lambda \in K\}$ is the cyclic subgroup of $\mathbb{F}^{\times}$ consisting of all $n_0$-th roots of unity in $\mathbb{F}$.

Let $Q_k$ be the set of all ordered pairs $(i, j)$ with $1 \leq i < j \leq k$, and let $\Delta_k$ be the set of all functions $\delta : Q_k \to K$. For each $\delta \in \Delta_k$ there is a group homomorphism $\chi_\delta : A_k' \to \mathbb{F}^{\times}$ determined by $\chi_\delta([x_i, x_j]) = \omega^{\delta(i,j)}$ for all $(i, j) \in Q_k$. Since the elements $[x_i, x_j]$ form a basis for $A_k'$, every homomorphism $A_k' \to \mathbb{F}^{\times}$ arises in this way from some $\delta$. We extend $\chi_\delta$ by linearity to a function $\chi_\delta : \mathbb{F}A_k' \to \mathbb{F}$. In the language of representation theory, the functions $\chi_\delta$ are the characters afforded by the irreducible representations of the abelian group $A_k'$ over $\mathbb{F}$, all of which are one-dimensional.

For each $\delta \in \Delta_k$, let $e_\delta$ be the element of $\mathbb{F}A_k'$ defined by

$$e_\delta = \frac{1}{|A_k'|} \sum_{a \in A_k'} \chi_\delta(a^{-1})a. \tag{5.1}$$

The elements $e_\delta$ have the following properties, which may be verified by elementary representation theory or direct calculation.

$$we_\delta = \chi_\delta(w)e_\delta \quad \text{for all} \quad \delta \in \Delta_k \quad \text{and all} \quad w \in \mathbb{F}A_k'. \tag{5.2}$$

$$\chi_\delta(e_\delta) = 1 \quad \text{and} \quad e_\delta^2 = e_\delta \quad \text{for all} \quad \delta \in \Delta_k. \tag{5.3}$$

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(5.4) $\chi_{\delta'}(e_\delta) = 0$ and $e_\delta e_{\delta'} = 0$ for all $\delta, \delta' \in \Delta_k$ with $\delta \neq \delta'$.

(5.5)
$$\sum_{\delta \in \Delta_k} e_\delta = 1.$$

Thus the elements $e_\delta$ are pairwise orthogonal idempotents. They form a basis of $\mathbb{F}A'_k$ and each $e_\delta$ spans a one-dimensional ideal of $\mathbb{F}A'_k$. Within the larger group algebras $\mathbb{F}A_k$ and $\mathbb{F}A$, the $e_\delta$ are central idempotents. For each $\delta$, let $I_\delta = (\mathbb{F}A_k)e_\delta$. Thus $I_\delta$ is the (two-sided) ideal of $\mathbb{F}A_k$ generated by $e_\delta$. By (5.3), (5.4) and (5.5),

(5.6)
$$\mathbb{F}A_k = \bigoplus_{\delta \in \Delta_k} I_\delta.$$

It follows from (5.6) and (5.2) that $\mathbb{F}A_k$ is spanned by all elements of the form $x_1^{\alpha_1} \cdots x_k^{\alpha_k} e_\delta$ with $\delta \in \Delta_k$ and $\alpha_i \in \mathbb{Z}/n\mathbb{Z}$ for $i = 1, \dots, k$. It is easily checked that there are exactly $|A_k|$ such elements. Hence they form a basis for $\mathbb{F}A_k$ and, for fixed $\delta$, the elements $x_1^{\alpha_1} \cdots x_k^{\alpha_k} e_\delta$ form a basis for $I_\delta$.

If $\psi : A_k \to A_l$ is a homomorphism, where $k, l \in \mathbb{N}$, then $\psi$ extends to a homomorphism $\mathbb{F}A_k \to \mathbb{F}A_l$, which we also denote by $\psi$. In particular, $\psi : A_k \to A_k$ extends to $\psi : \mathbb{F}A_k \to \mathbb{F}A_k$.

For each $k$, write $\tilde{A}_k = A_k/A'_k(A_k)^{n_0}$ and, for $a \in A_k$, write $\tilde{a} = aA'_k(A_k)^{n_0} \in \tilde{A}_k$. Thus $\tilde{A}_k$ is a free abelian group of exponent $n_0$ with basis $\{\tilde{x}_1, \dots, \tilde{x}_k\}$. We shall usually think of $\tilde{A}_k$ in additive notation: thus we may regard it as a free $K$-module.

If $\psi : A_k \to A_l$ is a homomorphism, we write $\tilde{\psi}$ for the induced homomorphism from $\tilde{A}_k$ to $\tilde{A}_l$. In particular, if $\eta \in \operatorname{Aut}(A_k)$ then $\tilde{\eta} \in \operatorname{Aut}(\tilde{A}_k)$.

For each $\delta \in \Delta_k$, let $\theta_\delta$ be the alternating $K$-form on $\tilde{A}_k$ satisfying $\theta_\delta(\tilde{x}_i, \tilde{x}_j) = \delta(i, j)$ for all $(i, j) \in Q_k$. Clearly every alternating $K$-form on $\tilde{A}_k$ arises in this way from some $\delta$. Since $\chi_\delta([x_i, x_j]) = \omega^{\delta(i,j)}$ it is straightforward to verify that

(5.7) $\chi_\delta([a_1, a_2]) = \omega^{\theta_\delta(\tilde{a}_1, \tilde{a}_2)}$ for all $a_1, a_2 \in A_k$.

LEMMA 5.1. *Let $\delta \in \Delta_k$ and $\eta \in \operatorname{Aut}(A_k)$. Then $e_\delta \eta = e_\varepsilon$ where $\varepsilon \in \Delta_k$ and $\theta_\varepsilon(\tilde{a}_1, \tilde{a}_2) = \theta_\delta(\tilde{a}_1\tilde{\eta}^{-1}, \tilde{a}_2\tilde{\eta}^{-1})$ for all $a_1, a_2 \in A_k$.*

PROOF. The map $a \mapsto \chi_\delta(a\eta^{-1})$ is a homomorphism from $A'_k$ to $\mathbb{F}^\times$. Hence there exists $\varepsilon \in \Delta_k$ such that $\chi_\varepsilon(a) = \chi_\delta(a\eta^{-1})$ for all $a \in A'_k$. By direct calculation we obtain $e_\delta \eta = e_\varepsilon$. Also, for all $a_1, a_2 \in A_k$, (5.7) gives $\omega^{\theta_\varepsilon(\tilde{a}_1, \tilde{a}_2)} = \chi_\varepsilon([a_1, a_2]) = \chi_\delta([a_1, a_2]\eta^{-1}) = \chi_\delta([a_1\eta^{-1}, a_2\eta^{-1}]) = \omega^{\theta_\delta(\tilde{a}_1\tilde{\eta}^{-1}, \tilde{a}_2\tilde{\eta}^{-1})}$. The result follows. □

LEMMA 5.2. *Let $\delta \in \Delta_k$ and $\varepsilon \in \Delta_l$, where $k, l \in \mathbb{N}$. Let $\psi : A_k \to A_l$ be a homomorphism which induces a homomorphism of $K$-forms from $(\tilde{A}_k, \theta_\delta)$ to $(\tilde{A}_l, \theta_\varepsilon)$ (that is, $\theta_\delta(\tilde{a}_1, \tilde{a}_2) = \theta_\varepsilon(\tilde{a}_1\tilde{\psi}, \tilde{a}_2\tilde{\psi})$ for all $a_1, a_2 \in A_k$). Then $(e_\delta\psi)e_\varepsilon = e_\varepsilon$.*

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PROOF. For all $a_1, a_2 \in A_k$,

$$\chi_\delta([a_1, a_2]) = \omega^{\theta_\delta(\tilde{a}_1, \tilde{a}_2)} = \omega^{\theta_\varepsilon(\tilde{a}_1\tilde{\psi}, \tilde{a}_2\tilde{\psi})} = \chi_\varepsilon([a_1\psi, a_2\psi]) = \chi_\varepsilon([a_1, a_2]\psi).$$

It follows that $\chi_\delta(w) = \chi_\varepsilon(w\psi)$ for all $w \in \mathbb{F}A_k'$. Therefore, by (5.2) and (5.3),

$$(e_\delta\psi)e_\varepsilon = \chi_\varepsilon(e_\delta\psi)e_\varepsilon = \chi_\delta(e_\delta)e_\varepsilon = e_\varepsilon. \qquad \square$$

For each $k$, we consider $\mathbb{F}(A_k \times A_k)$, identified with $\mathbb{F}A_k \otimes_{\mathbb{F}} \mathbb{F}A_k$. If $\psi : A_k \to A_l$ is a homomorphism, then $\psi$ yields homomorphisms $\psi : A_k \times A_k \to A_l \times A_l$ and $\psi : \mathbb{F}(A_k \times A_k) \to \mathbb{F}(A_l \times A_l)$. For $\delta, \delta' \in \Delta_k$, we write $e_\delta \otimes e_{\delta'}$ as $e_{\delta\delta'}$ and $I_\delta \otimes I_{\delta'}$ as $I_{\delta\delta'}$. Thus, by (5.6),

$$\mathbb{F}(A_k \times A_k) = \bigoplus_{\delta, \delta' \in \Delta_k} I_{\delta\delta'}. \tag{5.8}$$

Also, $I_{\delta\delta'}$ is the ideal of $\mathbb{F}(A_k \times A_k)$ generated by the central idempotent $e_{\delta\delta'}$, and $\sum_{\delta,\delta'} e_{\delta\delta'} = 1$.

For $\delta, \delta' \in \Delta_k$, let $\theta_{\delta\delta'}$ be the alternating $K \oplus K$-form on $\tilde{A}_k$ determined by $\theta_{\delta\delta'}(\tilde{x}_i, \tilde{x}_j) = (\theta_\delta(\tilde{x}_i, \tilde{x}_j), \theta_{\delta'}(\tilde{x}_i, \tilde{x}_j))$ for all $(i, j) \in Q_k$. Every alternating $K \oplus K$-form on $\tilde{A}_k$ arises in this way from some $\delta, \delta'$.

The following two results are easily deduced from Lemma 5.1 and Lemma 5.2, respectively.

LEMMA 5.3. *Let* $\delta, \delta' \in \Delta_k$ *and* $\eta \in \mathrm{Aut}(A_k)$. *Then* $e_{\delta\delta'}\eta = e_{\varepsilon\varepsilon'}$ *where* $\varepsilon, \varepsilon' \in \Delta_k$ *and* $\theta_{\varepsilon\varepsilon'}(\tilde{a}_1, \tilde{a}_2) = \theta_{\delta\delta'}(\tilde{a}_1\tilde{\eta}^{-1}, \tilde{a}_2\tilde{\eta}^{-1})$ *for all* $a_1, a_2 \in A_k$.

LEMMA 5.4. *Let* $\delta, \delta' \in \Delta_k$ *and* $\varepsilon, \varepsilon' \in \Delta_l$, *where* $k, l \in \mathbb{N}$. *Let* $\psi : A_k \to A_l$ *be a homomorphism which induces a homomorphism of* $K \oplus K$*-forms from* $(\tilde{A}_k, \theta_{\delta\delta'})$ *to* $(\tilde{A}_l, \theta_{\varepsilon\varepsilon'})$. *Then* $(e_{\delta\delta'}\psi)e_{\varepsilon\varepsilon'} = e_{\varepsilon\varepsilon'}$.

Let $N = n_0(2n_0^4 + n_0^2)$. By Lemma 3.3, every $K \oplus K$-form is $N$-regular with respect to some basis. For $\delta, \delta' \in \Delta_k$, we say that $\theta_{\delta\delta'}$ is *regular* if it is $N$-regular with respect to the basis $\{\tilde{x}_1, \dots, \tilde{x}_k\}$ of $\tilde{A}_k$.

LEMMA 5.5. *Let* $\delta, \delta' \in \Delta_k$. *Then there exists* $\eta \in \mathrm{Aut}(A_k)$ *such that* $e_{\delta\delta'}\eta = e_{\varepsilon\varepsilon'}$ *where* $\varepsilon, \varepsilon' \in \Delta_k$ *and* $\theta_{\varepsilon\varepsilon'}$ *is regular.*

PROOF. By Lemma 3.3, there is a basis $\{\tilde{a}_1, \dots, \tilde{a}_k\}$ of $\tilde{A}_k$ such that $(\tilde{A}_k, \theta_{\delta\delta'})$ is $N$-regular with respect to this basis. It is easily verified that there exists a generating set $\{y_1, \dots, y_k\}$ of $A_k$ such that $\tilde{y}_i = \tilde{a}_i$ for $i = 1, \dots, k$. Since $A_k$ is a finite relatively free group of rank $k$, it follows that $\{y_1, \dots, y_k\}$ is a free generating set. Let $\eta$ be the automorphism of $A_k$ satisfying $y_i\eta = x_i$ for $i = 1, \dots, k$. By Lemma 5.3, $e_{\delta\delta'}\eta = e_{\varepsilon\varepsilon'}$ where $\theta_{\varepsilon\varepsilon'}(\tilde{x}_i, \tilde{x}_j) = \theta_{\delta\delta'}(\tilde{y}_i, \tilde{y}_j)$ for all $(i, j) \in Q_k$. Thus $\theta_{\varepsilon\varepsilon'}$ is regular. $\square$

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LEMMA 5.6. *Let* $\delta, \delta' \in \Delta_k$ *and* $\varepsilon, \varepsilon' \in \Delta_l$, *where* $k, l \in \mathbb{N}$, *and consider* $I_{\delta\delta'}$ *and* $I_{\varepsilon\varepsilon'}$ *as subsets of* $\mathbb{F}(A \times A)$. *Then* $I_{\delta\delta'} \cap I_{\varepsilon\varepsilon'} = \{0\}$ *unless* $k = l$, $\delta = \delta'$ *and* $\varepsilon = \varepsilon'$.

PROOF. Suppose that $k < l$. It is easily verified that $([x_{l-1}, x_l] \otimes 1)w \notin \mathbb{F}(A_k \times A_k)$ for all $w \in \mathbb{F}(A_k \times A_k) \setminus \{0\}$. On the other hand, for all $v \in I_{\varepsilon\varepsilon'}$, the element $([x_{l-1}, x_l] \otimes 1)v$ is a scalar multiple of $v$ by (5.2). Thus $I_{\delta\delta'} \cap I_{\varepsilon\varepsilon'} = \{0\}$. If $k = l$ then $I_{\delta\delta'} \cap I_{\varepsilon\varepsilon'} \neq \{0\}$ implies $\delta = \delta'$ and $\varepsilon = \varepsilon'$ by (5.8). □

A non-zero element $w$ of $\mathbb{F}(A \times A)$ will be called *regular* if $w \in I_{\delta\delta'}$ for some $k$ and some $\delta, \delta' \in \Delta_k$ such that $\theta_{\delta\delta'}$ is regular. (By Lemma 5.6, $k$, $\delta$ and $\delta'$ are then unique.)

LEMMA 5.7. *Every* $\Psi$*-closed left ideal of* $\mathbb{F}(A \times A)$ *is generated, as a* $\Psi$*-closed vector space, by regular elements.*

PROOF. Let $J$ be a $\Psi$-closed left ideal of $\mathbb{F}(A \times A)$ and let $J_0$ be the vector space spanned by all elements $v\psi$ where $v$ is a regular element of $J$ and $\psi \in \Psi$. It suffices to show that $J = J_0$. Clearly $J_0 \subseteq J$. Let $w \in J$. Then $w \in \mathbb{F}(A_k \times A_k)$ for some $k$, and we have $w = \left(\sum_{\delta,\delta' \in \Delta_k} e_{\delta\delta'}\right)w = \sum_{\delta,\delta' \in \Delta_k} (e_{\delta\delta'}w)$, where $e_{\delta\delta'}w \in J \cap I_{\delta\delta'}$. It suffices to show that $e_{\delta\delta'}w \in J_0$. Clearly we may assume that $e_{\delta\delta'}w \neq 0$. By Lemma 5.5, there exists $\eta \in \mathrm{Aut}(A_k)$ such that $(e_{\delta\delta'}w)\eta$ is regular. But $(e_{\delta\delta'}w)\eta \in J$, since $\eta$ extends to an automorphism of $A$. Thus $e_{\delta\delta'}w = (e_{\delta\delta'}w)\eta\eta^{-1} \in J_0$. □

Let $\delta, \delta' \in \Delta_k$. Since the elements $x_1^{\alpha_1} \cdots x_k^{\alpha_k} e_\delta$ with $\alpha_i \in \mathbb{Z}/n\mathbb{Z}$ form a basis of $I_\delta$, the elements

$$(5.9) \qquad \left(x_1^{\alpha_1} \cdots x_k^{\alpha_k} \otimes x_1^{\alpha'_1} \cdots x_k^{\alpha'_k}\right) e_{\delta\delta'},$$

with $\alpha_i, \alpha'_i \in \mathbb{Z}/n\mathbb{Z}$, form a basis of $I_{\delta\delta'}$.

An element of $\mathbb{F}(A \times A)$ will be called a *monomial* if it has the form (5.9) for some $k$ and some $\delta, \delta' \in \Delta_k$, and a *regular monomial* if $\theta_{\delta\delta'}$ is regular. We write $\mathscr{M}$ for the set of all monomials, $\mathscr{M}^*$ for the set of all regular monomials, and $\mathscr{M}_{\delta\delta'}$ for the set of all monomials of $I_{\delta\delta'}$.

Let $T = \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$, that is, the Cartesian square of the set $\mathbb{Z}/n\mathbb{Z}$. With the monomial (5.9) we associate the $k$-tuple $(t_1, t_2, \ldots, t_k)$ where $t_i = (\alpha_i, \alpha'_i) \in T$ for $i = 1, \ldots, k$. Let $\leq$ be a total order on $T$ which is arbitrary except that $(0, 0) \leq t$ for all $t \in T$. Then the set of all $k$-tuples of elements of $T$ can be ordered lexicographically from the right: if $\mathbf{t} = (t_1, \ldots, t_k)$ and $\mathbf{t}' = (t'_1, \ldots, t'_k)$ are two such $k$-tuples, we set $\mathbf{t} < \mathbf{t}'$ if there exists $q \in \{1, \ldots, k\}$ such that $t_q < t'_q$ but $t_i = t'_i$ for $i = q + 1, \ldots, k$. Hence, for $\delta, \delta' \in \Delta_k$, we obtain an order $\leq$ on the finite set $\mathscr{M}_{\delta\delta'}$.

Each non-zero element $f$ of $I_{\delta\delta'}$ can be written (uniquely) in the form $f = \lambda_1 w_1 + \cdots + \lambda_r w_r$, where $w_1, \ldots, w_r \in \mathscr{M}_{\delta\delta'}$, $w_1 > \cdots > w_r$, and $\lambda_1, \ldots, \lambda_r \in \mathbb{F} \setminus \{0\}$. The largest monomial $w_1$ is called the *leading monomial* of $f$, and we write $w_1 = \mathrm{lead}(f)$.

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We shall now define a quasi-order on $\mathscr{M}$. Let $\delta, \delta' \in \Delta_k$ and $\varepsilon, \varepsilon' \in \Delta_l$. Let $v \in \mathscr{M}_{\delta\delta'}$ and $w \in \mathscr{M}_{\varepsilon\varepsilon'}$, where

$$v = \left(x_1^{\alpha_1} \cdots x_k^{\alpha_k} \otimes x_1^{\alpha_1'} \cdots x_k^{\alpha_k'}\right) e_{\delta\delta'}, \quad w = \left(x_1^{\beta_1} \cdots x_l^{\beta_l} \otimes x_1^{\beta_1'} \cdots x_l^{\beta_l'}\right) e_{\varepsilon\varepsilon'}.$$

We write $v \preccurlyeq w$ if there is a one-one order-preserving map $\phi : \{1, \ldots, k\} \to \{1, \ldots, l\}$ together with a homomorphism $\psi : A_k \to A_l$ with the following three properties.

(i) For $i = 1, \ldots, k$, we have $x_i\psi = z_i x_{i\phi}$ for some $z_i \in \langle x_1, \ldots, x_{i\phi - 1}\rangle$.
(ii) $\psi$ induces a homomorphism of $K \oplus K$-forms from $(\tilde{A}_k, \theta_{\delta\delta'})$ to $(\tilde{A}_l, \theta_{\varepsilon\varepsilon'})$.
(iii) For $i = 1, \ldots, k$, we have $\alpha_i = \beta_{i\phi}$ and $\alpha_i' = \beta_{i\phi}'$.

It is straightforward to check that $(\mathscr{M}, \preccurlyeq)$ is a quasi-ordered set. Thus $(\mathscr{M}^*, \preccurlyeq)$ is quasi-ordered. Let $\mathscr{X}$ be the set of all $N$-regular $(K \oplus K, T)$-forms as defined in Section 3 with $S = K \oplus K$. Thus, by Proposition 3.7, $(\mathscr{X}, \preccurlyeq)$ is well-quasi-ordered, where $\preccurlyeq$ is as defined in Section 3. Let $v \in \mathscr{M}^*$, where

$$v = \left(x_1^{\alpha_1} \cdots x_k^{\alpha_k} \otimes x_1^{\alpha_1'} \cdots x_k^{\alpha_k'}\right) e_{\delta\delta'},$$

with $\delta, \delta' \in \Delta_k$ and $\theta_{\delta\delta'}$ regular. Then we can define $Z(v) \in \mathscr{X}$ by

$$Z(v) = \left(\tilde{A}_k, \theta_{\delta\delta'}, \{\tilde{x}_1, \ldots, \tilde{x}_k\}, \mathbf{t}\right),$$

where $\mathbf{t} = ((\alpha_1, \alpha_1'), \ldots, (\alpha_k, \alpha_k'))$. It is straightforward to verify that if $v$ and $w$ are elements of $\mathscr{M}^*$ such that $Z(v) \preccurlyeq Z(w)$ then $v \preccurlyeq w$. Hence Proposition 3.7 gives the following result.

PROPOSITION 5.8. *The set $(\mathscr{M}^*, \preccurlyeq)$ is well-quasi-ordered.*

If $S$ is any set of elements of $\mathbb{F}(A \times A)$ we write $L_\Psi(S)$ for the $\Psi$-closed left ideal generated by $S$.

LEMMA 5.9. *Let $f \in I_{\delta\delta'} \setminus \{0\}$ and $g \in I_{\varepsilon\varepsilon'} \setminus \{0\}$ where $\delta, \delta' \in \Delta_k$ and $\varepsilon, \varepsilon' \in \Delta_l$. Suppose that* lead$(f) \preccurlyeq$ lead$(g)$. *Then there exists $g^* \in I_{\varepsilon\varepsilon'}$ such that $L_\Psi\{f, g\} = L_\Psi\{f, g^*\}$ and either $g^* = 0$ or* lead$(g^*) <$ lead$(g)$.

PROOF. Write $f = \lambda_1 v_1 + \cdots + \lambda_r v_r$, where $v_i \in \mathscr{M}_{\delta\delta'}$ and $\lambda_i \in \mathbb{F} \setminus \{0\}$ for all $i$, and where $v_i < v_1$ for all $i \geq 2$. Similarly, write $g = \mu_1 w_1 + \cdots + \mu_s w_s$, where $w_i \in \mathscr{M}_{\varepsilon\varepsilon'}$ and $\mu_i \in \mathbb{F} \setminus \{0\}$ for all $i$, and where $w_i < w_1$ for all $i \geq 2$. Write $v = v_1 = \text{lead}(f)$ and $w = w_1 = \text{lead}(g)$. Thus $v \preccurlyeq w$. We use the notation for $v$ and $w$ given in the definition of $\preccurlyeq$. Let $\phi$ and $\psi$ be as in that definition. Let $h_1$ and $h_2$ be the elements of $\mathbb{F}(A_l \times A_l)$ defined by

$$h_1 = \left(x_{1\phi}^{\alpha_1} \cdots x_{k\phi}^{\alpha_k} \otimes x_{1\phi}^{\alpha_1'} \cdots x_{k\phi}^{\alpha_k'}\right) \left(\left(x_1^{\alpha_1} \cdots x_k^{\alpha_k} \otimes x_1^{\alpha_1'} \cdots x_k^{\alpha_k'}\right)^{-1} \psi\right)$$

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and $h_2 = \prod_{j\in C} x_j^{\beta_j} \otimes \prod_{j\in C} x_j^{\beta'_j}$, where $C = \{1, \dots, l\} \setminus \{1\phi, \dots, k\phi\}$. Then

$$h_1(v\psi) = \left(x_{1\phi}^{\alpha_1} \cdots x_{k\phi}^{\alpha_k} \otimes x_{1\phi}^{\alpha'_1} \cdots x_{k\phi}^{\alpha'_k}\right)(e_{\delta\delta'}\psi),$$

and so, by Lemma 5.4,

$$h_1(v\psi)e_{\varepsilon\varepsilon'} = \left(x_{1\phi}^{\alpha_1} \cdots x_{k\phi}^{\alpha_k} \otimes x_{1\phi}^{\alpha'_1} \cdots x_{k\phi}^{\alpha'_k}\right)e_{\varepsilon\varepsilon'} = \left(x_{1\phi}^{\beta_{1\phi}} \cdots x_{k\phi}^{\beta_{k\phi}} \otimes x_{1\phi}^{\beta'_{1\phi}} \cdots x_{k\phi}^{\beta'_{k\phi}}\right)e_{\varepsilon\varepsilon'}.$$

Therefore $h_2h_1(v\psi)e_{\varepsilon\varepsilon'} = \left(x_1^{\beta_1} \cdots x_l^{\beta_l} \otimes x_1^{\beta'_1} \cdots x_l^{\beta'_l}\right)(a \otimes a')e_{\varepsilon\varepsilon'}$, where $a, a' \in A'_l$. By (5.2), $(a \otimes a')e_{\varepsilon\varepsilon'} = \lambda e_{\varepsilon\varepsilon'}$ where $\lambda \in \mathbb{F} \setminus \{0\}$. Hence

$$h_2h_1(v\psi)e_{\varepsilon\varepsilon'} = \lambda\left(x_1^{\beta_1} \cdots x_l^{\beta_l} \otimes x_1^{\beta'_1} \cdots x_l^{\beta'_l}\right)e_{\varepsilon\varepsilon'} = \lambda w.$$

Now let $u$ be an element of $\mathcal{M}_{\delta\delta'}$ such that $u < v$. Write $u = \left(x_1^{\gamma_1} \cdots x_k^{\gamma_k} \otimes x_1^{\gamma'_1} \cdots x_k^{\gamma'_k}\right)e_{\delta\delta'}$. Thus there exists $q \in \{1, \dots, k\}$ such that $(\gamma_q, \gamma'_q) < (\alpha_q, \alpha'_q)$ but $(\gamma_i, \gamma'_i) = (\alpha_i, \alpha'_i)$ for $i = q + 1, \dots, k$. We can write

$$\begin{aligned}\left(x_1^{\alpha_1} \cdots x_k^{\alpha_k} \otimes x_1^{\alpha'_1} \cdots x_k^{\alpha'_k}\right)^{-1}&\left(x_1^{\gamma_1} \cdots x_k^{\gamma_k} \otimes x_1^{\gamma'_1} \cdots x_k^{\gamma'_k}\right) \\ &= \left(x_1^{\gamma_1-\alpha_1} \cdots x_k^{\gamma_k-\alpha_k} \otimes x_1^{\gamma'_1-\alpha'_1} \cdots x_k^{\gamma'_k-\alpha'_k}\right)(b \otimes b')\end{aligned}$$

where $b, b' \in A'_k$. By (5.2), $(b \otimes b')\psi e_{\varepsilon\varepsilon'} = \nu e_{\varepsilon\varepsilon'}$ where $\nu \in \mathbb{F} \setminus \{0\}$. Hence

$$\begin{aligned}h_1(u\psi)e_{\varepsilon\varepsilon'} &= \nu(x_{1\phi}^{\alpha_1} \cdots x_{k\phi}^{\alpha_k} \otimes x_{1\phi}^{\alpha'_1} \cdots x_{k\phi}^{\alpha'_k})((x_1^{\gamma_1-\alpha_1} \cdots x_k^{\gamma_k-\alpha_k} \otimes x_1^{\gamma'_1-\alpha'_1} \cdots x_k^{\gamma'_k-\alpha'_k})\psi)e_{\varepsilon\varepsilon'} \\ &= \nu(x_{1\phi}^{\alpha_1} \cdots x_{k\phi}^{\alpha_k} \otimes x_{1\phi}^{\alpha'_1} \cdots x_{k\phi}^{\alpha'_k})((x_1^{\gamma_1-\alpha_1} \cdots x_q^{\gamma_q-\alpha_q} \otimes x_1^{\gamma'_1-\alpha'_1} \cdots x_q^{\gamma'_q-\alpha'_q})\psi)e_{\varepsilon\varepsilon'}.\end{aligned}$$

From the properties of $\psi$ we calculate that

$$\begin{aligned}\left(\left(x_1^{\gamma_1-\alpha_1} \cdots x_q^{\gamma_q-\alpha_q} \otimes x_1^{\gamma'_1-\alpha'_1} \cdots x_q^{\gamma'_q-\alpha'_q}\right)\psi\right) &e_{\varepsilon\varepsilon'} \\ &= \nu'\left(x_1^{\rho_1} \cdots x_{q\phi-1}^{\rho_{q\phi-1}} x_{q\phi}^{\gamma_q-\alpha_q} \otimes x_1^{\rho'_1} \cdots x_{q\phi-1}^{\rho'_{q\phi-1}} x_{q\phi}^{\gamma'_q-\alpha'_q}\right)e_{\varepsilon\varepsilon'}\end{aligned}$$

where $\nu' \in \mathbb{F} \setminus \{0\}$ and $\rho_1, \dots, \rho_{q\phi-1}, \rho'_1, \dots, \rho'_{q\phi-1} \in \mathbb{Z}/n\mathbb{Z}$. Hence $h_1(u\psi)e_{\varepsilon\varepsilon'}$ has the form

$$\nu''\left(x_1^{\sigma_1} \cdots x_{q\phi-1}^{\sigma_{q\phi-1}} x_{q\phi}^{\gamma_q} x_{(q+1)\phi}^{\alpha_{q+1}} \cdots x_{k\phi}^{\alpha_k} \otimes x_1^{\sigma'_1} \cdots x_{q\phi-1}^{\sigma'_{q\phi-1}} x_{q\phi}^{\gamma'_q} x_{(q+1)\phi}^{\alpha'_{q+1}} \cdots x_{k\phi}^{\alpha'_k}\right)e_{\varepsilon\varepsilon'}$$

where $\nu'' \in \mathbb{F} \setminus \{0\}$ and $\sigma_1, \dots, \sigma_{q\phi-1}, \sigma'_1, \dots, \sigma'_{q\phi-1} \in \mathbb{Z}/n\mathbb{Z}$. Therefore $h_2h_1(u\psi)e_{\varepsilon\varepsilon'}$ is a non-zero scalar multiple of a monomial of the form

$$\left(x_1^{\tau_1} \cdots x_{q\phi-1}^{\tau_{q\phi-1}} x_{q\phi}^{\gamma_q} x_{q\phi+1}^{\beta_{q\phi+1}} \cdots x_l^{\beta_l} \otimes x_1^{\tau'_1} \cdots x_{q\phi-1}^{\tau'_{q\phi-1}} x_{q\phi}^{\gamma'_q} x_{q\phi+1}^{\beta'_{q\phi+1}} \cdots x_l^{\beta'_l}\right)e_{\varepsilon\varepsilon'}$$

where $\tau_1, \dots, \tau_{q\phi-1}, \tau'_1, \dots, \tau'_{q\phi-1} \in \mathbb{Z}/n\mathbb{Z}$. Since $(\gamma_q, \gamma'_q) < (\alpha_q, \alpha'_q) = (\beta_{q\phi}, \beta'_{q\phi})$, this monomial is smaller than $w$.

Since $h_2h_1(f\psi)e_{\varepsilon\varepsilon'} = \lambda_1 h_2h_1(v_1\psi)e_{\varepsilon\varepsilon'} + \cdots + \lambda_r h_2h_1(v_r\psi)e_{\varepsilon\varepsilon'}$, we see that $h_2h_1(f\psi)e_{\varepsilon\varepsilon'}$ has leading monomial $w$ with coefficient $\lambda_1\lambda$. Also, since $\psi$ extends to an element of $\Psi$, we have $h_2h_1(f\psi)e_{\varepsilon\varepsilon'} \in L_\Psi\{f\}$. Let $g^* = g - \mu_1\lambda_1^{-1}\lambda^{-1}h_2h_1(f\psi)e_{\varepsilon\varepsilon'}$. Then $g^*$ has the required properties. □

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Now we are in a position to complete the proof of Theorem C. Let $J$ be a $\Psi$-closed left ideal of $\mathbb{F}(A \times A)$. It suffices to prove that $J$ is finitely generated as a $\Psi$-closed left ideal. By Proposition 5.8, there exists a finite set $S$ of regular elements of $J$ such that for every regular element $g$ of $J$ there exists $f \in S$ such that $\text{lead}(f) \preccurlyeq \text{lead}(g)$. We claim that $J = L_\Psi(S)$. By Lemma 5.7, it suffices to show that every regular element of $J$ belongs to $L_\Psi(S)$. Suppose, in order to get a contradiction, that this is not so, and let $g$ be a regular element of $J$ such that $g \notin L_\Psi(S)$. Suppose $g \in I_{\varepsilon\varepsilon'}$. Choose $g$ with the given properties such that $\text{lead}(g)$ is as small as possible in the finite set $(\mathscr{M}_{\varepsilon\varepsilon'}, \leq)$. There exists $f \in S$ such that $\text{lead}(f) \preccurlyeq \text{lead}(g)$. By Lemma 5.9, there exists $g^* \in I_{\varepsilon\varepsilon'}$ such that $L_\Psi\{f, g\} = L_\Psi\{f, g^*\}$ and either $g^* = 0$ or $\text{lead}(g^*) < \text{lead}(g)$. Since $g \notin L_\Psi\{f\}$, we have $g^* \neq 0$. Since $g^* \in L_\Psi\{f, g\} \subseteq J$, the choice of $g$ gives that $g^* \in L_\Psi(S)$. Hence $g \in L_\Psi\{f, g^*\} \subseteq L_\Psi(S)$ and we have the required contradiction.

## 6. Proof of Theorem D

Let $n$, $A$, $\Psi$, $\mathbb{F}$ and $R$ be as in Section 1, where $\mathbb{F}$ is a field of characteristic not dividing $n$. Let $\overline{\mathbb{F}}$ be the algebraic closure of $\mathbb{F}$. The subalgebra $\overline{\mathbb{F}} \otimes_{\mathbb{F}} R$ of $\overline{\mathbb{F}}(A \times A)$ corresponds to $R$ in $\mathbb{F}(A \times A)$. If $M$ is an $(R, \Psi)$-submodule of $\mathbb{F}(A \times A)$ which contains $R$, then $\overline{\mathbb{F}} \otimes_{\mathbb{F}} M$ is an $(\overline{\mathbb{F}} \otimes_{\mathbb{F}} R, \Psi)$-submodule of $\overline{\mathbb{F}}(A \times A)$ which contains $\overline{\mathbb{F}} \otimes_{\mathbb{F}} R$, and $M = \mathbb{F}(A \times A) \cap \overline{\mathbb{F}} \otimes_{\mathbb{F}} M$. Therefore, to prove Theorem D, we may assume that $\overline{\mathbb{F}} = \mathbb{F}$.

We shall use the notation of Section 5. If $n \leq 2$, then Theorem D follows from Proposition 4.5. Thus, as in Section 5, we assume that $n > 2$, so that $n_0 > 1$.

Let $P$ be the subgroup of $A \times A$ defined by $P = \{(c, c^{-1}) : c \in A'\}$ and let $H = (A \times A)/P$. Note that $P$ is a $\Psi$-closed subgroup of $A \times A$, so each element of $\Psi$ induces endomorphisms of $H$ and $\mathbb{F}H$. For $i, j \in \mathbb{N}$, let $c_{ij}$ be the element of $H$ given by $c_{ij} = ([x_i, x_j], 1)P = (1, [x_i, x_j])P$.

For each positive integer $k$, let $H_k$ be the subgroup of $H$ generated by the elements $(x_i, 1)P$ and $(1, x_i)P$ for $i = 1, \dots, k$. It is easily verified that $H_k'$ is a free abelian group of exponent $n_0$ with basis $\{c_{ij} : 1 \leq i < j \leq k\}$. Furthermore, there are isomorphisms from $A_k'$ to $H_k'$ and from $\mathbb{F}A_k'$ to $\mathbb{F}H_k'$ given by $[x_i, x_j] \mapsto c_{ij}$ for all $i, j$. If $\psi : A_k \to A_l$ is a homomorphism, where $k, l \in \mathbb{N}$, then the associated homomorphism $\psi : A_k \times A_k \to A_l \times A_l$ yields homomorphisms $\overline{\psi} : H_k \to H_l$ and $\overline{\psi} : \mathbb{F}H_k \to \mathbb{F}H_l$.

Let $K = \mathbb{Z}/n_0\mathbb{Z}$, and let $Q_k$ and $\Delta_k$ be as in Section 5. For each $\delta \in \Delta_k$, let $\chi_\delta$ and $e_\delta$ be defined as in Section 5, but with respect to $H_k'$ rather than $A_k'$. Thus $\chi_\delta$ is a character of $H_k'$ and $e_\delta$ is an idempotent of $\mathbb{F}H_k'$. Results (5.2)–(5.5) apply just as before. For $\delta \in \Delta_k$ we define $J_\delta = (\mathbb{F}H_k)e_\delta$. Thus $J_\delta$ is the ideal of $\mathbb{F}H_k$ generated by $e_\delta$, and we have $\mathbb{F}H_k = \bigoplus_{\delta \in \Delta_k} J_\delta$.

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For each $k$ we write $Q_k^0 = Q_k \setminus \{(1,2),(3,4),\dots\}$. An element $\delta$ of $\Delta_k$ will be called *standard* if $\delta(i,j) = 0$ (equivalently, $\chi_\delta(c_{ij}) = 1$) for all $(i,j) \in Q_k^0$. We write $\Delta_k^*$ for the set of all standard elements of $\Delta_k$ and $\Delta_k^0 = \Delta_k \setminus \Delta_k^*$.

For each $\delta \in \Delta_k$, let $\theta_\delta$ be the alternating $K$-form on $\tilde{A}_k$ defined as in Section 5. Thus $(\tilde{A}_k, \theta_\delta)$ is standard with respect to $\{\tilde{x}_1, \dots, \tilde{x}_k\}$ (in the terminology of Section 3) if and only if $\delta$ is standard, that is, $\delta \in \Delta_k^*$.

LEMMA 6.1. *Let $\delta \in \Delta_k$. Then there exists $\eta \in \mathrm{Aut}(A_k)$ such that, for the induced automorphism $\overline{\eta} : \mathbb{F}H_k \to \mathbb{F}H_k$, we have $e_\delta\overline{\eta} = e_\varepsilon$ where $\varepsilon \in \Delta_k^*$.*

PROOF. By Lemma 3.8 there is a basis $\{\tilde{a}_1, \dots, \tilde{a}_k\}$ of $\tilde{A}_k$ such that $(\tilde{A}_k, \theta_\delta)$ is standard with respect to this basis. As in the proof of Lemma 5.5, there is a free generating set $\{y_1, \dots, y_k\}$ of $A_k$ such that $\tilde{y}_i = \tilde{a}_i$ for $i = 1, \dots, k$. Let $\eta$ be the automorphism of $A_k$ satisfying $y_i\eta = x_i$ for $i = 1, \dots, k$. Note that $\eta$ acts on $A_k'$ just as $\overline{\eta}$ acts on $H_k'$. Thus Lemma 5.1 shows that $e_\delta\overline{\eta} = e_\varepsilon$, where $\varepsilon \in \Delta_k$ and $\theta_\varepsilon(\tilde{x}_i, \tilde{x}_j) = \theta_\delta(\tilde{y}_i, \tilde{y}_j)$ for all $i, j$. Thus $(\tilde{A}_k, \theta_\varepsilon)$ is standard with respect to $\{\tilde{x}_1, \dots, \tilde{x}_k\}$, that is, $\varepsilon \in \Delta_k^*$. □

Since $\mathbb{F}H'$ is a subalgebra of $\mathbb{F}H$, we may regard $\mathbb{F}H$ as a left $\mathbb{F}H'$-module. Following the terminology of Section 1, we shall consider $(\mathbb{F}H', \Psi)$-submodules of $\mathbb{F}H$. A non-zero element $w$ of $\mathbb{F}H$ will be called *standard* if $w \in J_\delta$ for some $k$ and some $\delta \in \Delta_k^*$.

LEMMA 6.2. *Every $(\mathbb{F}H', \Psi)$-submodule of $\mathbb{F}H$ is generated, as a $\Psi$-closed vector space, by standard elements.*

PROOF. This is similar to the proof of Lemma 5.7, with Lemma 6.1 taking the place of Lemma 5.5. □

Let $C$ be the subgroup of $H$ generated by all elements $c_{ij}$ for which $i < j$ and $(i,j) \notin \{(1,2),(3,4),\dots\}$. Let $\rho$ be the natural homomorphism $\rho : H \to H/C$. We also denote by $\rho$ the associated homomorphisms $H_k \to H/C$ and $\mathbb{F}H_k \to \mathbb{F}(H/C)$. Clearly the kernel of $\rho : H_k \to H/C$ is the subgroup of $H_k$ generated by all $c_{ij}$ for which $(i,j) \in Q_k^0$. Thus the kernel of $\rho : \mathbb{F}H_k \to \mathbb{F}(H/C)$ is the ideal generated by the elements $c_{ij} - 1$ for $(i,j) \in Q_k^0$. We write $(\mathbb{F}H_k)^* = \bigoplus_{\delta\in\Delta_k^*} J_\delta$ and $(\mathbb{F}H_k)^0 = \bigoplus_{\delta\in\Delta_k^0} J_\delta$.

LEMMA 6.3. *The kernel of $\rho : \mathbb{F}H_k \to \mathbb{F}(H/C)$ is $(\mathbb{F}H_k)^0$.*

PROOF. Let $\delta \in \Delta_k^0$. Then $\chi_\delta(c_{ij}) \neq 1$ for some $(i,j) \in Q_k^0$. By (5.2), $(c_{ij} - 1)e_\delta$ is a non-zero scalar multiple of $e_\delta$. But clearly $(c_{ij} - 1)e_\delta \in \ker(\rho)$. Thus $e_\delta \in \ker(\rho)$. It follows that $J_\delta \subseteq \ker(\rho)$ and so $(\mathbb{F}H_k)^0 \subseteq \ker(\rho)$.

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Let $(i, j) \in Q_k^0$. Then, for $\varepsilon \in \Delta_k^*$, we have $(c_{ij} - 1)e_\varepsilon = (\chi_\varepsilon(c_{ij}) - 1)e_\varepsilon = 0$. Hence $c_{ij} - 1 = (c_{ij} - 1)\sum_{\delta \in \Delta_k} e_\delta = (c_{ij} - 1)\sum_{\delta \in \Delta_k^0} e_\delta$. Hence $c_{ij} - 1$ belongs to the ideal $(\mathbb{F}H_k)^0$. Since this holds for all $(i, j) \in Q_k^0$ we obtain $\ker(\rho) \subseteq (\mathbb{F}H_k)^0$. □

For $k \in \mathbb{N}$, let $\psi_k$ be the endomorphism of $A$ defined by $x_i\psi_k = 1$ for $i > k$ and $x_i\psi_k = x_i$ for $i \leq k$. Also write $\psi_k$ for the induced endomorphisms of $H$ and $\mathbb{F}H$.

LEMMA 6.4. *Let $u \in (\mathbb{F}H_k)^*$ and let $l \geq k$. Then there exists $v \in (\mathbb{F}H_l)^*$ such that $v\psi_k = u$ and $v\rho = u\rho$.*

PROOF. Let $B$ be the subgroup of $H_l'$ generated by all elements $c_{ij}$ for $(i, j) \in Q_l^0 \setminus Q_k^0$. Let $v = u(|B|^{-1}\sum_{h \in B} h)$. Clearly $v\psi_k = u$ and $v\rho = u\rho$. To prove that $v \in (\mathbb{F}H_l)^*$ it is enough to show that $ve_\varepsilon = 0$ for all $\varepsilon \in \Delta_l^0$.

Let $\varepsilon \in \Delta_l^0$. Then there exists $(i, j) \in Q_l^0$ such that $\chi_\varepsilon(c_{ij}) \neq 1$. We consider two cases. Suppose first that $(i, j) \in Q_k^0$. Then the restriction of $\chi_\varepsilon$ to $H_k'$ has the form $\chi_{\delta'}$ for some $\delta' \in \Delta_k^0$. Then for all $\delta \in \Delta_k^*$ we have $e_\delta e_\varepsilon = \chi_{\delta'}(e_\delta)e_\varepsilon = 0$, by (5.2) and (5.4). Hence $ue_\varepsilon = 0$ and so $ve_\varepsilon = 0$. Suppose secondly that $(i, j) \in Q_l^0 \setminus Q_k^0$. Then $\sum_{h \in B} h$ can be written as $w(1 + c_{ij} + \cdots + c_{ij}^{n_0})$ for some $w \in \mathbb{F}H_l'$. Since $\chi_\varepsilon(c_{ij})$ is a non-trivial $n_0$-th root of unity, $\chi_\varepsilon(1 + c_{ij} + \cdots + c_{ij}^{n_0}) = 0$. Thus $(1 + c_{ij} + \cdots + c_{ij}^{n_0})e_\varepsilon = 0$ and so $ve_\varepsilon = 0$. □

LEMMA 6.5. *Suppose that $M_1$ and $M_2$ are $(\mathbb{F}H', \Psi)$-submodules of $\mathbb{F}H$ such that $M_1\rho = M_2\rho$. Then $M_1 = M_2$.*

PROOF. Suppose, in order to get a contradiction, that $M_1 \neq M_2$. Without loss of generality we may assume that $M_1 \not\subseteq M_2$. By Lemma 6.2 there exist $k$ and $\delta \in \Delta_k^*$ such that $M_1 \cap J_\delta \not\subseteq M_2$. Hence there exists $u \in (\mathbb{F}H_k)^*$ such that $u \in M_1 \setminus M_2$. By hypothesis there exists $w \in M_2$ such that $u\rho = w\rho$. Choose $l \geq k$ such that $w \in \mathbb{F}H_l$. Then $w = w^* + w^0$ where $w^* \in (\mathbb{F}H_l)^*$ and $w^0 \in (\mathbb{F}H_l)^0$. Since $M_2$ is an $\mathbb{F}H'$-module, $w^* \in M_2$. Also $u\rho = w\rho = w^*\rho$. By Lemma 6.4 there exists $v \in (\mathbb{F}H_l)^*$ such that $v\psi_k = u$ and $v\rho = u\rho$. Thus $v\rho = w^*\rho$. By Lemma 6.3, this gives $v = w^* \in M_2$. Hence $u = v\psi_k \in M_2$, which is a contradiction. □

Now we return to the group $H/C$. Recall that $H = (A \times A)/P$. For each $i \in \mathbb{N}$, let $G_i$ be the subgroup of $H/C$ generated by the four elements $((x_{2i-1}, 1)P)\rho$, $((1, x_{2i-1})P)\rho$, $((x_{2i}, 1)P)\rho$ and $((1, x_{2i})P)\rho$. Write $G = G_1$. Thus $G$ is a finite group. It is easily verified that $H/C$ is the direct product of the groups $G_i$, and, for each $i$, there is an obvious isomorphism from $G$ to $G_i$. Thus we may identify $H/C$ with the direct power $D$ of $G$ considered in Section 4. Let $\Xi$ be the set of endomorphisms of $D$ defined in Section 4.

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LEMMA 6.6. *Let $M$ be a $\Psi$-closed subspace of $\mathbb{F}H$. Then $M\rho$ is a $\Xi$-closed subspace of $\mathbb{F}D$.*

PROOF. Let $\xi \in \Xi$ and suppose that $\xi$ is associated with $\phi$, $X$ and $\sigma$ in the notation of Section 4. It suffices to show that there exists an endomorphism $\psi$ of $A$ such that the induced endomorphism of $H$ leaves $C$ invariant and induces $\xi$ on $H/C$. To simplify the notation we rewrite the generators of $A$ by setting $y_i = x_{2i-1}$ and $z_i = x_{2i}$ for all $i \in \mathbb{N}$. We define a homomorphism $\psi : A \to A$ by

$$y_i\psi = y_{i\phi} \prod_{\substack{j \in X \\ j\sigma = i\phi}} y_j, \quad z_i\psi = z_{i\phi} \prod_{\substack{j \in X \\ j\sigma = i\phi}} z_j,$$

for each $i$. The products are taken over all those values of $j$, if any, which lie in $X$ and satisfy $j\sigma = i\phi$, and the terms $y_j$ and $z_j$ are taken in increasing order of $j$ (this is an arbitrary choice). It is straightforward to verify that $\psi$ has the required properties. □

By Proposition 4.4 together with Lemma 6.5 and Lemma 6.6 we obtain

LEMMA 6.7. *The maximal condition holds for $(\mathbb{F}H', \Psi)$-submodules of $\mathbb{F}H$.*

Consider the natural homomorphism $\pi : A \times A \to H$ with kernel $P$, and let $I$ be the kernel of the corresponding homomorphism $\pi : \mathbb{F}(A \times A) \to \mathbb{F}H$.

LEMMA 6.8. *The maximal condition holds for $(R, \Psi)$-submodules of $\mathbb{F}(A \times A)$ which contain $I$.*

PROOF. By Lemma 6.7 it suffices to show that if $M$ is an $(R, \Psi)$-submodule of $\mathbb{F}(A \times A)$ which contains $I$ then $M\pi$ is an $(\mathbb{F}H', \Psi)$-submodule of $\mathbb{F}H$. It is clear that $M\pi$ is $\Psi$-closed, by definition of the action of $\Psi$ on $\mathbb{F}H$. Also, $M\pi$ is a left $R\pi$-submodule of $\mathbb{F}H$. Thus it suffices to show that $H' \subseteq R\pi$. Since $R\pi$ is an algebra, it suffices to show that $c_{ij} \in R\pi$ for all $i, j$. Note that $([x_i, x_j] \otimes [x_i, x_j])\pi = c_{ij}^2$ and $([x_i, x_j] \otimes 1 + 1 \otimes [x_i, x_j])\pi = 2c_{ij}$. Hence $c_{ij}^2 \in R\pi$ and $2c_{ij} \in R\pi$. If $n_0$ is odd then $c_{ij}^2 \in R\pi$ gives $c_{ij} \in R\pi$. But if $n_0$ is even then $\mathbb{F}$ does not have characteristic 2 and $2c_{ij} \in R\pi$ gives $c_{ij} \in R\pi$. □

In the notation of Section 5, we can write $\mathbb{F}(A_k \times A_k) = \bigoplus_{\delta,\delta' \in \Delta_k} I_{\delta\delta'}$. Note that

$$(e_\delta \otimes 1)\pi = (1 \otimes e_\delta)\pi = e_\delta \in \mathbb{F}H_k'.$$

Hence, for $\delta \neq \delta'$, we have $e_{\delta\delta'}\pi = e_\delta e_{\delta'} = 0$ and so $I_{\delta\delta'} \subseteq \ker(\pi) = I$. It is easily checked that $\bigoplus_{\delta \in \Delta_k} I_{\delta\delta}$ and $\mathbb{F}H_k$ have the same dimension. Hence

$$I \cap \mathbb{F}(A_k \times A_k) = \bigoplus_{\substack{\delta,\delta' \in \Delta_k \\ \delta \neq \delta'}} I_{\delta\delta'}. \tag{6.1}$$

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LEMMA 6.9. *Let $M$ be an $(R, \Psi)$-submodule of $\mathbb{F}(A \times A)$ such that $R \cap I \subseteq M \subseteq I$, and let $T$ be the largest $\Psi$-closed left ideal of $\mathbb{F}(A \times A)$ contained in $M$. Then $M = T + (R \cap I)$.*

PROOF. Let $L$ be the subspace of $M$ spanned by all elements of $M$ which have the form $we_{\delta\delta'}$ where $w \in \mathbb{F}(A \times A)$ and $\delta, \delta' \in \Delta_k$ for some $k$, with $\delta \neq \delta'$. Let $we_{\delta\delta'}$ be such an element of $M$. Let $\psi \in \Psi$ and $a, a' \in A$. Choose $l \geq k$ so that $e_\delta\psi, e_{\delta'}\psi \in \mathbb{F}A'_l$. Since $e_\delta\psi$ and $e_{\delta'}\psi$ are idempotents, we can write $e_\delta\psi = \sum_{\lambda\in\Lambda} e_\lambda$ and $e_{\delta'}\psi = \sum_{\lambda'\in\Lambda'} e_{\lambda'}$ where $\Lambda, \Lambda' \subseteq \Delta_l$. But $(e_\delta\psi)(e_{\delta'}\psi) = (e_\delta e_{\delta'})\psi = 0$. Thus $\Lambda$ and $\Lambda'$ are disjoint. For $\varepsilon \in \Lambda$ and $\varepsilon' \in \Lambda'$,

$$((a \otimes a')e_{\varepsilon\varepsilon'} + (a' \otimes a)e_{\varepsilon'\varepsilon})(w\psi)(e_{\delta\delta'}\psi) \in M,$$

because $M$ is an $(R, \Psi)$-module. However, $e_{\delta\delta'}\psi = \sum_{\lambda,\lambda'} e_{\lambda\lambda'}$. Hence $e_{\varepsilon'\varepsilon}(e_{\delta\delta'}\psi) = 0$ and $e_{\varepsilon\varepsilon'}(e_{\delta\delta'}\psi) = e_{\varepsilon\varepsilon'}$. Therefore $(a \otimes a')(w\psi)e_{\varepsilon\varepsilon'} \in M$, and so $(a \otimes a')(w\psi)e_{\varepsilon\varepsilon'} \in L$. Since this holds for all $\varepsilon, \varepsilon'$, we have $(a \otimes a')(w\psi)(e_{\delta\delta'}\psi) \in L$. Therefore $L$ is a $\Psi$-closed left ideal of $\mathbb{F}(A \times A)$. We next prove that $M = L + (R \cap I)$, which will give the required result.

Let $u \in M$ and choose $k$ so that $u \in \mathbb{F}(A_k \times A_k)$. Since $M \subseteq I$ we can use (6.1) to write $u = \sum w_{\delta\delta'}e_{\delta\delta'}$, where the sum is over all $\delta, \delta' \in \Delta_k$ with $\delta \neq \delta'$ and each $w_{\delta\delta'}$ belongs to $\mathbb{F}(A_k \times A_k)$. Let $\delta, \delta' \in \Delta_k$ with $\delta \neq \delta'$. Since $M$ is an $R$-module,

$$(e_{\delta\delta'} + e_{\delta'\delta})u = w_{\delta\delta'}e_{\delta\delta'} + w_{\delta'\delta}e_{\delta'\delta} \in M.$$

Write $v = w_{\delta\delta'}$ and $v' = w_{\delta'\delta}$. Then it suffices to show that $ve_{\delta\delta'} + v'e_{\delta'\delta} \in L + (R \cap I)$.

Let $\tau$ be the involutory automorphism of $\mathbb{F}(A \times A)$ satisfying $(a \otimes a')\tau = a' \otimes a$ for all $a, a' \in A$. Then $w + w\tau \in R$ for all $w \in \mathbb{F}(A \times A)$. We can write

$$ve_{\delta\delta'} + v'e_{\delta'\delta} = (v - v'\tau)e_{\delta\delta'} + v'e_{\delta'\delta} + (v'\tau)e_{\delta\delta'}. \tag{6.2}$$

Here

$$v'e_{\delta'\delta} + (v'\tau)e_{\delta\delta'} = v'e_{\delta'\delta} + (v'e_{\delta'\delta})\tau \in R \cap I.$$

Since $R \cap I \subseteq M$, (6.2) gives $(v - v'\tau)e_{\delta\delta'} \in M$, and so $(v - v'\tau)e_{\delta\delta'} \in L$. Therefore, by (6.2), $ve_{\delta\delta'} + v'e_{\delta'\delta} \in L + (R \cap I)$, as required. □

To complete the proof of Theorem D, let $M_1 \subseteq M_2 \subseteq \dots$ be an ascending chain of $(R, \Psi)$-submodules of $\mathbb{F}(A \times A)$ which contain $R$. By Lemma 6.8 the chain $M_1 + I \subseteq M_2 + I \subseteq \cdots$ becomes stationary. Thus it suffices to show that the chain $M_1 \cap I \subseteq M_2 \cap I \subseteq \cdots$ becomes stationary. For each $i$, let $T_i$ be the largest $\Psi$-closed left ideal of $\mathbb{F}(A \times A)$ contained in $M_i \cap I$. By Lemma 6.9 it suffices to show that the chain $T_1 \subseteq T_2 \subseteq \cdots$ becomes stationary. But this holds by Theorem C.

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