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INVARIANT MEANS ON DENSE SUBSEMIGROUPS OF TOPOLOGICAL GROUPS

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1. Introduction. Let S be a topological semigroup (i.e., S is a semigroup with a Hausdorff topology such that the mapping from $S \times S$ to S defined by $(s, t) \rightarrow s \cdot t$ for all s, t in S is continuous when $S \times S$ has the product topology) and $C(S)$ be the space of bounded continuous real valued functions on S. For each f in $C(S)$ and a in S, define $||f|| = \sup \{|f(s)|: s \in S\}$ (sup norm of f); $r_a f(s) = f(sa)$ and $l_a f(s) = f(as)$ for all s in S. If X is a sup norm closed subspace of $C(S)$ which is translation invariant (i.e., $r_a(X) \subseteq X$ and $l_a(X) \subseteq X$ for all a in S) and contains the constant one function 1_S, then an element ϕ in X^*, the conjugate space of X, is a LIM (*left invariant mean*) if $\phi(1_S) = ||\phi|| = 1$ and $\phi(l_a f) = \phi(f)$ for all f in X and a in S. (See [**2**].)

A function $f \in C(S)$ is *left* (*right*) *uniformly continuous* if whenever $\{s_i\}$ is a net in S and s_i converges to some s in S, then $||l_{s_i} f - l_s f|| \rightarrow 0$ ($||r_{s_i} f - r_s f|| \rightarrow 0$); furthermore, f is *uniformly continuous* if f is left and right uniformly continuous. As known, LUC(S) and UC(S), the space of left uniformly continuous functions on S and the space of uniformly continuous functions on S respectively, are translation invariant, sup norm closed subalgebras of $C(S)$ containing 1_S. Furthermore, if S is compact, then $C(S) =$ LUC$(S) =$ UC(S) (see [**10**, pp. 64–65]).

Let G be a topological semigroup and S be a dense subsemigroup of G. It is easy to see that

(*) if UC(S) has a LIM, then UC(G) also has a LIM.

Indeed, if ϕ is a LIM on UC(S), define $\tilde{\phi} \in$ UC(S) by $\tilde{\phi}(f) = \tilde{\phi}(f|_S)$, where $f|_S$ is the restriction of f to S; then $\tilde{\phi}$ is a LIM on UC(G) (see [**9**, Theorem 8)]. However, the converse of (*) is false in general even when G is compact. (Consider the following example of [**9**, pp. 640–641]: let S be the free semigroup on two generators with the discrete topology and $G = S \cup \{z\}$ be the one point compactification of S, where $tz = zt = z$ for all $t \in G$. It is easy to see that $C(G)$ has a LIM and yet UC(S) does not.)

The main purpose of this paper is to establish a partial converse of (*). Let G be a topological group and S be a dense subsemigroup of G. We show in § 3 that:

(1) If $C(G)$ has a LIM, then UC(S) has a LIM.

(2) If LUC(G) has a LIM and S has finite intersection property for right ideals, then UC(S) has a LIM.

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(3) If UC(G) has a LIM and S has finite intersection property for right ideals and finite intersection property for left ideals, then UC(S) has a LIM.

Unfortunately, we know of no example which shows that the condition "S has finite intersection property for right ideals" in (2) (and the corresponding condition on S in (3)) cannot be entirely dropped.

2. Extension of uniformly continuous functions on subsemigroups of topological groups. Recently, S. Wiley [**12**] has considered pairs S, T, where T is a topological semigroup and S is a subsemigroup of T such that each function in LUC(S) has an extension to a function in LUC(T). In this section, we consider extension properties of the similar type. Results in this section are essential tools for our main work in § 3. Proof of the early results of this section are adaptations of the proofs of [**12**].

For the rest of this paper, G will denote a topological group. If S is a topological semigroup and $a \in S$, then $S[a]$ will denote the subsemigroup $\{aS \cap Sa\} \cup \{a\}$ of S.

LEMMA 2.1. *Let S be a dense subsemigroup of G and $a \in S$. If $\{s_i\}$ and $\{t_j\}$ are two nets in $S[a]$ which converge to some $g \in G$, then* $\lim_i f(s_i)$ *and* $\lim_j f(t_j)$ *exist and are equal for each $f \in$ UC(S).*

Proof. Let $f \in$ UC(S) be arbitrary and fixed. We first assume that $\lim_i f(s_i) = L_1$, $\lim_j f(t_j) = L_2$ and $L_1 \neq L_2$. Let $\epsilon = |L_1 - L_2|$. For each $n \in N$, where N is the family of neighbourhoods of g, we can find elements, s_n and t_n from $\{s_i\}$ and $\{t_j\}$, respectively, with the property that

$$|f(s_n) - f(t_n)| \geqq \epsilon.$$

Clearly, the nets $\{s_n: n \in N\}$ and $\{t_n: n \in N\}$ also converge to g. For each $n \in N$, pick p_n, q_n in S such that $s_n = ap_n$ and $t_n = q_n a$. Then the nets $\{p_n : n \in N\}$, $\{q_n: n \in N\}$ converge to $a^{-1}g$ and ga^{-1}, respectively. Let $\{w_k : k \in D\}$ be a net in S which converges to $w = ag^{-1}a$. Then for each $n \in N$, $k \in D$,

$$\begin{aligned} \epsilon &\leqq |f(ap_n) - f(q_n a)| \\ &\leqq |f(ap_n) - f(q_n w_k p_n)| + |f(q_n w_k p_n) - f(q_n a)| \\ &\leqq ||l_a f - l_{q_n w_k} f|| + ||r_{w_k p_n} f - r_a f||, \end{aligned}$$

which is impossible since the nets $\{q_n w_k : (n, k) \in N \times D\}$ and $\{w_k p_n : (n, k) \in N \times D\}$ (where $N \times D$ denote the product directed set of N and D) in S converge to $a \in S$, and $f \in$ UC(S). Hence $L_1 = L_2$.

It remains to show that $\lim_i f(s_i)$ and $\lim_j f(t_j)$ exist. If $\lim_i f(s_i)$ does not exist, say, then we may find subnets $\{f(s_{i'})\}$ and $\{f(s_{i''})\}$ of the net $\{f(s_i)\}$ which converge to two distinct real numbers L_1' and L_2' in the closed interval $[-||f||, ||f||]$. However, the subnets $\{s_{i'}\}$ and $\{s_{i''}\}$ of the net $\{s_i\}$ also

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converge to g. Consequently, it follows from what we have proved that $L_1' = L_2'$, contradicting our assumption that L_1' and L_2' are distinct.

LEMMA 2.2. *Let θ be a mapping from a Hausdorff space X into a metric space (Y, d). Then the following are equivalent:*

(a) *There exists a dense subspace T of X with the property that whenever $\{t_i\}$ is a net in T which converges to a point x in X, then $d(\theta(t_i), \theta(x))$ converges to* 0.

(b) *θ is continuous.*

Proof. (b) $\Rightarrow$ (a) is trivial. Conversely, if (a) holds, and θ is not continuous, then there exist $\epsilon > 0$ and a net $\{x_i : i \in D\}$ in X which converges to a point x_0 in X such that $d(\theta(x_i), \theta(x_0)) \geqq \epsilon$. For each $i \in D$, let $\{t_{(i,j)} : j \in E_i\}$ be a net in T which converges to x_i. Let P be the product directed set $\times \{E_i : i \in D\}$. If $(i, h) \in D \times P$, define $t_{(i,h)} = t_{(i,h(i))}$. Then the net $\{t_{(i,h)} : (i, h) \in D \times P\}$ in T converges to $\lim_i \lim_j t_{(i,j)} = x_0$ (see [**8**, p. 69]). By assumption, we may choose $(i_0, h_0) \in D \times P$ such that whenever $(i, h) \geqq (i_0, h_0)$, $d(\theta(t_{(i,h)}) - \theta(x_0)) < \epsilon/2$. Furthermore, we can also choose $j_0 \in E_{i_0}$ such that $j_0 \geqq h_0(i_0)$ and $d(\theta(x_{i_0}), \theta(t_{(i_0,j_0)}) < \epsilon/2$. Define $h_1 \in P$ by $h_1(i) = h_0(i)$ if $i \neq i_0$ and $h_1(i_0) = j_0$. Then $(i_0, h_1) \geqq (i_0, h_0)$ and $t_{(i_0,j_0)} = t_{(i_0,h_1)}$. Consequently,

$$\begin{aligned}\epsilon &\leqq d(\theta(x_{i_0}), \theta(x_0)) \\ &\leqq d(\theta(x_{i_0}),\ \theta(t_{(i_0,j_0)})) + d(\theta(t_{(i_0,h_1)},\ \theta(x_0)) \\ &< \epsilon,\end{aligned}$$

which is impossible. Hence, θ is continuous.

THEOREM 2.3. *If S is a subsemigroup of G such that $\bar{S}$ is a group, then for each $f \in \mathrm{UC}(S)$, there exists $f \in C(G)$ such that $F|_S = f$.*

Proof. Using Tietze's extension theorem, we may assume that $\bar{S} = G$. Let $a \in S$ be fixed. It is easy to see that $S[a]$ is also dense in G. For each $g \in G$, define $F(g) = \lim_i f(s_i)$ where $\{s_i\}$ is a net in $S[a]$ converging to g. It follows from Lemmas 2.1 and 2.2 that F is well defined, $F \in C(G)$, and $F|_S = f$.

If G is a compact group, then $\mathrm{UC}(G) = C(G)$ and $\bar{S}$ is a group for *any* subsemigroup of G (see, for example, [**6**, p. 99]). Hence we have:

COROLLARY 2.4. *If G is compact and S is a subsemigroup of G, then for each $f \in \mathrm{UC}(S)$, there exists $F \in \mathrm{UC}(G)$ such that $F|_S = f$.*

Remark 2.5. Corollary 2.4 was proved by Wiley [**12**, Theorem 4.6] for the case when S is abelian.

The next lemma, due to Wiley [**12**], follows immediately from [**7**, Theorem 3] and the observation that if G is a topological group, then $\mathrm{LUC}(G)$ is precisely the uniformly continuous bounded real valued functions on (G, R), where R is the right uniformity on G (see, for example, [**6**, p. 21]).

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LEMMA 2.6 (Wiley [**12**, Lemma 3.5]). *If* G_0 *is a subgroup of* G *and* $f \in \mathrm{LUC}(G_0)$, *then there exists* $F \in \mathrm{LUC}(G)$ *such that* $F|_S = f$.

LEMMA 2.7. *If* S *is a dense subsemigroup of* G *with finite intersection property for right ideals and* $F \in C(G)$ *such that* $F|_S \in \mathrm{LUC}(S)$, *then* $F \in \mathrm{LUC}(G)$.

Proof. We first note that since S has finite intersection property for right ideals, $G_0 = SS^{-1}$ is a subgroup of G containing S (see, for example, [**1**, p. 36]). Let F_0 denote the restriction of F to G_0. If we can show that $F_0 \in \mathrm{LUC}(G_0)$, then by Lemma 2.6 there exists $\tilde{F}_0 \in \mathrm{LUC}(G)$ which extends F_0. Since $\tilde{F}_0(s) = F(s)$ for all $s \in S$ and S is dense in G, it follows that $F_0 = F$.

It remains to show that $F_0 \in \mathrm{LUC}(G_0)$. If $\{s_i\}$ is a net in S converging to some $g \in G_0$, let $a, b \in S$ such that $g = ab^{-1}$. Then

$$\begin{aligned}||l_{s_i}F_0 - l_gF_0|| &= ||l_b(l_{s_i}F_0 - l_gF_0)|| \\ &= \sup\{|F_0(s_ibt) - F_0(gbt)| : t \in S\},\end{aligned}$$

which converges to 0 since the net $\{s_ib\}$ converges to $gb = a \in S$ and $F_0|_S \in \mathrm{LUC}(S)$. It follows from Lemma 2.2 that the mapping $\theta : G_0 \to C(G_0)$ defined by $\theta(g) = l_gF_0$ for each $g \in G_0$ is continuous when $C(G_0)$ has the sup norm topology, i.e., $F \in \mathrm{LUC}(G_0)$.

THEOREM 2.8. *Let* S *be a subsemigroup of* G *with finite intersection property for right ideals. If* $\bar{S}$ *is a group, then for each* $f \in \mathrm{UC}(S)$, *there exists* $F \in \mathrm{LUC}(G)$, *such that* $F|_S = f$.

Proof. By Lemma 2.6, we may assume that $\bar{S} = G$. The theorem now follows from Theorem 2.3 and Lemma 2.7.

COROLLARY 2.9. *If* S *is a subsemigroup of* G *with finite intersection property for right ideals and finite intersection property for left ideals, then for each* $f \in \mathrm{UC}(S)$ *there exists* $F \in \mathrm{UC}(G)$ *such that* $F|_S = f$.

Proof. It follows from Theorem 2.8 that there exists $F \in \mathrm{LUC}(G)$ such that $F|_S = f$. Furthermore, an application of Lemma 2.6 (and interchanging "left" and "right") shows that F is also right uniformly continuous.

3. Main results. We are now ready to state and prove our main results.

THEOREM 3.1. *Let* S *be a dense subsemigroup of* G.

(a) *If* $C(G)$ *has a* LIM, *then* $\mathrm{UC}(S)$ *has a* LIM.

(b) *If* S *has finite intersection property for right ideals and* $\mathrm{LUC}(S)$ *has a* LIM, *then* $\mathrm{UC}(S)$ *has a* LIM.

(c) *If* S *has finite intersection property for right ideals and finite intersection property for left ideals, and* $\mathrm{UC}(G)$ *has a* LIM, *then* $\mathrm{UC}(S)$ *has a* LIM.

Proof. We will prove (a). The proofs for (b) and (c) are similar.

If ψ is a LIM on $C(G)$, define $\tilde{\psi} \in \mathrm{UC}(S)^*$ by $\tilde{\psi}(f) = \psi(\tilde{f})$, for each $f \in C(S)$, and $\tilde{f}$ is the *unique* function in $C(G)$ extending f (Theorem 2.3). It

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is easy to see that $\|\tilde{\psi}\| = \tilde{\psi}(1_S) = 1$. Now if $a \in S$ and $f \in C(S)$, then $l_a\tilde{f} \in C(G)$ and $l_a\tilde{f}(s) = (l_af)(s)$ for each $s \in S$. Consequently, $l_a\tilde{f} = (l_af)^\sim$. Hence $\tilde{\psi}(l_af) = \psi(l_af)^\sim = \psi(l_a\tilde{f}) = \psi(\tilde{f}) = \tilde{\psi}(f)$.

COROLLARY 3.2. *If G is a locally compact group such that* UC(G) *has a* LIM, *and S is a subsemigroup of G such that $\bar{S}$ is a group, then* UC(S) *has a* LIM.

Proof. UC(G) has a LIM implies that $C(\bar{S})$ has a LIM (see [**5**,Theorem 2.3.2]), and hence UC(S) has a LIM.

COROLLARY 3.3. *If G is compact, then* UC(S) *has a unique* LIM *for each subsemigroup S of G.*

Proof. Since UC(G) has a LIM (see [**13**, p. 224]) and $\bar{S}$ is a group [**6**, p. 99], it follows from Corollary 3.2 that UC(S) also has a LIM. If ϕ_1 and ϕ_2 are distinct LIM on $UC(S)$, there exists $f_0 \in$ UC(S) such that $\phi_1(f_0) \neq \phi_2(f_0)$. For each $F \in$ UC(G_0), $G_0 = \bar{S}$, define $\tilde{\phi}_i(F) = \phi_i(F|_S)$, $i = 1, 2$. Then as is readily checked, $\tilde{\phi}_i(l_sF) = \tilde{\phi}_i(F)$ for each $s \in S$.

Since S is dense in G_0 and the mapping $s \rightarrow l_sF$, $s \in S$, is continuous, it follows that $\tilde{\phi}_1$ and $\tilde{\phi}_2$ are LIM on UC(G_0). Let $F_0 \in$ UC(G_0) such that $F_0|_S = f_0$ (Corollary 2.4), then $\tilde{\phi}_1(F_0) \neq \tilde{\phi}_2(F_0)$, which is impossible by the uniqueness of LIM on UC(G_0) (see [**11**]).

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