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ON FINITE INVARIANT MEASURE FOR SEMIGROUPS OF OPERATORS

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Introduction. Let Σ be a left amenable semigroup, and let $\{T_\sigma : \sigma \in \Sigma\}$ be a representation of Σ as a semigroup of positive linear contraction operators on $L_1(X, \mathscr{A}, p)$. This paper is devoted to the study of existence of a finite equivalent invariant measure for such semigroups of operators. Various necessary and sufficient conditions have, at times, been given by different authors for the existence of a finite equivalent invariant measure for a positive linear operator of norm ≤ 1, i.e. for semigroups generated by one operator. The theorems presented in this paper extend to left amenable semigroups the results already known for the particular semigroup generated by one operator. Theorem 3.1 is an extension of a result of Dean and Sucheston [2]. This theorem uses the identification of the functionals M and m, the supremum and infimum respectively of all the left invariant means on Σ. The identification of M, and therefore trivially of m, is due to Granirer [4]. Theorem 3.2 of this paper was proved for powers of a point transformation by Sucheston [11], and was extended to left amenable semigroups of operators by Lloyd [7]. Using the method of Arens products, Lloyd [7] obtained something more than this theorem. Here we obtain an exact generalization of Sucheston's theorem by another, simple method. The equivalence of conditions (i), (ii), and (iii) of Theorem 3.3 for powers of an operator was obtained in Dean and Sucheston [2] and in Neveu [10]. (i) and (iv) of the same theorem, for a point transformation, were shown to be equivalent by Hajian and Kakutani [6] and Sucheston [12], and for a left amenable semigroup of nonsingular and measurable transformations, by Natarajan [8]; see also Hajian and Ito [5]. Part of the proof of Theorem 3.3 resembles the proof of Lemma 3 of Neveu [10], with a modification suggested by Granirer's approach [4]. Finally, in Theorem 4.1, we extend to Markov kernels the results proved by Granirer [4] for amenable semigroups of point transformations not necessarily null-preserving.

1. **Definitions and preliminaries.** If $\mathscr{S}$ is a semigroup, $B(\mathscr{S})$ denotes the Banach space of all bounded real-valued functions on $\mathscr{S}$, with supremum norm $\|f\| = \sup_{s \in \mathscr{S}} |f(s)|$. A linear functional φ on $B(\mathscr{S})$ is a *mean* iff $\inf_s f(s) \leq \varphi(f) \leq \sup_s f(s)$,

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$f \in B(\mathscr{S})$. A mean φ is called a *finite mean* iff $\varphi = \sum_{i=1}^{n} \alpha_i \cdot 1_{s_i}$ for some $\alpha_i \geq 0$, $\sum_{i=1}^{n} \alpha_i = 1$, and $s_i \in \mathscr{S}$, where 1_s, $s \in \mathscr{S}$, are the evaluation functionals defined by $1_s f = f(s)$. Let $l_1(\mathscr{S})$ denote the set of all functions θ on $\mathscr{S}$ such that $\sum_s |\theta(s)| < \infty$. Clearly, if Q is the natural embedding of $l_1(\mathscr{S})$ into its second conjugate space $B(\mathscr{S})^*$, then $Q(\theta) = \sum_s \theta(s) 1_s$. Thus a finite mean φ can always be written in the form $\varphi = \sum_s \theta(s) 1_s = Q(\theta)$ for some $\theta \in l_1(\mathscr{S})$ such that $\theta(s) \geq 0$, $\sum_s \theta(s) = 1$, and $\theta(s) = 0$ except for a finite number of $s \in \mathscr{S}$. The set of all finite means will be denoted by F. A mean φ is said to be *left invariant* iff $\varphi(L_a f) = \varphi(f)$ for all $f \in B(\mathscr{S})$ and $a \in \mathscr{S}$, where L_a is a *left shift* defined by $(L_a f)(s) = f(as)$. The right shift R_a and the right invariant means are defined analogously. *LIM* resp. *RIM* denote the sets of all left invariant respectively right invariant means on $\mathscr{S}$. $\mathscr{S}$ is *left amenable* iff $LIM \neq \varnothing$; right amenable iff $RIM \neq \varnothing$; amenable iff $LIM \cap RIM \neq \varnothing$. If $\mathscr{S}$ is left amenable, we denote, for $f \in B(\mathscr{S})$,

$$M(f) = \sup\{\varphi(f) : \varphi \in LIM\}$$

and

$$m(f) = \inf\{\varphi(f) : \varphi \in LIM\}.$$

If $\psi \in B(\mathscr{S})^*$ is of the form $\psi = \sum_{i=1}^{n} \beta_i 1_{s_i}$, define $L_\psi f = \sum_{i=1}^{n} \beta_i L_{s_i} f$. Clearly, $\|L_\psi f\| \leq \|\psi\| \cdot \|f\|$, where $\|\psi\| = \sum_{i=1}^{n} |\beta_i|$ is the l_1-norm, and $\|f\|$ is the supremum norm. We note that $\varphi(L_\psi f) = \varphi(f)$ for every $\varphi \in LIM$ and $\psi \in F$. A net $\psi_\alpha \in F$ is said to be *convergent in norm to left invariance* iff $\lim_\alpha \|\psi_\alpha L_a - \psi_\alpha\| = 0$ for all $a \in \mathscr{S}$, where $\psi_\alpha L_a$ is the mean defined by $(\psi_\alpha L_a)(f) = \psi_\alpha(L_a f)$.

Let $(X, \mathscr{A}, p, \mathscr{S})$ be given, where $\mathscr{A}$ is a σ-algebra of subsets of a nonvoid set X, p is a probability measure on $(X, \mathscr{A})$ and $\mathscr{S} = \{T_\sigma : \sigma \in \Sigma\}$ is a representation of a left amenable semigroup Σ as a semigroup of positive linear operators on $L_1(X, \mathscr{A}, p)$, such that $\|T_\sigma\| \leq 1$. Multiplication in $\mathscr{S}$ is defined by $T_{\sigma_1} \cdot T_{\sigma_2} = T_{\sigma_1 \sigma_2}$. A measure $\mu \ll p$ is said to be *T-invariant* iff $T\mu = \mu$, where $T\mu$ is the measure defined by $(T\mu)(A) = \int_A T(d\mu/dp)\, dp$, $A \in \mathscr{A}$. μ is *$\mathscr{S}$-invariant* iff μ is T_σ-invariant for each $\sigma \in \Sigma$. An $f \in L_1$ is said to be a *positive fixed point* for $\mathscr{S}$ iff $T_\sigma f = f$ for all $\sigma \in \Sigma$, and $p\{f > 0\} = 1$. Thus μ, equivalent to p, is a finite $\mathscr{S}$-invariant measure iff $d\mu/dp$ is a positive fixed point for $\mathscr{S}$.

2. **Identification of M, m; and a theorem of Day.** Theorems 2.1 and 2.2, due to Granirer [4] and to Day [1] respectively, are stated here without proof.

THEOREM 2.1 (*Granirer*). *Let $\mathscr{S}$ be a left amenable semigroup, and let $\psi_\alpha \in F$ be a net converging in norm to left invariance, i.e.* $\lim \|\psi_\alpha L_a - \psi_\alpha\| = 0$ *for all $a \in \mathscr{S}$, then*

$$M(f) = \inf_{\psi \in F} \sup_s (L_\psi f)(s) = \lim_\alpha \sup_s (L_{\psi_\alpha} f)(s).$$

COROLLARY 2.1. *Under the same assumptions as in Theorem 1.1, we have*

$$m(f) = \sup_{\psi \in F} \inf_s (L_\psi f)(s) = \lim_\alpha \inf_s (L_{\psi_\alpha} f)(s).$$

THEOREM 2.2 (*Day*). *Let $\mathscr{S}$ be a left amenable semigroup. Then there exists a net $\psi_\alpha \in F$ such that* $\lim_\alpha \|\psi_\alpha L_a - \psi_\alpha\| = 0$ *for all $a \in \mathscr{S}$.*

THEOREM 2.3. *Let $\mathscr{S}$ be a countably generated left amenable semigroup. Then there exists a sequence $\psi_n \in F$ such that* $\lim_n \|\psi_n L_a - \psi_n\| = 0$ *for all $a \in \mathscr{S}$.*

Proof. Let $\{s_i : i=1, 2, 3, \ldots\}$ be a countable set of generators for the semigroup $\mathscr{S}$. Since $\|\psi L_{a_1 a_2} - \psi\| \le \|\psi L_{a_2} L_{a_1} - \psi L_{a_2}\| + \|\psi L_{a_2} - \psi\| \le \|\psi L_{a_1} - \psi\| + \|\psi L_{a_2} - \psi\|$ for each $\psi \in F$ and $a_1, a_2 \in \mathscr{S}$ and each $a \in \mathscr{S}$ is a product of s_i's, it suffices to prove that there exists a sequence $\psi_n \in F$ such that $\lim_n \|\psi_n L_{s_i} - \psi_n\| = 0$ for $i=1, 2, 3, \ldots$. By the previous theorem, there exists a net $\varphi_\alpha \in F$ such that $\lim_\alpha \|\varphi_\alpha L_{s_i} - \varphi_\alpha\| = 0$ for each i. Therefore there exist $\alpha_{n,i}$ such that $\alpha \ge \alpha_{n,i}$ implies that $\|\varphi_\alpha L_{s_i} - \varphi_\alpha\| < 1/n$. Let β_n be such that $\beta_n > \alpha_{n,i}$ for $1 \le i \le n$. Then $\|\varphi_{\beta_n} L_{s_i} - \varphi_{\beta_n}\| < 1/n$ for $1 \le i \le n$. Therefore $\lim_n \|\varphi_{\beta_n} L_{s_i} - \varphi_{\beta_n}\| = 0$ for $i=1, 2, 3, \ldots$. That concluded the proof of the theorem.

3. **Finite invariant measures for left amenable semigroups of L_1-operators.** In this section, Σ is assumed to be a left amenable semigroup and $\mathscr{S} = \{T_\sigma : \sigma \in \Sigma\}$ is a representation of Σ as a semigroup of positive linear contraction operators on $L_1(X, \mathscr{A}, p)$.

THEOREM 3.1. *Let $\mathscr{S} = \{T_\sigma : \sigma \in \Sigma\}$ be a countably generated left amenable semigroup of positive linear contraction operators on $L_1(X, \mathscr{A}, p)$. If there exists a positive fixed point for $\mathscr{S}$, then the T_σ are all conservative and for each $A \in \mathscr{A}$, all the left invariant means on $\int_A T_\sigma 1\, dp$ coincide. Conversely, if $T_\sigma^* 1 = 1$ for each $\sigma \in \Sigma$, and if for each set $A \in \mathscr{A}$, all the left invariant means on $\int_A T_\sigma 1\, dp$ coincide, then there is a positive fixed point for $\mathscr{S}$.*

Proof. Assume that f_0 is a positive fixed point for $\mathscr{S}$. Then T_σ are all conservative, since $\{\sum_k T_\sigma^k f_0 = \infty\} = \{f_0 > 0\} = X$ for all σ. We are to show that $\varphi_1\left(\int_A T_\sigma 1\, dp\right) = \varphi_2\left(\int_A T_\sigma 1\, dp\right)$ for every pair $\varphi_1, \varphi_2 \in LIM$. Let $\mu_i(A) = \varphi_i\left(\int_A T_\sigma 1\, dp\right)$, $i=1, 2$. Then, as in the case of cyclic semigroup (see [2, p. 8]), we obtain that μ_1, μ_2 are $\mathscr{S}$-invariant measures. Clearly $\mu_i \ll p$; to show that $p \ll \mu_i$, we let $f_i = d\mu_i/dp$. Let $\mathscr{I}_\sigma = \{A : T_\sigma^* 1_A = 1_A\}$, $\mathscr{I} = \bigcap_{\sigma \in \Sigma} \mathscr{I}_\sigma = \{A : T_\sigma^* 1_A = 1_A \text{ for all } \sigma \in \Sigma\}$. Then the sets in $\mathscr{I}_\sigma$ are of the form $\{\sum_k T_\sigma^k f = \infty\}$, $f \in L_1^+$ (see [9, p. 196]). Also, $A \in \mathscr{I}$ implies that $\mu_1(A) = p(A) = \mu_2(A)$. If $A_i = \{f_i = 0\}$, then $A_i^c = \{f_i > 0\} = \{\sum_k T_\sigma^k f_i = \infty\} \in \mathscr{I}_\sigma$ for all $\sigma \in \Sigma$. It follows that $T_\sigma^* 1_{A_i^c} = 1_{A_i^c}$ and thus $T_\sigma^* 1_{A_i} = T_\sigma^* 1 - T_\sigma^* 1_{A_i^c} = 1 - 1_{A_i^c}$ for all $\sigma \in \Sigma$, i.e. $A_i \in \mathscr{I}$. Therefore $p(A_i) = \mu_i(A_i) = \int_{A_i} f_i\, dp = 0$, i.e. $p\{f_i > 0\} = 1$. Hence f_1, f_2 are positive fixed points. By Chacon–Ornstein theorem,

$$\frac{f_1}{f_2} = \frac{\sum_{k=0}^{n-1} T_\sigma^k f_1}{\sum_{k=0}^{n-1} T_\sigma^k f_2} \longrightarrow \frac{E(f_1 \mid \mathscr{I}_\sigma)}{E(f_2 \mid \mathscr{I}_\sigma)} \quad \text{a.e.}$$

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Therefore f_1/f_2 is $\mathscr{I}_\sigma$-measurable for all $\sigma \in \Sigma$ and hence $\mathscr{I}$-measurable. Consider the space $L_1(X, \mathscr{A}, \mu_2)$;

$$\int \frac{f_1}{f_2}\, d\mu_2 = \int \frac{d\mu_1/dp}{d\mu_2/dp}\, d\mu_2 = \mu_1(X) < \infty$$

implies that $f_1/f_2 \in L_1(X, \mathscr{A}, \mu_2)$. This and the fact that f_1/f_2 is $\mathscr{I}$-measurable further imply that $E(f_1/f_2 \mid \mathscr{I}) = f_1/f_2$, where μ_2 is the measure in view while taking conditional expectations. But $A \in \mathscr{I}$ implies that $\mu_1(A) = \mu_2(A)$, thus

$$\int_A \frac{f_1}{f_2}\, d\mu_2 = \int_A \frac{d\mu_1/dp}{d\mu_2/dp}\, d\mu_2 = \mu_1(A) = \mu_2(A) = \int_A 1\, d\mu_2,$$

which shows that $f_1/f_2 = 1$ a.e. (μ_2). Since $\mu_2 \sim p$, we obtain that $f_1 = f_2$ a.e. (p); therefore $\mu_1(A) = \mu_2(A)$ for all $A \in \mathscr{A}$. Hence all left invariant means on the bounded function $\int_A T_\sigma 1\, dp$ of σ coincide, and their common value is a finite equivalent invariant measure.

Conversely, assume that for each $A \in \mathscr{A}$, all the left invariant means on $\int_A T_\sigma 1\, dp$ coincide. Then setting $f(\sigma) = \int_A T_\sigma 1\, dp$, we have $Mf = mf$. From Theorems 1.1 and 1.3 and from Corollary 1.1, it follows that there exists a sequence $\psi_n \in F$ such that $\lim_n \sup_\sigma (L_{\psi_n} f)(\sigma) = \lim_n \inf_\sigma (L_{\psi_n} f)(\sigma)$. This implies that $\lim_n (L_{\psi_n} f)(\sigma)$ exists uniformly in σ, and the limit is independent of σ. Fix $\sigma_0 \in \Sigma$, and let $\mu(A) = \lim_n (L_{\psi_n} f)(\sigma_0)$, $A \in \mathscr{A}$. If $\psi_n = \sum_\sigma \theta_n(\sigma) 1_\sigma$, we have

$$\mu(A) = \lim_n \Big\{\sum_\sigma \theta_n(\sigma) L_\sigma f\Big\}(\sigma_0)$$

$$\lim_n \int_A \sum_\sigma \theta_n(\sigma) T_\sigma T_{\sigma_0} 1\, dp.$$

By Vitali–Hahn–Saks theorem, μ is a measure. (The idea of using the Vitali–Hahn–Saks theorem at this point is due to Mrs. Y. N. Dowker [3].) The arguments used in the first part of the theorem can be repeated here to show that μ is a finite $\mathscr{S}$-invariant measure equivalent with p.

THEOREM 3.2. *Let $LM(X)$ be the set of probability measures on $(X, \mathscr{A})$, invariant under $\mathscr{S}$. Then the following conditions on a probability measure μ are equivalent*:

(i) *For some $\varphi \in LIM$, $\varphi\left(\int f \cdot T_\sigma g\, d\mu\right) = \int f\, d\mu \int g\, d\mu$ for every pair $f, g \in L_\infty$.*

(ii) *$\varphi\left(\int f \cdot T_\sigma g\, d\mu\right) = \int f\, d\mu \int g\, d\mu$ for all $\varphi \in LIM$ and for every pair $f, g \in L_\infty$.*

(iii) *μ is an extreme point of $LM(X)$.*

Proof (ii) implies (i) is obvious.

(i) implies (iii): First, we show that $\mu \in LM(X)$, i.e. μ is $\mathscr{S}$-invariant. We are to show that $\int T^*_{\sigma_0} f\, d\mu = \int f\, d\mu$ for every $f \in L_\infty$, and for every $\sigma_0 \in \Sigma$. Putting $g = 1$

in (i), we get $\int f\,d\mu=\varphi(\int T_\sigma^* f\,d\mu)$ for every $f\in L_\infty$. Therefore, replacing f by $T_{\sigma_0}^* f$, we have

$$\int T_{\sigma_0}^* f\,d\mu = \varphi\left(\int T_\sigma^* T_{\sigma_0}^* f\,d\mu\right) = \varphi\left(\int (T_{\sigma_0}T_\sigma)^* f\,d\mu\right)$$

$$= \varphi\left(\int T_\sigma^* f\,d\mu\right) = \int f\,d\mu.$$

Now we assert that μ is an extreme point of $LM(X)$. Assume to the contrary; then there exists α, $0<\alpha<1$, and $\mu_1,\mu_2\in LM(X)$, $\mu_1\neq\mu\neq\mu_2$, such that $\mu=\alpha\mu_1+(1-\alpha)\mu_2$. Since

$$\frac{d\mu_1}{d\mu}=\frac{1}{\alpha}\left(\alpha\frac{d\mu_1}{d\mu}\right)\leq\frac{1}{\alpha}\frac{d\mu}{d\mu}=\frac{1}{\alpha}\in L_\infty,$$

putting $g=d\mu_1/d\mu$ in (i), we obtain

$$\int f\,d\mu\int\frac{d\mu_1}{d\mu}\,d\mu = \varphi\left(\int T_\sigma^* f\cdot\frac{d\mu_1}{d\mu}\,d\mu\right) = \varphi\left(\int T_\sigma^* f\,d\mu_1\right)\quad\text{for } f\in L_\infty.$$

Therefore

$$\int f\,d\mu = \varphi\left(\int T_\sigma^* f\,d\mu_1\right) = \varphi\left(\int f\,d\mu_1\right) = \int f\,d\mu_1,$$

the second equality following from the fact that $\mu_1\in LM(X)$ and hence $\mathscr{S}$-invariant. But this contradicts the assumptions that $\mu_1\neq\mu$. Hence μ is an extreme point of $LM(X)$, as asserted.

(iii) implies (ii): We first show that (iii) implies the validity of (ii) for all functions g of the form 1_A, $A\in\mathscr{A}$, that is,

$$\varphi\left(\int_A T_\sigma^* f\,d\mu\right) = \mu(A)\int f\,d\mu\quad\text{for } f\in L_\infty,\ A\in\mathscr{A}\text{ and }\varphi\in LIM.$$

Assume that this is not true. Then there exist f, A, and φ such that $\varphi(\int_A T_\sigma^* f\,d\mu)\neq\mu(A)\int f\,d\mu$. Let $\gamma(A)=\varphi(\int_A T_\sigma^* f\,d\mu)$. Then γ is a finitely additive set function. Also, γ is μ-continuous; indeed, given $\epsilon>0$, there exists $\delta=\epsilon/\|f\|_\infty$, such that $\mu(A)<\delta$ implies that

$$\gamma(A) = \varphi\left(\int_A T_\sigma^* f\,d\mu\right) \leq \|f\|_\infty\,\varphi\left(\int_A 1\,d\mu\right) = \|f\|_\infty\,\mu(A) < \|f\|_\infty\delta = \epsilon.$$

Therefore, given a sequence A_n of sets with $A_n\downarrow\emptyset$, one has $\mu(A_n)\downarrow 0$ and hence $\gamma(A_n)\downarrow 0$. Hence γ is a measure, and by the same arguments as in Theorem 3.1, γ is $\mathscr{S}$-invariant. Now, $\mu=\frac{1}{2}[(\mu+k\gamma)+(\mu-k\gamma)]$ where we will choose k so as to make $\mu-k\gamma$ positive. Since

$$(\mu-k\gamma)(B) = \mu(B)-k\varphi\left(\int_B T_\sigma^* f\,d\mu\right) \geq \mu(B)-k\,\|f\|_\infty\mu(B),$$

any choice of k satisfying $k\,\|f\|_\infty<1$ will make $\mu-k\gamma$ positive. Having chosen such a k, we choose α so that $(\mu+k\gamma)/2\alpha$ is a normalized measure. Since

$$(\mu+k\gamma)(X) = \mu(X)+k\varphi\left(\int T_\sigma^* f\,d\mu\right) \le \mu(X)+k\|f\|_\infty\mu(X) < 2\mu(X) = 2,$$

we have $\alpha<1$. Also, such a choice of α normalizes the measure $(\mu-k\gamma)/[2(1-\alpha)]$, since

$$\frac{\mu-k\gamma}{2(1-\alpha)}(X) = \frac{2\mu(X)}{2(1-\alpha)}-\frac{\mu+k\gamma}{2(1-\alpha)}(X) = \frac{1}{1-\alpha}-\frac{2\alpha}{2(1-\alpha)}\,\frac{\mu+k\gamma}{2\alpha}(X)$$
$$= \frac{1}{1-\alpha}-\frac{\alpha}{1-\alpha} = 1.$$

Thus we have shown that

$$\mu = \alpha\frac{\mu+k\gamma}{2\alpha}+(1-\alpha)\frac{\mu-k\gamma}{2(1-\alpha)},$$

where the measures $(\mu+k\gamma)/2\alpha$ and $(\mu-k\gamma)/[2(1-\alpha)]$ are in $LM(X)$; this will contradict the assumption that μ is an extreme point of $LM(X)$, provided we can show that $\mu+k\gamma$ and $\mu-k\gamma$ are not any multiples of μ, i.e. γ is not a multiple of μ. If $\gamma=c\mu$ for some constant c, then

$$\varphi\left(\int_A T_\sigma^* f\,d\mu\right) = \gamma(A) = c\cdot\mu(A) = \mu(A)c\mu(X) = \mu(A)\gamma(X)$$
$$= \mu(A)\varphi\left(\int T_\sigma^* f\,d\mu\right) = \mu(A)\varphi\left(\int f\,d\mu\right)$$
$$= \mu(A)\int f\,d\mu;$$

this contradicts the choice of φ, f, and A. Thus (ii) is proved for all indicator functions, and hence for all simple functions. The validity of (ii) for arbitrary $g\in L_\infty$ follows by approximation.

THEOREM 3.3. *The following conditions on the left amenable semigroup* $\mathscr{S}=\{T_\sigma\colon \sigma\in\Sigma\}$ *are equivalent*:

(i) *There exists* $f_0\in L_1$ *with* $0<f_0=T_\sigma f_0$ *for all* $\sigma\in\Sigma$.

(ii) $p(A)>0$ *implies that* $\inf_\sigma \int_A T_\sigma 1\,dp>0$.

(iii) $p(A)>0$ *implies that* $M\left(\int_A T_\sigma 1\,dp\right)>0$.

(iv) $h\in L_\infty^+$, $\sum_n T_{\sigma_n}^* h\in L_\infty$ *for some sequence* σ_n *from* Σ *implies that* $h\equiv 0$.

Proof. We will prove that (ii) ⇒ (iii) ⇒ (iv) ⇒ (ii) ⇒ (i) ⇒ (iv). (ii) implies (iii) is obvious.

The proofs of (iii)⇒(iv), (ii)⇒(i) and (i)⇒(iv) are very similar to the proofs of these implications in the case of cyclic semigroups; therefore we omit these proofs.

(iv)⇒(ii): The proof of this part resembles, to some extent, the proof of Lemma 3 of Neveu [10]. Suppose (ii) does not hold; then there exists a set A with $p(A)>0$ and $\inf_\sigma \int_A T_\sigma 1\, dp=0$. This implies that $\inf_\sigma \int_A T_\sigma f\, dp=0$ for all $f\in L_1^+$. Indeed, since $f\in L_1^+$, given $\epsilon>0$, we can choose an integer j such that $\int_{\{f>j\}} f\, dp<\epsilon$. Thus

$$\begin{aligned}\int_A T_\sigma f\, dp &= \int_{\{f>j\}} 1_A\cdot T_\sigma f\, dp+\int_{\{f<j\}} f\cdot T_\sigma^* 1_A\, dp\\ &\le \int_{\{f>j\}} f\, dp+j\int_A T_\sigma 1\, dp\\ &< \epsilon+j\cdot\int_A T_\sigma 1\, dp\end{aligned}$$

Therefore

$$\inf_\sigma \int_A T_\sigma f\, dp \le \epsilon+j\cdot\inf_\sigma\int_A T_\sigma 1\, dp;$$

it follows that $\inf_\sigma \int_A T_\sigma f\, dp=0$, since ϵ is arbitrary.

Let $0<\epsilon<p(A)$; we choose a sequence σ_n from Σ by induction on n. Since $\inf_\sigma \int_A T_\sigma 1\, dp=0$, there exists $\sigma_1\in\Sigma$ such that $\int_A T_{\sigma_1} 1\, dp<\epsilon/2$. Assume that $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ have been chosen; since

$$\inf_\sigma \int_A T_\sigma \left(\sum_{i=1}^{n-1} T_{\sigma_{n-1}}T_{\sigma_{n-2}}\cdots T_{\sigma_i}1+1\right) dp = 0,$$

we can choose $\sigma_n\in\Sigma$ such that

$$\sum_{i=1}^{n}\int_A T_{\sigma_n}T_{\sigma_{n-1}}\cdots T_{\sigma_i}1\, dp = \int_A T_{\sigma_n}\left(\sum_{i=1}^{n-1} T_{\sigma_{n-1}}\cdots T_{\sigma_i}1+1\right) dp < \epsilon/2^n.$$

Let

$$h = \left(1_A-\sum_{n=1}^{\infty}\sum_{i=1}^{n}(T_{\sigma_n}T_{\sigma_{n-1}}\cdots T_{\sigma_i})^* 1_A\right)^+;$$

we assert that h violates condition (iv). Clearly $0\le h\le 1_A$, and

$$\begin{aligned}\int (1_A-h)\, dp &\le \int \sum_{n=1}^{\infty}\sum_{i=1}^{n}(T_{\sigma_n}T_{\sigma_{n-1}}\cdots T_{\sigma_i})^* 1_A\, dp\\ &\le \sum_{n=1}^{\infty}\sum_{i=1}^{n}\int_A T_{\sigma_n}\cdots T_{\sigma_i}1\, dp \le \sum_{n=1}^{\infty}\epsilon/2^n\\ &= \epsilon < p(A) = \int 1_A\, dp,\end{aligned}$$

which shows that $h\not\equiv 0$. It will be proved that $\sum_n (T_{\sigma_n}T_{\sigma_{n-1}}\cdots T_{\sigma_i})^* h\in L_\infty$. Define the operators S_{ij}, $j\ge i\ge 0$, as follows:

$$S_{ij} = \begin{cases} T_{\sigma_j}T_{\sigma_{j-1}}\cdots T_{\sigma_{i+1}} & \text{if } j>i\ge 0\\ I & \text{if } j=i\ge 0.\end{cases}$$

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It suffices to show that

$$(*)\qquad \sum_{j=i}^{i+k} S_{ij}{}^*h \le 1 \quad \text{a.e.} \quad \text{for all } i, k \ge 0.$$

The sufficiency of (∗) follows by putting $i=0$ and letting $k\uparrow\infty$. We prove (∗) by induction on k. For $k=0$, (∗) is obvious. Assume that $\sum_{j=i}^{i+k} S_{ij}{}^*h \le 1$ a.e. for a fixed k and for all i. To show that $\sum_{j=i}^{i+k+1} S_{ij}^*h \le 1$ a.e. for all i, we consider separately the sets $A_1=\{x: h(x)=0\}$ and $A_2=\{x: h(x)>0\}$. On A_1,

$$\sum_{j=i}^{i+k+1} S_{ij}^*h = h+\sum_{j=i+1}^{i+1+k} S_{ij}^*h = 0+T_{\sigma_{i+1}}^*\left(\sum_{j=i+1}^{i+1+k} S_{i+1j}^*h\right) \le 1 \quad \text{a.e.}$$

by induction hypothesis. If $x\in A_2$, i.e. $h(x)>0$, then by definition of h, we have

$$\begin{aligned}\sum_{j=i}^{i+k+1} S_{ij}^*h &= h+\sum_{j=i+1}^{i+1+k} S_{ij}^*h = h+\sum_{j=i}^{i+k} S_{ij+1}^*h\\ &\le h+\sum_{j=0}^{\infty}\sum_{i=0}^{j} S_{ij+1}^*1_A = h+\sum_{j=1}^{\infty}\sum_{i=1}^{j}(T_{\sigma_j}\dots T_{\sigma_i})^*1_A\\ &= 1_A \le 1.\end{aligned}$$

That completes the proof of the theorem.

4. **Finite invariant measure for amenable semigroups of Markov kernels.** Let $\mathscr{S}=\{P_\sigma(x, A): \sigma\in\Sigma\}$ be a representation of a semigroup Σ as a semigroup of Markov transition probability functions: $P_\sigma(x,\cdot)$ are probabilities for fixed x, and $P_\sigma(\cdot, A)$ are measurable functions in x for each fixed A. Multiplication in $\mathscr{S}$ is defined by $P_{\sigma_1\sigma_2}(x, A)=\int P_{\sigma_1}(y, A)P_{\sigma_2}(x, dy)$. A measure μ is $\mathscr{S}$-invariant iff $\int P_\sigma(x, A)\mu(dx)=\mu(A)$ for all $A\in\mathscr{A}$ and for all $\sigma\in\Sigma$. $B(X)$ will denote the Banach space of all bounded measurable functions on X, with supremum norm $\|f\|=\sup|f(x)|$. In the following theorem, $P_\sigma(x, A)$ are not assumed to be null-preserving: $p(A)=0$ does not necessarily imply that $P_\sigma(x, A)=0$ a.e. (p).

THEOREM 4.1. *Let $\mathscr{S}=\{P_\sigma: \sigma\in\Sigma\}$ be a representation of an amenable semigroup Σ as an amenable semigroup of Markov transition probability functions. Then the following conditions are equivalent*:

(i) *There exists an $\mathscr{S}$-invariant finite measure $\mu\gg p$.*
(ii) *$A\in\mathscr{A}$, $\sum_n P_{\sigma_n}(x, A)\in B(X)$ for some sequence σ_n from Σ implies that $p(A)=0$.*

Proof. (i) implies (ii): Assume (i); let $A\in\mathscr{A}$ and let σ_n be a sequence from Σ such that $\Sigma_n P_{\sigma_n}(x, A)\le C$ for all $x\in X$. Then for every $N\ge 1$, we have

$$N\cdot\mu(A) = \sum_{n=1}^{N}\mu(A) = \sum_{n=1}^{N}\int P_{\sigma_n}(x, A)\mu(dx) \le C\mu(X);$$

it follows that $\mu(A)=0$ and therefore $p(A)=0$.

(ii) implies (i): To the semigroup $\mathscr{S}$, we associate two operator semigroups $\{T_\sigma : \sigma \in \Sigma\}$ and $\{S_\sigma : \sigma \in \Sigma\}$ defined below, the first of which operates on the space $\mathfrak{M}$ of all bounded measures on $(X, \mathscr{A})$ and the second operates on the space $B(X)$:

$$(T_\sigma \mu)(A) = \int P_\sigma(x, A)\mu\,(dx)$$

$$(S_\sigma h)(x) = \int h(y) P_\sigma(x, dy).$$

Then for any $\sigma_1, \sigma_2 \in \Sigma$, we have

$$(T_{\sigma_1\sigma_2}\mu)(A) = (T_{\sigma_1}T_{\sigma_2}\mu)(A)$$

and

$$(S_{\sigma_1\sigma_2}h)(x) = (S_{\sigma_2}S_{\sigma_1}h)(x).$$

We observe that if μ is a measure on $(X, \mathscr{A})$, then $(T_\sigma\mu)(g) = \mu(S_\sigma g)$ for $g \in B(X)$ and $\sigma \in \Sigma$:

$$\begin{aligned}(T_\sigma\mu)(g) &= \int g(x)(T_\sigma\mu)\,(dx) = \int g(x) \int P_\sigma(y, dx)\mu(dy)\\ &= \iint g(x)P_\sigma(y, dx)\mu(dy) = \int (S_\sigma g)(y)\mu(dy)\\ &= \mu(S_\sigma g).\end{aligned}$$

Let $\varphi \in IM = LIM \cap RIM$, and for $g \in B(X)$, define $\lambda(g) = \varphi(\int S_\sigma g\, dp)$; and for $A \in \mathscr{A}$, let $\lambda(A) = \lambda(1_A)$. Then for $\sigma_0 \in \Sigma$,

$$\lambda(S_{\sigma_0}g) = \varphi\Big(\int S_\sigma S_{\sigma_0} g\, dp\Big) = \varphi\Big(\int S_{\sigma_0\sigma} g\, dp\Big) = \varphi\Big(\int S_\sigma g\, dp\Big) = \lambda(g),$$

i.e. $\lambda \cdot S_{\sigma_0} = \lambda$. Regarding λ as a finitely additive set function, we write $\lambda = \mu + \gamma$, where μ is a measure, and γ is purely finitely additive. Then, for $\sigma \in \Sigma$,

$$\lambda = \lambda \cdot S_\sigma = \mu \cdot S_\sigma + \gamma \cdot S_\sigma = T_\sigma \mu + \gamma \cdot S_\sigma$$

and $T_\sigma\mu$ is a measure. But μ is the largest measure dominated by λ, therefore, $T_\sigma\mu \le \mu$. This and the fact that $(T_\sigma\mu)(X) = \mu(X)$ imply that the inequality $T_\sigma\mu \le \mu$ cannot be strict; hence $T_\sigma\mu = \mu$. Therefore, μ is $\mathscr{S}$-invariant. We will show that $\mu \gg p$. If this is not the case, then, as in the proof of the implication (ii)⇒(i) of Theorem 3.3, there exists a set C such that $p(C) > 0$, and $\varphi(\int S_\sigma 1_C\, dp) = \lambda(C) = 0$. As observed by Granirer [4], if $\delta > 0$ and $a_1, a_2, \ldots, a_n \in \Sigma$ are given, then

$$\inf_\sigma \sum_{i=1}^n \int S_{\sigma a_i} 1_C\, dp \le \varphi\Big(\sum_{i=1}^n \int S_{\sigma a_i} 1_C\, dp\Big) = \sum_{i=1}^n \varphi\Big(\int S_\sigma 1_C\, dp) = 0,$$

and therefore there exists $\sigma_0 \in \Sigma$ with $\sum_{i=1}^{n} \int S_{\sigma_0 a_i} 1_C \, dp < \delta$. Let $0 < \epsilon < p(C)$; then we can choose a sequence by induction on n, such that

$$\sum_{i=1}^{n} \int S_{\sigma_n} \cdots {}_{\sigma_i} 1_C \, dp < \frac{\epsilon}{2^{n+1}}.$$

Then proceeding exactly as in the proof of the part (iv)$\Rightarrow$(ii) of Theorem 3.3, and using S_σ in place of T_σ^*, we obtain an $h \in B(X)$, $h \not\equiv 0$, such that $\sum_n S_{\sigma_n \sigma_{n-1} \ldots \sigma_1} h \leq 1$. Choose $D \in \mathscr{A}$, $p(D) > 0$, such that $1_D \leq c \cdot h$ for some constant $c > 0$. Then

$$\sum_n P_{\sigma_n \sigma_{n-1} \ldots \sigma_1}(x, D) = \sum_n S_{\sigma_n \ldots \sigma_1} 1_D(x) \leq c \quad \text{for all } x;$$

this contradicts (ii). Hence the theorem.

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