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ON THE COHOMOLOGY OF EXTENSIONS BY A HEISENBERG LIE ALGEBRA

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This article describes the cohomology spaces of any Lie algebra containing a Lie algebra of Heisenberg type (whose cohomology was studied by Santharoubane) as an ideal of codimension 1. For instance, the twisted standard filiform Lie algebras are of this kind. We give an explicit formula for the Betti numbers of this Lie algebra, and use this to describe new families of algebras whose Betti numbers do not behave unimodally.

1. INTRODUCTION

Although defined in linear algebra terms, the cohomology of a Lie algebra remains difficult to describe. Even in determining the Betti numbers $\beta_i = H^i(\mathfrak{g}_n, \mathbb{R})$ of the Lie algebras in a family $\{\mathfrak{g}_n\}_n$, one meets serious combinatorial problems (see for instance [**2, 3, 7**]). To handle these, we use spectral sequence type arguments, thus relating the cohomology of the Lie algebra we are interested in, to a Lie algebra whose cohomology has already been studied.

Let $\mathfrak{h}_n$ be a (real) Heisenberg Lie algebra with a basis $\{x_1, \ldots, x_n, y_1, \ldots y_n, w\}$ and corresponding Lie bracket

$$[x_i, y_i] = w \tag{1}$$

for all $1 \leqslant i \leqslant n$. The following theorem is the central result of this article.

THEOREM. (With notations as above.) *Let $\mathfrak{g}$ be an extension of the one-dimensional Lie algebra $\langle z \rangle$ by $\mathfrak{h}_n$, for some n,*

$$1 \longrightarrow \mathfrak{h}_n \longrightarrow \mathfrak{g} \longrightarrow \langle z \rangle \longrightarrow 0$$

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such that $\mathfrak{g}$ acts trivially on the centre $\mathfrak{z} = \langle w \rangle$ of $\mathfrak{h}_n$. Let $\mathfrak{f} = \mathfrak{g}/\mathfrak{z}$. Then

$$\beta_p(\mathfrak{g}) = \begin{cases} \beta_p(\mathfrak{f}) & \text{for } p = 0 \text{ or } p = 1, \\ \beta_p(\mathfrak{f}) - \beta_{p-2}(\mathfrak{f}) & \text{for } 2 \leqslant p \leqslant n, \\ 2\left[\beta_{n+1}(\mathfrak{f}) - \beta_{n-1}(\mathfrak{f})\right] & \text{for } p = n+1, \\ \beta_{p-1}(\mathfrak{f}) - \beta_{p+1}(\mathfrak{f}) & \text{for } n+2 \leqslant p \leqslant 2n, \\ \beta_{p-1}(\mathfrak{f}) & \text{for } p = 2n+1 \text{ or } p = 2n+2. \end{cases}$$

The condition that z acts trivially on the centre of $\mathfrak{h}_n$ is equivalent to the condition that $\mathfrak{g}$ is unimodular. In case $\mathfrak{g}$ is nilpotent, the action of $\mathfrak{g}$ on w is automatically trivial. In general, this need not be the case; an easy counterexample is the extension of $\mathfrak{h}_1$ by $\langle z \rangle$ determined by

$$[z, x_1] = x_1, \quad \text{and} \quad [z, w] = w.$$

Picking an extension of $\langle z \rangle$ by $\mathfrak{h}_n$ corresponds to choosing a derivation of $\mathfrak{h}_n$; any derivation of $\mathfrak{h}_n$ has a matrix representation with respect to the basis given in (1) of the form

$$\left(\begin{array}{ccc|ccc|c} & & & & & & 0 \\ & A & & & B & & \vdots \\ & & & & & & 0 \\ \hline & & & & & & 0 \\ & C & & & a\mathbb{I} - A^t & & \vdots \\ & & & & & & 0 \\ \hline a_1 & \dots & a_n & b_1 & \dots & b_n & a \end{array}\right),$$

where $A, B, C \in \mathbb{R}^{n \times n}$ with B and C symmetric; $\mathbb{I}$ is the identity matrix of dimension n. In the setting of the main theorem, the parameter a equals zero.

We apply our theory to two families of Lie algebras. First of all, we have a look at the algebras $\mathfrak{b}_n$ with a basis $\{x_1, \dots, x_n, y_1, \dots y_n, w, z\}$ and Lie bracket

$$[z, x_i] = y_i, \quad [x_i, y_i] = w$$

for all $1 \leqslant i \leqslant n$. We compute its Betti numbers in terms of the Betti numbers of the algebra $\mathfrak{a}_n$ determined by

$$[z, x_i] = y_i$$

with respect to a basis $\{x_1, \dots, x_n, y_1, \dots, y_n, z\}$. The Betti numbers of the algebras $\mathfrak{a}_n$ have been studied in [**2**]. An elementary analysis shows that, for any $n \geqslant 5$, the Betti numbers of $\mathfrak{b}_n$ do *not* behave unimodally (that is, $\beta_i(\mathfrak{b}_n) \leqslant \beta_{i+1}(\mathfrak{b}_n)$ for all $0 \leqslant i \leqslant n$), in contrast to, for instance, Lie algebras containing an Abelian ideal of codimension 1 (see [**1**]).

The second type of Lie algebras we study, is the family of the twisted standard filiform Lie algebras $\mathfrak{g}_n$ given by

$$(2)\qquad \begin{aligned} &[z, x_i] = x_{i+1}, && \text{for all } 1 \leqslant i \leqslant 2n-1 \\ &[z, x_{2n}] = w, && \\ &[x_i, x_{2n-i+1}] = (-1)^i w, && \text{for all } 1 \leqslant i \leqslant n \end{aligned}$$

with respect to a basis $\{x_1, \dots x_{2n}, w, z\}$. Again, examples indicate that the Betti numbers of these algebras do not follow a unimodal distribution. The heart of the proof of this fact would involve studying a special type of partitions of integers, which is in itself an interesting problem, but is beyond the scope of this article.

2. Cohomology of extensions by the one-dimensional Lie algebra

Let $\mathfrak{g}$ be any finite dimensional Lie algebra with a basis $\{x_1, \dots, x_n\}$, and suppose x_n is central. Let $\mathrm{pr} : \mathfrak{g} \to \mathfrak{f} = \mathfrak{g}/\langle x_n \rangle$ be the projection (also denoted by $^-$). Then pr induces a map $\mathrm{pr}_* : (\Lambda^*\mathfrak{g}, d_\mathfrak{g}) \to (\Lambda^*\mathfrak{f}, d_\mathfrak{f})$ of differential graded algebras, with $\Lambda^{*-1}\langle x_1, \dots, x_{n-1}\rangle \wedge x_n \cong \Lambda^{*-1}\mathfrak{f}$ as kernel. Applying the $\mathrm{Hom}(-, \mathbb{R})$-functor to the short exact sequence of differential graded algebras

$$0 \longrightarrow \Lambda^{*-1}\mathfrak{f} \longrightarrow \Lambda^*\mathfrak{g} \longrightarrow \Lambda^*\mathfrak{f} \longrightarrow 0$$

and taking cohomology, induces a long exact sequence,

$$(3)\qquad \begin{aligned} \cdots \longrightarrow H^{p-2}(\mathfrak{f}) &\xrightarrow{\mathrm{d}^{p-2}} H^p(\mathfrak{f}) \xrightarrow{i^p} H^p(\mathfrak{g}) \\ &\xrightarrow{j^p} H^{p-1}(\mathfrak{f}) \xrightarrow{\mathrm{d}^{p-1}} H^{p+1}(\mathfrak{f}) \longrightarrow \cdots \end{aligned}$$

Lemma 1. (With notations as above.) *Let $u \in \Lambda^2\mathfrak{f}$ such that $i^2(u) = d^1_\mathfrak{g} x^*_n$. Then the connecting homomorphism $\mathrm{d}^p : H^p(\mathfrak{f}) \to H^{p+2}(\mathfrak{f})$ is given by*

$$\mathrm{d}^p([f]) = [f \wedge u].$$

Proof: First of all, note that $j^2(d^1_\mathfrak{g} x^*_n)(\overline{x}_i) = d^1_\mathfrak{g} x^*_n(x_i \wedge x_n) = 0$, such that $d^1 x^*_n$ is contained in the image of i^2, and u does exist. The connecting homomorphism d^p is induced on cohomology by following the dotted arrows in the diagram

$$\begin{array}{ccccc} \Lambda^{p+1}\mathfrak{f}^* & \xrightarrow{\;i^{p+1}\;} & \Lambda^{p+1}\mathfrak{g}^* & \underset{\cdots\cdots}{\overset{j^{p+1}}{\rightleftarrows}} & \Lambda^p\mathfrak{f}^* \\ \downarrow & & \downarrow d^{p+1}_\mathfrak{g} \;\vdots & & \downarrow \\ \Lambda^{p+2}\mathfrak{f}^* & \underset{\cdots\cdots}{\overset{i^{p+2}}{\rightleftarrows}} & \Lambda^{p+2}\mathfrak{g}^* & \xrightarrow{\;j^{p+2}\;} & \Lambda^{p+1}\mathfrak{f}^*. \end{array}$$

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Let $f \in \Lambda^p \mathfrak{f}^*$ such that $d_{\mathfrak{f}}^p f = 0$. Then f lifts to an element $\overline{f} \in \Lambda^{p+1}\mathfrak{g}^*$ by defining

$$\overline{f}(x_{i_1} \wedge \ldots \wedge x_{i_{p+1}}) = \begin{cases} f(x_{i_1} \wedge \ldots \wedge x_{i_p}) & \text{if } i_{p+1} = n, \\ 0 & \text{otherwise} \end{cases}$$

for any $1 \leqslant i_1 < \cdots < i_{p+1} \leqslant n$. In other words, $\overline{f} = i^p(f) \wedge x_n^*$. Therefore,

$$d_{\mathfrak{g}}^{p+1}\overline{f} = i^p(f) \wedge d_{\mathfrak{g}}^1 x_n^* = i^{p+2}(f \wedge u).$$

□

EXAMPLE. Let $\mathfrak{g}_1$ be the twisted standard filiform Lie algebra of dimension 4 (see (2) for the description of a basis and its corresponding Lie brackets). We compute the cohomology of $\mathfrak{g}_1$ in terms of the cohomology of $\mathfrak{f}_2 = \mathfrak{g}_1/\langle w\rangle$, the standard filiform Lie algebra of dimension 3 given by

$$[z, x_1] = x_2.$$

To do this, we need to compute the connecting homomorphisms d^0 and d^1. Since, for any $c \in H^0(\mathfrak{f}_2) = \mathbb{R}$, we have $\mathrm{d}^0 c = c(z^* \wedge x_2^* - x_1^* \wedge x_2^*)$, d^0 is injective. In degree 1, a class $[c_1 z^* + c_2 x_1^*] \in H^1(\mathfrak{f}_2)$ maps to $\mathrm{d}^1([c_1 z^* + c_2 x_1^*]) = [-(c_2 + c_1) z^* \wedge x_1^* \wedge x_2^*]$, so d^1 is surjective. Now we know the connecting homomorphisms in the long exact sequence (3) relating the cohomology of $\mathfrak{g}_1$ to the cohomology of $\mathfrak{f}_2$, we know that

$$\beta_0(\mathfrak{g}_1) = 1, \quad \beta_1(\mathfrak{g}_1) = 2, \quad \beta_2(\mathfrak{g}_1) = 2, \quad \beta_3(\mathfrak{g}_1) = 2, \quad \beta_4(\mathfrak{g}_1) = 1.$$

It turns out this injectivity or surjectivity of the connecting homomorphisms in the long exact sequence (3) is typical for an extension of the one-dimensional Lie algebra by a Heisenberg Lie algebra $\mathfrak{h}_n$.

3. Cohomology of extensions by a Heisenberg Lie algebra

Let $\mathfrak{g}$ be an extension of $\langle z\rangle$ by the Heisenberg Lie algebra $\mathfrak{h}_n$,

$$0 \longrightarrow \mathfrak{h}_n \longrightarrow \mathfrak{g} \longrightarrow \langle z\rangle \longrightarrow 0$$

such that the adjoined action of z on the centre $\langle w\rangle$ of $\mathfrak{h}_n$ is trivial (see (1) for the description of the basis we use and its related bracket structure). Then w is central in $\mathfrak{g}$. Write $\mathfrak{f} = \mathfrak{g}/\langle w\rangle$ for the quotient algebra. The long exact sequence (3) relates the cohomology of $\mathfrak{g}$ to the cohomology of $\mathfrak{f}$. In this long exact sequence, the connecting homomorphism d^p is given by the wedge product with

$$u = z^* \wedge z'^* + \sum_{i=1}^{n} x_i^* \wedge y_i^*,$$

for some $z' \in \mathfrak{f}$. Since there is no confusion here about the fact that we are working in the quotient algebra $\mathfrak{f}$, we do not mention the projections explicitly.

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Proposition 2. (With notations as above.) *For each p, the connecting homomorphism d^p has maximal rank, that is, d^p is injective for all $p \leqslant n-1$, and d^p is surjective for all $n \leqslant p$.*

We first of all use this proposition to prove the main theorem of this paper, postponing the proof of the proposition itself to the following section.

Proof of the Main Theorem: First of all, since the five-term sequence

$$H^{p-2}(\mathfrak{f}) \xrightarrow{\mathrm{d}^{p-2}} H^p(\mathfrak{f}) \xrightarrow{i^p} H^p(\mathfrak{g}) \xrightarrow{j^p} H^{p-1}(\mathfrak{f}) \xrightarrow{\mathrm{d}^{p-1}} H^{p+1}(\mathfrak{f})$$

coming from (3), is exact, we know that

$$\dim H^p(\mathfrak{g}) = \dim \operatorname{Im} i^p + \dim \operatorname{Im} j^p = \beta_p(\mathfrak{f}) - \dim \operatorname{Im} \mathrm{d}^{p-2} + \dim \ker \mathrm{d}^{p-1}.$$

Now let $p \leqslant n$. Then both d^{p-2} and d^{p-1} are injective, so

$$\dim H^p(\mathfrak{g}) = \beta_p(\mathfrak{f}) - \beta_{p-2}(\mathfrak{f}).$$

For $p \geqslant n+2$, both d^{p-2} and d^{p-1} are surjective, so

$$\dim H^p(\mathfrak{g}) = \beta_{p-1}(\mathfrak{f}) - \beta_{p+1}(\mathfrak{f}).$$

In case $p = n+1$ we know that d^{n-1} is injective and d^n surjective, so

$$\begin{aligned} \dim H^{n+1}(\mathfrak{g}) &= \beta_{n+1}(\mathfrak{f}) - \beta_{n-1}(\mathfrak{f}) + \beta_n(\mathfrak{f}) - \beta_{n+2}(\mathfrak{f}) \\ &= 2\Big[\beta_{n+1}(\mathfrak{f}_{n-1}) - \beta_{n-1}(\mathfrak{f}_{n-1})\Big] \end{aligned}$$

by Poincaré duality. ∎

4. A proof of proposition 2

In this section, we show that the map $\mathrm{d}^p = \mathrm{d}^p_{\mathfrak{f}} : H^p(\mathfrak{f}) \to H^{p+2}(\mathfrak{f})$ on the cohomology of $\mathfrak{f}$, induced by the wedge product with

$$u = z^* \wedge z'^* + \sum_{i=1}^{n} x_i^* \wedge y_i^*,$$

has maximal rank.

Let $\mathfrak{a}$ be the Abelian ideal of $\mathfrak{f}$ generated by $x_1, \ldots, x_n, y_1, \ldots, y_n$. Then, according to [4], the cohomology of $\mathfrak{f}$ fits into a long exact sequence

$$\begin{aligned} \cdots \longrightarrow H^{p-1}(\mathfrak{a}) \xrightarrow{\delta^{p-1}} H^{p-1}(\mathfrak{a}) \xrightarrow{i^p} H^p(\mathfrak{f}) \\ \xrightarrow{j^p} H^p(\mathfrak{a}) \xrightarrow{\delta^p} H^p(\mathfrak{a}) \longrightarrow \ldots, \end{aligned}$$

where the connecting homomorphism $\delta^p : H^p(\mathfrak{a}) \to H^p(\mathfrak{a})$ is induced by the adjoined action of z on $\mathfrak{a}$. ∎

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Lemma 3. (With notations as above.) *Let $\mathrm{d}_{\mathfrak{a}}^p : H^p(\mathfrak{a}) \to H^{p+2}(\mathfrak{a})$ be the map induced by the wedge product with*

$$v = \sum_{i=1}^{n} x_i^* \wedge y_i^*.$$

Then the diagram

$$
\begin{array}{ccccccccc}
H^{p-1}(\mathfrak{a}) & \xrightarrow{\delta^{p-1}} & H^{p-1}(\mathfrak{a}) & \xrightarrow{i^p} & H^p(\mathfrak{f}_{n-1}) & \xrightarrow{j^p} & H^p(\mathfrak{a}) & \xrightarrow{\delta^p} & H^p(\mathfrak{a}) \\
\downarrow \mathrm{d}_{\mathfrak{a}}^{p-1} & & \downarrow \mathrm{d}_{\mathfrak{a}}^{p-1} & & \downarrow \mathrm{d}_{\mathfrak{f}}^{p} & & \downarrow \mathrm{d}_{\mathfrak{a}}^{p} & & \downarrow \mathrm{d}_{\mathfrak{a}}^{p} \\
H^{p+1}(\mathfrak{a}) & \xrightarrow{\delta^{p+1}} & H^{p+1}(\mathfrak{a}) & \xrightarrow{i^{p+2}} & H^{p+2}(\mathfrak{f}_{n-1}) & \xrightarrow{j^{p+2}} & H^{p+2}(\mathfrak{a}) & \xrightarrow{\delta^{p+2}} & H^{p+2}(\mathfrak{a})
\end{array} \tag{4}
$$

is commutative.

Although the lemma below is well-known (see for instance [8]), we include a proof of this result for completeness.

Lemma 4. *Let $\mathfrak{a}_n$ be the $2n$-dimensional Abelian Lie algebra generated by $x_1, \ldots, x_n, y_1, \ldots, y_n$. Let $v = \sum_{i=1}^{n} x_i \wedge y_i$. Then, for any $p \geqslant 0$, the map*

$$\mathrm{d}_n^p : \Lambda^p \mathfrak{a}_n^* \to \Lambda^{p+2} \mathfrak{a}_n^* : f \mapsto f \wedge v$$

has maximal rank, that is, d_n^p is injective for all $0 \leqslant p \leqslant n-1$, and surjective for all $n-1 \leqslant p \leqslant 2n$.

From these two lemmas, proposition 2 follows at once. For $p \leqslant n-2$, both $\mathrm{d}_{\mathfrak{a}}^{p-1}$ and $\mathrm{d}_{\mathfrak{a}}^p$ are injective, so the Five Lemma (see [5]) applied to the diagram (4) assures that also $\mathrm{d}_{\mathfrak{f}}^p$ is injective. For $p \geqslant n-1$, both $\mathrm{d}_{\mathfrak{a}}^{p-1}$ and $\mathrm{d}_{\mathfrak{a}}^p$ are surjective, and therefore, according to the Five Lemma again, $\mathrm{d}_{\mathfrak{f}}^p$ is also surjective.

Proof of lemma 3. First of all, we show that $\mathrm{d}_{\mathfrak{f}}^{p+1}\, i^p = i^{p+2}\, \mathrm{d}_{\mathfrak{a}}^p$. Let $f \in \Lambda^p \mathfrak{a}^*$. Then

$$\mathrm{d}_{\mathfrak{f}}^p i^p(f) = i^p(f) \wedge z^* \wedge z'^* + i^p(f) \wedge \sum_{i=1}^{n} x_i \wedge y_i.$$

But $i^p(f)(x_{i_1} \wedge \ldots \wedge x_{i_s} \wedge y_{j_1} \wedge \ldots \wedge y_{j_{p-s+1}}) = 0$ for all $s \leqslant p+1$, $1 \leqslant i_1 < \cdots < i_s \leqslant n$ and $1 \leqslant j_1 < \cdots < j_{p-s+1} \leqslant n$. Therefore,

$$i^p(f) \wedge z^* \wedge z'^* = 0.$$

Now

$$\mathrm{d}_{\mathfrak{f}}^p i^p(f) = i^p(f) \wedge \left(\sum_{i=1}^{n} x_i^* \wedge y_i^* \right) = i^{p+2}(f \wedge v) = i^{p+2} \mathrm{d}_{\mathfrak{a}}^p f.$$

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Next, we show that $j^{p+2}\, \mathrm{d}_{\mathfrak{f}}^p = \mathrm{d}_{\mathfrak{a}}^p\, j^p$. Let f be a representative of a cohomology class in $H^p(\mathfrak{f}_{n-1})$. Since j^* just means restricting the domain to $\Lambda^*\mathfrak{a}$, we have

$$j^{p+2}\mathrm{d}_{\mathfrak{f}}^p(f) = j^{p+2}(f \wedge u) = j^p f \wedge j^2 u = -j^p \wedge v = -\mathrm{d}_{\mathfrak{a}}^p j^p f.$$

Finally, we show that $\delta^{p+2}\, \mathrm{d}_{\mathfrak{a}}^p = \mathrm{d}_{\mathfrak{a}}^p\, \delta^p$. Let $f \in \Lambda^p\mathfrak{a}^*$. Since δ^p is nothing but the action induced by the adjoined action of z (see [**4**]), we have

$$\delta^{p+2}\mathrm{d}_{\mathfrak{a}}^p(f) = z \cdot (f \wedge v) = \mathrm{d}_{\mathfrak{a}}^{p+2}\delta^p f + f \wedge (z \cdot v).$$

But $z \cdot v = 0$, and therefore $\delta^{p+2}\mathrm{d}_{\mathfrak{a}}^p = \mathrm{d}_{\mathfrak{a}}^{p+2}\delta^p$. ∎

4.1. Proof of lemma 4. We prove lemma 4 by induction on the dimension of $\mathfrak{a}_n$.

For $n = 1$, the image of $1 \in \Lambda^0\mathfrak{a}_1^*$ under d_1^0 is just $x_1^* \wedge y_1^*$, so d_1^0 is injective. Clearly, since $\Lambda^3\mathfrak{a}_1 = \Lambda^4\mathfrak{a}_1 = 0$, d_1^1 and d_1^2 are surjective.

Now suppose this lemma has been proved for even dimensional Abelian Lie algebras up to dimension $2(n-1)$. Fix p. When decomposing $\Lambda^p\mathfrak{a}_n^*$ as

$$x_1^* \wedge \Lambda^{p-1}\mathfrak{a}_{n-1}^* + y_1^* \wedge \Lambda^{p-1}\mathfrak{a}_{n-1} + x_1^* \wedge y_1^* \wedge \Lambda^{p-2}\mathfrak{a}_{n-1}^* + \Lambda^p\mathfrak{a}_{n-1} \tag{5}$$

we see that

$$\mathrm{d}_n^p(x_1^* \wedge f) = x_1^* \wedge \mathrm{d}_{n-1}^{p-1} f \qquad \text{for } f \in \Lambda^{p-1}\mathfrak{a}_{n-1}^* \tag{6}$$

$$\mathrm{d}_n^p(y_1^* \wedge f) = y_1^* \wedge \mathrm{d}_{n-1}^{p-1} f \qquad \text{for } f \in \Lambda^{p-1}\mathfrak{a}_{n-1}^* \tag{7}$$

$$\mathrm{d}_n^p(x_1^* \wedge y_1^* \wedge f) = x_1^* \wedge y_1^* \wedge \mathrm{d}_{n-1}^{p-2} f \qquad \text{for } f \in \Lambda^{p-2}\mathfrak{a}_{n-1}^* \tag{8}$$

$$\mathrm{d}_n^p(f) = \mathrm{d}_{n-1}^p f + x_1^* \wedge y_1^* \wedge f \qquad \text{for } f \in \Lambda^p\mathfrak{a}_{n-1}^* \tag{9}$$

Therefore, d_n^p is injective (or surjective) if and only if both d_{n-1}^{p-1} and

$$\alpha^p = \mathrm{d}_n^p|_{x_1^* \wedge y_1^* \wedge \Lambda^{p-2}\mathfrak{a}_{n-1}^* + \Lambda^p\mathfrak{a}_{n-1}}$$

are. Note that α^p is injective (or surjective) in case both d_{n-1}^p and d_{n-1}^{p-2} are.

For $p \leqslant n-2$, the induction hypothesis implies that d_{n-1}^{p-2}, d_{n-1}^{p-1} and d_{n-1}^p all are injective, so d_n^p is as well. Analogously, for $p \geqslant n$, we know from the induction hypothesis that d_{n-1}^{p-2}, d_{n-1}^{p-1} and d_{n-1}^p are surjective, so d_n^p is too. ∎

For $p = n-1$, the induction hypothesis states that the maps d_{n-1}^{n-3} and d_{n-1}^{n-2} are injective, whereas d_{n-1}^{n-1} is surjective. To settle this last case, we need to show that α^{n-1} is injective (or, equivalently, surjective). This is done by showing that

$$\ker \mathrm{d}_{n-1}^{n-3} \cap \operatorname{Im} \mathrm{d}_{n-1}^{n-1} = 0.$$

In fact, we prove slightly more than this in

Lemma 5. (With notations as above.) *For each* $2 \leqslant k \leqslant n$, *let*

$$D_n^k = d_n^{n+k-2} \dots d_n^{n-k+2}.$$

Then $\operatorname{Im} d_n^{n-k} \cap \ker D_n^k = 0$.

Since the proof of lemma 4 for a Lie algebra of dimension $2n$ only needs lemma 5 for a Lie algebra of dimension $2(n-1)$, we may safely use lemma 4 for Lie algebras of dimension $2n$ while proving lemma 5 for algebras of dimension $2n$.

Proof of lemma 5: We proceed by induction on n. For $n = 2$ we only need to check the case $k = 2$. In this case

$$\operatorname{Im} d_2^0 = \langle x_1 \wedge y_1 + x_2 \wedge y_2 \rangle$$

while

$$d_2^2(x_1 \wedge y_1 + x_2 \wedge y_2) = -2x_1 \wedge x_2 \wedge y_1 \wedge y_2 \neq 0.$$

Now suppose the lemma has been proven for Abelian Lie algebras up to dimension $2(n-1)$. Let $2 \leqslant k \leqslant n$. We decompose Λ^{n-k} and Λ^{n-k+2} again as in (5), and use the expressions (6), (7), (8) and (9) for d_n^p with respect to these decompositions.

Any element of $\operatorname{Im} d_n^{n-k}$ is of the form

$$x_1^* \wedge d_{n-1}^{n-k-1} f_1 + y_1^* \wedge d_{n-1}^{n-k-1} f_2 + x_1^* \wedge y_1 \wedge (f_4 + d_{n-1}^{n-k-2} f_3) + d_{n-1}^{n-k} f_4,$$

where $f_1, f_2 \in \Lambda^{n-k-1}\mathfrak{a}_{n-1}$, $f_3 \in \Lambda^{n-k-2}\mathfrak{a}_{n-1}$ and $f_4 \in \Lambda^{n-k}\mathfrak{a}_{n-1}$. For the map D_n^k we find

$$D_n^k(x_1^* \wedge g_1) = x_1^* \wedge (d_{n-1}^{n+k-3} \dots d_{n-1}^{n-k+1} g_1) = x_1^* \wedge D_{n-1}^k g_1$$

for any $g_1 \in \Lambda^{n-k+1}\mathfrak{a}_{n-1}$, and analogously

$$D_n^k(y_1^* \wedge g_2) = y_1^* \wedge (d_{n-1}^{n+k-3} \dots d_{n-1}^{n-k+1} g_2) = y_1^* \wedge D_{n-1}^k g_2$$

for any $g_2 \in \Lambda^{n-k+1}\mathfrak{a}_{n-1}$. For D_n^k applied to the last two parts in the decomposition of $\Lambda^{n-k+2}\mathfrak{a}_n$ one proves easily that

$$D_n^k(x_1^* \wedge y_1^* \wedge g_3 + g_4) = x_1^* \wedge y_1^* \wedge \left((k-1)D_{n-1}^{k-1} g_4 + D_{n-1}^{k-1} d_{n-1}^{n-k} g_3\right) + d_{n-1}^{n+k-2} D_{n-1}^{k-1} g_4$$

for any $g_3 \in \Lambda^{n-k}\mathfrak{a}_{n-1}$ and any $g_4 \in \Lambda^{n-k+2}\mathfrak{a}_{n-1}$.

Now suppose $g = x_1^* \wedge g_1 + y_1^* \wedge g_2 + x_1^* \wedge y_1^* \wedge g_3 + g_4$ is contained in $\operatorname{Im} d_n^{n-k} \cap \ker D_n^k$. Then we have

$$D_{n-1}^k g_1 = 0, \tag{10}$$

$$D_{n-1}^k g_2 = 0, \tag{11}$$

$$(k-1)D_{n-1}^{k-1} g_4 + D_{n-1}^{k-1} d_{n-1}^{n-k} g_4 = 0, \tag{12}$$

$$d_{n-1}^{n+k-2} D_{n-1}^{k-1} g_4 = 0, \tag{13}$$

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and there exists a $f = x_1^* \wedge f_1 + y_1^* \wedge f_2 + x_1^* \wedge y_1^* \wedge f_3 + f_4 \in \Lambda^{n/2-2}\mathfrak{a}_n$ such that

$$d_{n-1}^{n-k-1} f_1 = g_1, \tag{14}$$

$$d_{n-1}^{n-k-1} f_2 = g_2, \tag{15}$$

$$f_4 + d_{n-1}^{n-k-2} f_3 = g_3, \tag{16}$$

$$d_{n-1}^{n-k} f_4 = g_4. \tag{17}$$

The induction hypothesis implies that the systems of equations (10), (14) and (11), (15) do not have a solution except the trivial one. From equations (12) and (17) it follows that

$$\mathrm{D}_{n-1}^{k-1}\mathrm{d}_{n-1}^{n-k}\big((k-1)f_4 + g_3\big) = 0.$$

Stated otherwise, $\mathrm{d}_{n-1}^{n-k}\big((k-1)f_4 + g_3\big) \in \ker D_{n-1}^{k-1}$. The induction hypothesis now states that $\mathrm{d}_{n-1}^{n-k}\big((k-1)f_4 + g_3\big) = 0$, and thus $g_3 = -(k-1)f_4$ by the injectivity of d_{n-1}^{n-k}. Substituting g_3 in (16) yields

$$d_{n-1}^{n-k-2} f_3 = k f_4$$

while (17) combined with (13) gives

$$\mathrm{d}_{n-1}^{n+k-2} D_{n-1}^{k-1} d_{n-1}^{n-k} f_4 = D_{n-1}^{k+1} f_4 = 0$$

The induction hypothesis again yields $f_4 = 0$, and therefore $g_3 = g_4 = 0$. □

5. Examples

5.1. A family of three-step nilpotent Lie algebras. Let $n \geqslant 1$ and $\mathfrak{b}_n$ the Lie algebra of dimension $2n+2$ with a basis

$$\{x_1, \ldots, x_n, y_1, \ldots y_n, w, z\}$$

and Lie bracket

$$[z, x_i] = y_i, \quad [x_i, y_i] = w$$

for all $1 \leqslant i \leqslant n$. Then $\mathfrak{b}_n$ is an extension of $\langle z \rangle$ by the Heisenberg Lie algebra $\mathfrak{h}_n$. The Betti numbers of the quotient algebra $\mathfrak{f}_n = \mathfrak{b}_n / \langle w \rangle$ are given by

$$\beta_p(\mathfrak{f}_n) = \binom{n+1}{\left\lfloor \frac{1+p}{2} \right\rfloor} \binom{n}{\left\lfloor \frac{p}{2} \right\rfloor},$$

where $\lfloor x \rfloor$ denotes the integer part of x (see [**2**]). The main theorem allows us to compute the Betti numbers of $\mathfrak{b}_n$ in terms of the cohomology of $\mathfrak{f}_n$.

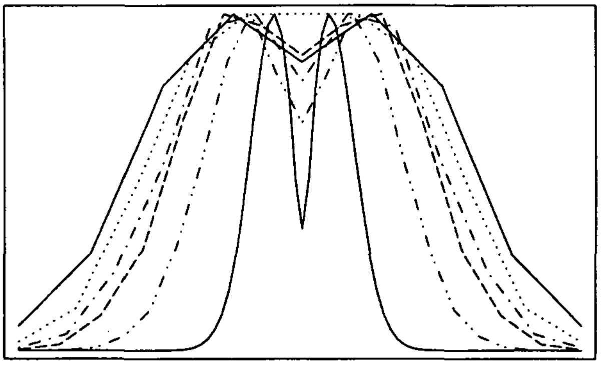
The table below lists the Betti numbers (or, at least, half of them) up to dimension 26.

$\mathfrak{b}_1$	1	2	2											
$\mathfrak{b}_2$	1	3	5	6										
$\mathfrak{b}_3$	1	4	11	14	12									
$\mathfrak{b}_4$	1	5	19	35	40	40								
$\mathfrak{b}_5$	1	6	29	69	120	125	100							
$\mathfrak{b}_6$	1	7	41	119	273	399	385	350						
$\mathfrak{b}_7$	1	8	55	188	532	980	1372	1274	980					
$\mathfrak{b}_8$	1	9	71	279	936	2064	3696	4704	4116	3528				
$\mathfrak{b}_9$	1	10	89	395	1530	3915	8460	13320	16380	14112	10584			
$\mathfrak{b}_{10}$	1	11	109	539	2365	6875	17325	32175	49500	57420	47124	38808		
$\mathfrak{b}_{11}$	1	12	131	714	3498	11374	32670	69575	127050	179685	202554	165528	121968	
$\mathfrak{b}_{12}$	1	13	155	923	4992	17940	57772	138424	291005	479765	665379	722007	566280	453024

As illustrated in the figure below (showing the distribution of the Betti numbers of β_n for $n = 3, 4, 5, 6, 10$ and 40), these Betti numbers do not have a unimodal distribution. It is easy to prove that

$$\beta_n(\mathfrak{b}_n)-\beta_{n+1}(\mathfrak{b}_n) = \begin{cases} \dfrac{n!(n+1)!}{(n/2)!(n/2+1)!(n/2+1)!(n/2+2)!}\left(\dfrac{n}{2}-2\right) & \text{for } n \text{ even,} \\ \dfrac{n!(n+1)!}{((n-1)/2)!((n+1)/2)!((n+3)/2)!((n+3)/2)!}\,\dfrac{n-1}{2} & \text{for } n \text{ odd.} \end{cases}$$

The sign of this difference is determined by a linear polynomial; for $n = 1$ or $n = 4$ we see that $\beta_n(\mathfrak{b}_n) = \beta_{n+1}(\mathfrak{b}_n)$, for $n = 2$ we have $\beta_n(\mathfrak{b}_n) < \beta_{n+1}(\mathfrak{b}_n)$ and for $n = 3$ or $n \geqslant 5$ we have $\beta_n(\mathfrak{b}_n) > \beta_{n+1}(\mathfrak{b}_n)$, thus implying non-unimodality. An analysis along the same lines allows us to determine exactly the number p such that $\beta_{p-1}(\mathfrak{b}_n) < \beta_p(\mathfrak{b}_n)$ but $\beta_p(\mathfrak{b}_n) > \beta_{p+1}(\mathfrak{b}_n)$.



5.2. Cohomology of the twisted standard filiform Lie algebras. Let $\mathfrak{g}_n$ be the twisted standard filiform Lie algebra of dimension $n+2$ given in (2). Then $\mathfrak{g}_n$ is an extension of $\langle z \rangle$ by the Heisenberg algebra $\mathfrak{h}_n$, and $\mathfrak{g}_n/\langle w \rangle$ is just the standard filiform Lie algebra $\mathfrak{f}_n$ of dimension $n+1$. According to [**3**], the Betti numbers of this Lie algebra are given by the formula

$$\beta_p(\mathfrak{f}_n) = \kappa_p + \kappa_{p-1}$$

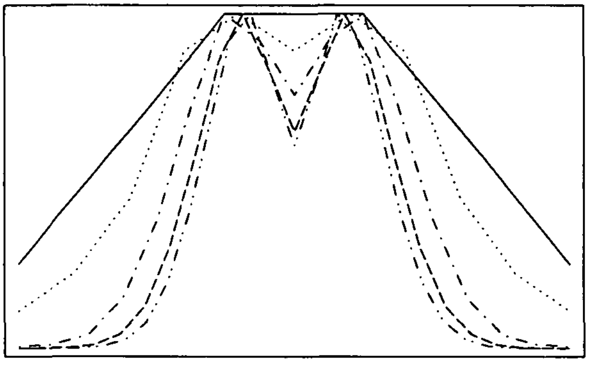
where

$$\kappa_p = \#\left\{(\alpha_1, \ldots, \alpha_p) \in \mathbb{Z}^p \mid 1 \leqslant \alpha_1 < \cdots < \alpha_p \leqslant n \text{ and } \sum_i \alpha_i = \left\lfloor \frac{i(n+1)}{2} \right\rfloor \right\}.$$

Again, the main theorem tells us how to compute the Betti numbers of $\mathfrak{g}_n$ in terms of the Betti numbers of $\mathfrak{f}_n$. For the Betti numbers (at least, half of them) of the algebras $\mathfrak{g}_n$ up to dimension 26 we find

$\mathfrak{g}_4$	1	2	2											
$\mathfrak{g}_6$	1	2	2	2										
$\mathfrak{g}_8$	1	2	3	4	4									
$\mathfrak{g}_{10}$	1	2	4	8	9	8								
$\mathfrak{g}_{12}$	1	2	5	13	22	23	20							
$\mathfrak{g}_{14}$	1	2	6	19	41	61	59	50						
$\mathfrak{g}_{16}$	1	2	7	26	68	129	177	163	134					
$\mathfrak{g}_{18}$	1	2	8	34	105	240	414	530	466	376				
$\mathfrak{g}_{20}$	1	2	9	43	152	406	839	1342	1630	1388	1100			
$\mathfrak{g}_{22}$	1	2	10	53	211	643	1541	2929	4410	5129	4243	3320		
$\mathfrak{g}_{24}$	1	2	11	64	284	970	2636	5773	10252	14657	16430	13278	10260	
$\mathfrak{g}_{26}$	1	2	12	76	371	1405	4267	10553	21474	35980	49232	53456	42307	32384

Again, we see that the Betti numbers of these low-dimensional examples follow a non-unimodal distribution. The figure below shows the distribution of the Betti numbers of $\mathfrak{g}_n$ for $n = 3, 4, 7, 10$ and 12. However, a proof of this fact would probably involve partition functions and, in particular, the way their coefficients grow with respect to one another, which is far beyond the scope of this text.



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