Bull. Austral. Math. Soc. 22E60, 22A05, 22E05
Vol. 48 (1993) [13–22]

# ENLARGEABLE BANACH-LIE ALGEBRAS AND FREE TOPOLOGICAL GROUPS

Vladimir G. Pestov

We characterise in terms of free topological groups those Banach-Lie algebras with finite-dimensional centre coming from Lie groups.

## Introduction

A Banach-Lie algebra $\mathfrak{g}$ is called *enlargeable* if it comes from a Banach-Lie group [**6, 10, 20–23, 24, 25**]. The Lie-Cartan theorem [**4**] says that finite-dimensional Lie algebras are enlargeable; a similar statement is no longer true for infinite-dimensional Banach-Lie algebras [**24, 20–23**]. There exist various criteria of enlargeability, mainly in cohomological terms [**24, 23, 7**].

It seems that the question on existence of a "natural" (that is, functorial) proof of the Lie-Cartan theorem, which would be independent of both known proofs (the cohomological one by Cartan [**4**] and the representation-theoretic one by Ado [**1**]), is still open; for a discussion see [**18**]. In this note we reshape criteria for enlargeability of a given Banach-Lie algebra $\mathfrak{g}$ in terms of closedness of a certain subgroup of the free topological group, $F(\mathfrak{g})$, over $\mathfrak{g}$; this is done with an idea of paving a way towards a conjectural "direct" proof of the Lie-Cartan theorem.

Let $\mathfrak{g}$ be a Banach-Lie algebra. To be precise, by this term we understand a Lie algebra endowed with a complete submultiplicative norm. The Hausdorff series $H(x,y)$ converges for all $x, y$ from a sufficiently small neighbourhood of zero, $U$. The resulting binary operation $(x,y) \mapsto x.y$ makes $U$ into a local analytic Lie group, and therefore a Lie group germ, $loc(\mathfrak{g})$ (in the sense of [**19, 23**]), associated to $\mathfrak{g}$, comes into being; the functorial nature of the correspondence $\mathfrak{g} \mapsto loc(\mathfrak{g})$ (which turns out to be an equivalence of categories) is well known [**3, 24, 20**].

According to the Świerczkowski's Theorem on Extension of Analytic Structure [**20**], if a local Banach-Lie group $U$ can be embedded into a topological group $G$ as a local topological subgroup, then $G$ can be given a structure of an analytical Banach-Lie

Received 16th July 1992
This investigation was stimulated by the paper [8] by Elyahu Katz (which the author studied in 1983) and partially supported by 1992 research grant V212/451/ RGNT/594/153 from the Internal Grant Committee of the Victoria University of Wellington.

group extending the structure of the original local Lie group. Therefore, the problem of enlarging a Banach-Lie algebra is reduced to the problem of embedding the corresponding local Lie group into a topological group as a local topological subgroup. Ever since the paper [**11**] by Mal'cev it has been known that not every local topological group admits extension to a global group; the same is true relative to local Banach-Lie groups [**24, 22, 23**].

It is natural to consider a *universal morphism*, $i_g$, from a Lie group germ, $g$, to a topological group, say $G_g$; this mapping is a morphism of group germs such that any other morphism $f : g \to G$ of this kind, where $G$ is a topological group, factors through $i_g$ uniquely, that is, for some unique continuous homomoprhism $\widehat{f} : G_g \to G$ one has $i_g \circ \widehat{f} = f$. This construction is but one example of what in category theory is referred to as *universal arrows*, see [**13**]. From Świerczkowski's theorem the following result is an easy corollary. (We shall write $i_\mathfrak{g}$ instead of $i_{loc(\mathfrak{g})}$, and $G_\mathfrak{g}$ instead of $G_{loc(\mathfrak{g})}$.)

**Theorem 1.** *A Banach-Lie algebra* $\mathfrak{g}$ *is enlargeable if and only if the restriction of the universal morphism* $i_\mathfrak{g}$ *to an appropriately small local Lie group* $V$ *representative of the Lie group germ* $loc(\mathfrak{g})$ *is one-to-one. In this case the topological group* $G_\mathfrak{g}$ *can be given the structure of a simply connected Banach-Lie group in such a way that the restriction of the universal morphism* $i_\mathfrak{g}$ *to* $V$ *is a local diffeomorphism.*

The universal character of the construction of $i_\mathfrak{g}$ makes it a very convenient but highly non-constructive device. We shall show that this construction can be performed by means of free topological groups, and it leads to a new criterion of enlargeability for Banach-Lie algebras with finite-dimensional centre in terms of the closedness of a certain subgroup of the free topological group over the underlying topological space of $\mathfrak{g}$ (Theorem 8). This criterion fails in the case where $dim\ \mathfrak{c}(\mathfrak{g}) = +\infty$ (Example). In order to give an independent proof of an auxiliary result on the structure of the topological group $G_\mathfrak{g}$ (Theorem 5), which can be also deduced from earlier results by Świerczkowski and van Est [**21**], we invoke another kind of universal arrows — the free Banach-Lie algebras [**17**].

We prove in passing the following curious result: a Banach-Lie algebra is enlargeable if and only if all its separable Banach-Lie subalgebras are (Theorem 7).

Since the structure of free topological groups $F(\mathbb{R}^n)$ is well understood [**12, 9**], then there is some hope that a "direct" proof of the Lie-Cartan theorem is within reach.

## Main construction

If $X$ is a Tychonoff topological space then by $F(X)$ we denote the free topological group over $X$ in the sense of Markov. (Our construct makes sense for free topological groups in the sense of Graev as well.) For an overview of the theory of free topological groups and a relevant bibliography, see [**2, 5, 9, 12**].

Let $\mathfrak{g}$ be an arbitrary Banach-Lie algebra. Fix a neighbourhood of zero, $U$, which is "small" in the following sense:

(∗) *the Hausdorff series $H(x,y)$ converges for every $x,y \in U$.*

(For example, $U$ may be an open or closed ball of radius less than $(1/3)\log(3/2)$ [3].) Denote by $\mathcal{N}_{\mathfrak{g}}$ a normal subgroup generated by all elements of the form $x^{-1}[x.(-y)]y$, $x,y \in U$. The following statement admits a straightforward verification.

**Assertion 1.** *The subgroup $\mathcal{N}_{\mathfrak{g}}$ does not depend on the particular choice of a neighbourhood $U$ with property (∗).*

Denote by $G_{\mathfrak{g}}$ the topological group quotient of $F(\mathfrak{g})$ by $\mathcal{N}_{\mathfrak{g}}$, and by $\phi_{\mathfrak{g}} : \mathfrak{g} \to G_{\mathfrak{g}}$ the restriction of the quotient homomorphism $\pi_g : F(\mathfrak{g}) \to G_{\mathfrak{g}}$ to $\mathfrak{g}$. Let $i_{\mathfrak{g}}$ stand for the morphism of group germs from $loc(\mathfrak{g})$ to $G_{\mathfrak{g}}$ of which a representative is $\phi_{\mathfrak{g}}$.

**Theorem 2.** *The pair $(i_{\mathfrak{g}}, G_{\mathfrak{g}})$ is a universal morphism for group germ morphisms from the Lie group germ associated to a Banach-Lie algebra $\mathfrak{g}$ to topological groups. In other words, if $G$ is a topological group (not necessarily Hausdorff) and $i : loc(\mathfrak{g}) \to G$ is a morphism of group germs, then there exists a unique continuous homomorphism $\tilde{i} : G_{\mathfrak{g}} \to G$ making the diagram commutative:*

$$\begin{array}{ccc} loc(\mathfrak{g}) & \xrightarrow{\ i_{\mathfrak{g}}\ } & G_{\mathfrak{g}} \\ \| & & \downarrow{\scriptstyle \tilde{i}} \\ loc(\mathfrak{g}) & \xrightarrow{\ i\ } & G \end{array}$$

Proof: Let $U$ be an open convex neighbourhood of zero in $\mathfrak{g}$, a local Lie group representative of the Lie group germ $loc(\mathfrak{g})$. Then the morphism $i$ may be thought of as a continuous local group homomorphism from $U$ to $G$. For every $x \in \mathfrak{g}$ there is an $n \in \mathbb{N}$ such that $x/n$ is in $U$; by putting $i(x) =_{def} [i(x/n)]^n$ one obtains a mapping from the whole of $\mathfrak{g}$ to the group $G$ which we shall still denote by $i$. It does not depend on the choice of $n$ and is continuous at every point $x \in \mathfrak{g}$ as a composition of three continuous mappings: $x \mapsto x/n \mapsto i(x/n) \mapsto [i(x/n)]^n$, if the number $n$ has been chosen uniformly in a neighbourhood of $x$. It extends to a continuous homomorphism of topological groups $\hat{i} : F(\mathfrak{g}) \to G$. Obvisouly, the kernel of $\hat{i}$ contains the subgroup $\mathcal{N}_{\mathfrak{g}}$ and therefore $\hat{i}$ factors through $\pi_{\mathfrak{g}}$ giving rise to a continuous homomorphism $\tilde{i} : G_{\mathfrak{g}} \to G$. Since clearly $\hat{i}|U = i$ then the above diagram commutes. ∎

**Theorem 3.** *The following conditions are equivalent for a Banach-Lie aglebra $\mathfrak{g}$.*

(i) *$\mathfrak{g}$ is enlargeable;*

(ii) *the intersection* $\mathcal{N}_{\mathfrak{g}} \cap \mathfrak{g}$ *is discrete in* $\mathfrak{g}$*;*

(iii) *the restriction of* $\phi_{\mathfrak{g}}$ *to a neighbourhood of zero in* $\mathfrak{g}$ *is one-to-one;*

(iv) *the topological group* $G_{\mathfrak{g}}$ *can be given a structure of an analytical Banach-Lie group in such a way that* $\phi_{\mathfrak{g}}$ *is a local analytical diffeomorphism. In this case Lie* $(G_{\mathfrak{g}}) \cong \mathfrak{g}$, $\phi_{\mathfrak{g}} = exp_{G_{\mathfrak{g}}}$, *and* $G_{\mathfrak{g}}$ *is simply connected.*

PROOF: $(i) \Rightarrow (iii)$: Let $G$ be a simply connected Banach-Lie group associated to $\mathfrak{g}$, and let $exp_G$ be the corresponding exponential mapping. Since obviously $\mathcal{N}_{\mathfrak{g}} \subset \{x \in \mathfrak{g} : exp_G(x) = e_G\}$, then for all $x, y$ in a small enough neighbourhood of zero in $\mathfrak{g}$ one has: if $\phi_{\mathfrak{g}}(x) = \phi_{\mathfrak{g}}(y)$ then $exp_G(x) = exp_G(y)$. Since $exp_G$ is locally one-to-one, then so is $\phi_{\mathfrak{g}}$.
$(ii) \Leftrightarrow (iii)$: an immedeate check.
$(iii) \Rightarrow (iv)$: see Theorem 1.
$(iv) \Rightarrow (i)$: obvious. ∎

As a matter of fact, all these characterisations were discovered decades ago, perhaps in a different form [**4, 24, 20–23, 25, 6**]. For example, the set $\mathcal{N}_{\mathfrak{g}} \cap \mathfrak{g}$ forms an additive subgroup of $\mathfrak{g}$ isomorphic to the *period group Per* $(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ [**24**]. What is new, is our suggestion to consider topology on the group $F(\mathfrak{g})$. Traditionally, the abstract free group over an arbitrarily small neighbourhood of zero in $\mathfrak{g}$ was given full attention (see [**22, 23**]) rather than the *free topological group over the Lie algebra* $\mathfrak{g}$. Free topological groups over neighbourhoods of the identity in finite-dimensional Lie groups were first considered in [**8**].

If a Banach-Lie algebra $\mathfrak{g}$ is enlargeable, then the subgroup $\mathcal{N}_{\mathfrak{g}}$ of the free topological group $F(\mathfrak{g})$ is easily checked to be closed; indeed, it is the kernel of the homomoprhism $\pi_{\mathfrak{g}} : F(\mathfrak{g}) \to G_{\mathfrak{g}}$, and according to Theorem 2 and item *(iv)* of Theorem 3, $\pi_{\mathfrak{g}}$ is continuous while $G_{\mathfrak{g}}$ is endowed with a Hausdorff topology.

It turns out that the closedness of $\mathcal{N}_{\mathfrak{g}}$ is not sufficent for enlargeability of $\mathfrak{g}$, as the following example shows.

EXAMPLE. Let $\mathfrak{g}$ be a Banach-Lie algebra with the following properties.

(a) the extension

$$0 \to \mathfrak{c}(\mathfrak{g}) \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{c}(\mathfrak{g}) \to 0$$

is topologically split ($\mathfrak{c}(\mathfrak{g})$ is the centre of $\mathfrak{g}$);

(b) $\mathfrak{g}$ is enlargeable, and the centre of the corresponding connected simply connected Banach-Lie group $G$ contains a circle, $T$.

For examples of such Lie algebras, see [**24, 25, 3, 6**]. Fix a submultiplicative norm on $\mathfrak{g}$ and define for every $n = 1, 2, \ldots$ a submultiplicative norm $\|\cdot\|_n$ on $\mathfrak{g}$ by

$$\| x + y \|_n =_{def} \| x \|_{\mathfrak{g}/\mathfrak{c}(\mathfrak{g})} + \| y \|_{\mathfrak{c}(\mathfrak{g})} \ (x \in \mathfrak{g}/\mathfrak{c}(\mathfrak{g}),\ y \in \mathfrak{c}(\mathfrak{g}))$$

https://doi.org/10.1017/S0004972700015409 Published online by Cambridge University Press

Denote by $\mathfrak{g}_n$ the Lie algebra $\mathfrak{g}$ with the norm $\|\cdot\|_n$, and by $\mathfrak{g}_\infty$ the $l_1$-type sum of the Lie algebras $\mathfrak{g}_n$, that is, the completion of the direct sum $\oplus_{n=1}^{\infty}\mathfrak{g}_n$ with respect to the submultiplicative norm

$$\| (x_n)_{n\in\mathbb{N}} \| =_{def} \sum_{n\in\mathbb{N}} \| x_n \|_n$$

Denote for every $n = 1,2,\ldots$ by $p_n$ the canonical projection $\mathfrak{g}_\infty \to \mathfrak{g}_n$, and by $\widehat{p}_n$ its continuous homomorphic extension $F(\mathfrak{g}_\infty) \to F(\mathfrak{g}_n)$. We shall show now that

$$\mathcal{N}_{\mathfrak{g}_\infty} = \bigcap_{n=1}^{\infty} \widehat{p_n}^{-1}(\mathcal{N}_{\mathfrak{g}_n}) \tag{1}$$

Since all the Lie subalgebras $\mathfrak{g}_n$ of $\mathfrak{h}$ commute with each other, then (1) is derived from the following simple observation: the subgroup $\mathcal{N}_{\mathfrak{g}_\infty}$ is generated by a subset with the property $(*)$ of the form $\bigcap_{n=1}^{\infty} \widehat{p_n}^{-1}(U_n)$; just pick for every $n$ an $U_n \subset \mathfrak{g}_n$ satisfying $(*)$.

The property (1) implies that the subgroup $\mathcal{N}_{\mathfrak{g}_\infty}$ is closed. Another corollary of (1) is the fact that the period group $Per\ (\mathfrak{g}_\infty) \cong F(\mathfrak{g}_\infty)/\mathcal{N}_{\mathfrak{g}_\infty}$ of the Lie algebra $\mathfrak{g}_\infty$ is the (completed) infinite direct sum of circles $T_n$ sitting in the centres of the Lie algebras $\mathfrak{g}_n$. Since the radii of those circles approach zero as $n \to \infty$ then the group $Per\ (\mathfrak{g}_\infty)$ is non-discrete and therefore $G_{\mathfrak{g}_\infty}$ is non-enlargeable.

However, things are different for Banach-Lie algebras $\mathfrak{g}$ with finite-dimensional centre, $\mathfrak{c}(\mathfrak{g})$, in which case the discreteness of the interesection $\mathcal{N}_\mathfrak{g} \cap \mathfrak{g}$ can be deduced from closedness of $\mathcal{N}_\mathfrak{g}$ in $F(\mathfrak{g})$.

**Lemma 1.** *Let $V$ be a closed local subgroup of a finite-dimensional local Lie group $U$ such that the intersections of $V$ with all one-parameter Lie subgroups are discrete. Then $V$ is discrete.*

Proof: Obviously follows from compactness of a unit sphere in a finite-dimensional Euclidean space. ∎

**Theorem 4.** *Let $\mathfrak{g}$ be a Banach-Lie algebra. Then $\mathcal{N}_\mathfrak{g} \cap \mathfrak{g} \subset \mathfrak{c}(\mathfrak{g})$.*

Proof: Actually, this is a classical observation, and the proof of it is "functorial" (see [**24**]). It stems from the fact that the group $\mathcal{N}_\mathfrak{g}$ is contained in the (closed!) kernel of the continuous homomorphism $ad_\mathfrak{g} \widehat{\circ\, exp_{Aut\ \mathfrak{g}}} : G_\mathfrak{g} \to Aut\ \mathfrak{g}$ where $ad_\mathfrak{g}$ is the adjoint representation $\mathfrak{g} \to Der\ \mathfrak{g}$. Therefore, $\mathcal{N}_\mathfrak{g} \cap \mathfrak{g} \subset ker\ ad_\mathfrak{g} = \mathfrak{c}(\mathfrak{g})$. ∎

**Theorem 5.** *Let $\mathfrak{g}$ be a separable Banach-Lie algebra. Then $\mathcal{N}_\mathfrak{g} \cap \mathfrak{g}$ contains no one-parameter local subgroups of $loc(\mathfrak{g})$.*

Proof: First of all, we shall recall some of our earlier results [**17**].

https://doi.org/10.1017/S0004972700015409 Published online by Cambridge University Press

**Theorem 6.** *Let $E$ be a normed space. There exist a complete normed Lie algebra $\mathcal{FL}(E)$ and a linear isometrical embedding $i_E : E \hookrightarrow \mathcal{FL}(E)$ with the following properties.*

(1) *$i_E(E)$ topologically generates $\mathcal{FL}(E)$.*
(2) *For an arbitrary complete normed Lie algebra $\mathcal{L}$ and any linear operator $f: E \to \mathcal{L}$ of norm $\leqslant 1$, there exists a Lie algebra homomorphism $\widehat{f}: \mathcal{FL}(E) \to \mathcal{L}$ of norm $\leqslant 1$ such that $\widehat{f} \circ i_E = f$.*

*The pair $(\mathcal{FL}(E), i_E)$ with the properties (1) and (2) is essentially unique.* □

The Banach-Lie algebra $\mathcal{FL}(E)$ is termed the *free Banach-Lie algebra over a normed space* $E$. It turns out that if $dim\ E \geqslant 2$ then $\mathcal{FL}(E)$ is centreless and therefore, for any normed space $E$, the free Banach-Lie algebra $\mathcal{FL}(E)$ is enlargeable, the corresponding Banach-Lie group in the case $dim\ E \geqslant 2$ being a Banach-Lie subgroup of the automorphism group of $\mathcal{FL}(E)$ generated by the image of the latter Lie algebra under the exponential mapping. As a result, every Banach-Lie algebra, $\mathfrak{g}$, is a quotient algebra of an enlargeable Banach-Lie algebra, and in a "functorial way" indeed: the identity mapping $id_{\mathfrak{g}}$ from $\mathfrak{g}$ to its underlying Banach (= complete normed) space extends to a quotient Banach-Lie algebra homomorphism $\widehat{id_{\mathfrak{g}}}: \mathcal{FL}(\mathfrak{g}) \to \mathfrak{g}$, which is easily verified to be an open Lie algebra morphism onto. The Banach-Lie algebra $\mathcal{FL}(E)$ is separable if and only if $E$ so is.

Now suppose $\mathfrak{g}$ is a separable Banach-Lie algebra. Denote the kernel of $\widehat{id_{\mathfrak{g}}}$ by $J_{\mathfrak{g}}$; it is a closed Lie ideal. Now let $\mathcal{G}_{\mathfrak{g}}$ be a connected simply connected Banach-Lie group corresponding to the enlargeable Lie algebra $\mathcal{FL}(\mathfrak{g})$ and let $exp_{(\mathfrak{g})}$ be the exponential mapping $\mathcal{FL}(\mathfrak{g}) \to \mathcal{G}_{\mathfrak{g}}$. Let $J_{\mathfrak{g}}^{\times}$ stand for a subgroup in $\mathcal{G}_{\mathfrak{g}}$ algebraically generated by the image of $J_{\mathfrak{g}}$ under $exp_{(\mathfrak{g})}$. This group is normal but not necessarily closed. Denote by $exp\ \mathfrak{g}$ the quotient topological group $\mathcal{G}_{\mathfrak{g}}/J_{\mathfrak{g}}^{\times}$.

A natural continuous mapping $exp_{\mathfrak{g}}: \mathfrak{g} \to exp\ \mathfrak{g}$ is obtained by factoring the exponential map $exp_{(\mathfrak{g})}: \mathcal{FL}(\mathfrak{g}) \to \mathcal{G}_{\mathfrak{g}}$ through $\widehat{id_{\mathfrak{g}}}$. In view of the universality of the mapping $i_{\mathfrak{g}}$ (Theorem 2), it suffices to show that for an arbitrary $x \in \mathfrak{g}\backslash\{0\}$, the image under the mapping $exp_{\mathfrak{g}}$ of the one-dimensional linear space spanned by $x$ is non-degenerate in $exp\ \mathfrak{g}$, that is, for an arbitrary $y \in \mathcal{FL}(\mathfrak{g})\backslash J_{\mathfrak{g}}$, the one-parameter subgroup of $\mathcal{G}_{\mathfrak{g}}$ tangent to $y$ is not entirely contained in the subgroup $J_{\mathfrak{g}}^{\times}$. But this follows from [**3**, Chapter III, Section 6.2 Corollary 2]. □

**Corollary 1.** *Let $\mathfrak{g}$ be a separable Banach-Lie algebra such that the subgroup $\mathcal{N}_{\mathfrak{g}}$ is closed in $F(\mathfrak{g})$. Then the intersection of $\mathcal{N}_{\mathfrak{g}}$ with every one-dimensional local subgroup (one-dimensional linear subspace) of $\mathfrak{g}$ is discrete.*

Proof: Every proper closed subgroup of $\mathbb{R}$ is discrete; now apply Theorem 5. □

**Corollary 2.** *Let $\mathfrak{g}$ be a separable Banach-Lie algebra with finite-dimensional*

https://doi.org/10.1017/S0004972700015409 Published online by Cambridge University Press

*centre. Then* $\mathfrak{g}$ *is enlargeable if and only if the subgroup* $\mathcal{N}_{\mathfrak{g}}$ *is closed in* $F(\mathfrak{g})$*. In this case the quotient topological group* $G_{\mathfrak{g}}$ *carries a natural structure of a Banach-Lie group associated to* $\mathfrak{g}$*.*

The separability restriction is removed with the help of the following curious result, which is also new and of interest by itself.

THEOREM 7. *A Banach-Lie algebra* $\mathfrak{g}$ *is enlargeable if and only if every separable Banach-Lie subalgebra of* $\mathfrak{g}$ *is enlargeable.*

PROOF: Denote by $\mathfrak{H}$ the family of all separable Banach-Lie subalgebras of $\mathfrak{g}$ partially ordered by inclusion. Obviously, $\cup\mathfrak{H} = \mathfrak{g}$. One can assume that $\mathfrak{g}$ is inseparable and therefore $\mathfrak{H}$ has no upper bound. For every $\mathfrak{h} \in \mathfrak{H}$ denote by $G_{\mathfrak{h}}$ a simply connected Lie group associated to $\mathfrak{h}$ and by $exp_{\mathfrak{h}} : \mathfrak{h} \to G_{\mathfrak{h}}$ the corresponding exponential mapping. Put for each $\mathfrak{h} \in \mathfrak{H}$

$$\varepsilon(\mathfrak{h}) =_{def} min\{1,\ max\{r \in \mathbb{R}_+ :\ exp_{\mathfrak{h}}|B_r(0) \text{ is one-to-one }\}\}$$

where $B_r(0)$ is the open ball of radius $r$ centred at zero. If $\mathfrak{h}_1 \subset \mathfrak{h}_2$ then the inclusion mapping $i_{\mathfrak{h}_1}^{\mathfrak{h}_2} : \mathfrak{h}_1 \to \mathfrak{h}_2$ gives rise to a Lie group morphism $\widehat{i}_{\mathfrak{h}_1}^{\mathfrak{h}_2} : G_{\mathfrak{h}_1} \to G_{\mathfrak{h}_2}$ which commutes with the exponential mappings. One concludes that if $\mathfrak{h}_1 \subset \mathfrak{h}_2$ then $\varepsilon(\mathfrak{h}_1) \geqslant \varepsilon(\mathfrak{h}_2)$, and therefore $\varepsilon$ is a non-increasing mapping from the directed partially ordered set $\mathfrak{H}$ to $\mathbb{R}$. Put $\varepsilon_0 = inf\ \{\varepsilon(\mathfrak{h}) : \mathfrak{h} \in \mathfrak{H}\}$. Clearly, $\varepsilon_0 \geqslant 0$, and the case $\varepsilon_0 = 0$ is impossible because $\mathfrak{H}$ contains unions of countable subfamilies. Therefore, for every $\mathfrak{h} \in \mathfrak{H}$, the restriction of the exponential mapping $exp_{\mathfrak{h}}$ to the open ball of radius $\varepsilon_0 > 0$ about zero is one-to-one. Now we apply the Local Theorem on Enlargeability of Banach-Lie Algebras, [**15, 16**]. ▯

Finally we come to the central

THEOREM 8. *A Banach-Lie algebra* $\mathfrak{g}$ *with finite-dimensional center is enlargeable if and only if the subgroup* $\mathcal{N}_{\mathfrak{g}}$ *is closed in* $F(\mathfrak{g})$*. In this case the quotient topological group* $G_{\mathfrak{g}}$ *carries a natural structure of a Banach-Lie group associated to* $\mathfrak{g}$*.*

PROOF: Necessity was pointed out at just before our Example, and sufficiency follows from Corollary 2 and Theorem 7. ▯

## FINAL DISCUSSION

1. All the structures participating in our construct come into being in a functorial

https://doi.org/10.1017/S0004972700015409 Published online by Cambridge University Press

way as the objects of the following commutative diagram

$$
\begin{array}{ccccccccc}
 & & & & F(\mathfrak{g}) & \xleftarrow[\supset]{} & \mathfrak{g} & & \\
 & & & & \| & & \downarrow{\scriptstyle \phi_{\mathfrak{g}}} & & \\
e & \longrightarrow & \mathcal{N}_{\mathfrak{g}} & \xrightarrow{\subset} & F(\mathfrak{g}) & \xrightarrow{\pi_{\mathfrak{g}}} & G_{\mathfrak{g}} & \longrightarrow & e \\
 & & & & & & \downarrow{\scriptstyle \widehat{exp_{\mathfrak{g}}}} & & \\
e & \longrightarrow & J_{\mathfrak{g}}^{\times} & \xrightarrow{\subset} & \mathcal{G}_{\mathfrak{g}} & \longrightarrow & exp\ \mathfrak{g} & \longrightarrow & e \\
 & & {\scriptstyle exp_{(\mathfrak{g})}}\uparrow & & \uparrow{\scriptstyle exp_{(\mathfrak{g})}} & & \uparrow{\scriptstyle exp_{\mathfrak{g}}} & & \\
0 & \longrightarrow & J_{\mathfrak{g}} & \xrightarrow{\subset} & \mathcal{FL}(\mathfrak{g}) & \xrightarrow{\widehat{id_{\mathfrak{g}}}} & \mathfrak{g} & \longrightarrow & 0
\end{array}
$$

(The arrow $\mathfrak{g} \overset{=}{\longrightarrow} \mathfrak{g}$ is understood.)

A direct proof of the Lie-Cartan theorem would result from applying corollary 2 to a finite-dimensional Banach-Lie algebra, $\mathfrak{g}$. The only missing link is the verification of closedness of $\mathcal{N}_{\mathfrak{g}}$ in $F(\mathfrak{g})$ which should rely solely on topological structure of the free topological group $F(\mathfrak{g})$. Since the structure of free topological groups over $k_\omega$-spaces, and in particular over $\mathbb{R}^n$, is deeply understood now ([**12**]; see also [**9**] and references therein), then the problem of recovering this link does not seem entirely hopeless. We conjecture that $\mathcal{N}_{\mathfrak{g}}$ is the free topological group over a $k_\omega$ space, and thence it is complete and closed.

2. The arguments of the kind discussed in this note are no longer solid beyond the setting of Banach-Lie algebras and groups. For example, one cannot in general expect the Hausdorff series to converge in a neighbourhood of zero of a Fréchet-Lie algebra; the exponential mapping may not be even $C^\infty$; moreover, it is not known yet whether every smooth Fréchet-Lie group possesses an exponential mapping, [**10, 14**]. In general, we anticipate the techniques of universal arrows to forgetful functors to lead to "pathological" examples of Fréchet-Lie groups rather than some positive results. Nevertheless, our construct makes perfect sense for the so-called *Baker-Campbell-Hausdorff* Lie groups modeled over locally convex spaces [**14**].

3. In connection with a result [**26**] that the free topological group $F(\mathbb{R}^n)$ is topologically a manifold modeled over the locally convex space $\mathbb{R}^\omega \cong \varinjlim\{\mathbb{R}^k : k \in \mathbb{N}\}$, one wonders whether $F(\mathbb{R}^n)$ can be given a structure of an (at least, $C^1$) Lie group modeled over the same $(LB)$-space $\mathbb{R}^\omega$. What may its Lie algebra look like?

## References

[1] I.D. Ado, 'The representation of Lie algebras by matrices', *Amer. Math. Soc. Transl. Ser. 1* **9** (1962), 308–327.

[2] A.V. Arhangel'skiĭ, 'Classes of topological groups', *Russian Math. Surveys* **36** (1981), 151–174.

[3] N. Bourbaki, *Lie groups and Lie algebras* (Springer-Verlag, Berlin, Heidelberg, New York, 1989).

[4] E. Cartan, *La topologie des groupes de Lie* (Hermann, Paris, 1936).

[5] W.W. Comfort, K.H. Hofmann and D. Remus, 'A survey on topological groups and semigroups', in *General Topology — Recent Developments* (North-Holland, Amsterdam, 1992) (to appear).

[6] J. Eells, jr., 'A setting for global analysis', *Bull. Amer. Math. Soc.* **72** (1966), 751–807.

[7] D.B. Fuks, *Cohomology of infinite-dimensional Lie algebras*, Translated from Russian by A.B. Sosinski (Consultants Bureau, New York, London, 1986).

[8] E. Katz, 'Free topological groups and principal fiber bundles', *Duke Math. J.* **42** (1975), 83–90.

[9] E. Katz and S.A. Morris, 'Free Abelian topological groups on countable CW-complexes', *Bull. Austral. Math. Soc.* **41** (1990), 451–456.

[10] O. Kobayashi, A. Yoshioka, Y. Maeda and H. Omori, 'The theory of infinite-dimensional Lie groups and its applications', *Acta Appl. Math.* **3** (1985), 71–106.

[11] A.I. Mal'cev, 'Sur les groupes topologiques locaux et complets', *C.R. (Doklady) Acad. Sci. URSS (N.S.)* **32** (1941), 606–608.

[12] J. Mack, S.A. Morris and E.T. Ordman, 'Free topological groups and the projective dimension of a locally compact abelian group', *Proc. Amer. Math. Soc.* **40** (1973), 399–402.

[13] S. Mac Lane, *Categories for the working mathematician*, Graduate Texts in Mathematics **5** (Springer-Verlag, Berlin, Heidelberg, New York, 1971).

[14] J. Milnor, 'Remarks on infinite-dimensional Lie groups', in *Relativité, groupes et topologie II. Les Houches, Session XL* (Elsevier Sci. Publ., Amsterdam, 1984), pp. 1007–1058.

[15] V.G. Pestov, 'Fermeture nonstandard des algèbres et groupes de Lie banachiques', *C.R. Acad. Sci. Paris Ser. 1 Math.* **306** (1988), 643-645.

[16] V.G. Pestov, 'Nonstandard hulls of Banach-Lie groups and algebras', *Nova Journal of Algebras and Geometries* (to appear).

[17] V.G. Pestov, 'Free Banach-Lie algebras, couniversal Banach-Lie groups, and more', *Pacific J. Math.* **157** (1993), 137-144.

[18] M.M. Postnikov, *Lie groups and Lie algebras*, Lectures in geometry. Semester V, Translated from Russian by V. Shokurov (Mir, Moscow, 1986).

[19] A. Robinson, 'Germs', in *Applications of model theory to algebra, analysis and probability* (Holt, Rinehart and Winston, New York, 1969).

[20] S. Świerczkowski, 'Embedding theorems for local analytic groups', *Acta Math.* **114** (1965), 207–235.

[21] S. Świerczkowski, 'The path-functor on Banach-Lie algebras', *Indag. Math.* **33** (1971), 235–239.

[22] S. Świerczkowski, 'On the cohomology of local groups', *J. Austral. Math. Soc.* **12** (1971), 249–255.

[23] S. Świerczkowski, 'Cohomology of group germs and Lie algebras', *Pacific J. Math.* **39** (1971), 471–482.

[24] W.T. van Est and T.J. Korthagen, 'Non-enlargible Lie algebras', *Nederl. Akad. Wetensch. Proc.* **A26** (1964), 15–31.

[25] W.T. van Est and S. Świerczkowski, 'The path functor and faithful representability of Banach Lie algebras', *J. Austral. Math. Soc.* **16** (1973), 471–482.

[26] M.M. Zarichnyĭ, 'Free topological groups of absolute neighbourhood retracts and infinite-dimensional manifolds', *Soviet Math. Dokl.* **26** (1982), 367–371.

Department of Mathematics
Victoria University of Wellington
Wellington
New Zealand
E-mail: vova@kauri.vuw.ac.nz