

On soluble just-non-Cross varieties of groups

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We prove a partial confirmation of Kovács and Newman's conjecture that a just-non-Cross variety is soluble if and only if it is in the following list: \underline{A} , $\underline{\underline{A}}_{p=p}$, $\underline{\underline{A}}_{p=q}$, $\underline{\underline{A}}_{p=q,r}$, where p , q and r are any three distinct primes.

1. Introduction

The Oates-Powell Theorem enables us to call a variety (of groups) Cross if it can be generated by a single finite group. L.G. Kovács and M.F. Newman [7, Theorem 1] have pointed out that a variety is non-Cross if and only if it contains a non-Cross variety whose proper subvarieties are all Cross: a so-called jnC (just-non-Cross) variety. In [6], Kovács and Newman prove that the decomposable (that is, nontrivially factorisable) jnC varieties are precisely the $\underline{\underline{A}}_{p=p}$, $\underline{\underline{A}}_{p=q}$ and $\underline{\underline{A}}_{p=q,r}$, where p , q and r are any three distinct primes, and they propose the following conjecture [6, p. 222].

CONJECTURE 1.1. *Every soluble jnC variety of finite exponent is decomposable.*

Our aim in this paper is to substantially reduce Conjecture 1.1 by proving:

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THEOREM 1.2. *If there is a soluble jnC variety \underline{V} of finite exponent which is not decomposable, then one can find (distinct) primes p and q and an integer n (all three depending on \underline{V}), such that \underline{V} is a subvariety of $\underline{A}_p(\underline{V} \wedge \underline{B}_q^n)$, where $\underline{V} \wedge \underline{B}_q^n$ is nilpotent of class at least three.*

Theorem 1.2 is phrased so as to emphasise the fact that its hypotheses can be satisfied (if and) only if Conjecture 1.1 is false. Consider the following special case of Conjecture 1.1.

CONJECTURE 1.3. *If \underline{N} is a nilpotent variety of class at least two and exponent a power of the prime q , and if p is a prime unequal to q , then $\underline{A}_{p=q}^T$ is the unique jnC subvariety of $\underline{A}_{p=q}^N$.*

COROLLARY 1.4. *Conjecture 1.1 is true if and only if Conjecture 1.3 is.*

To date, the only significant progress we have made with Conjecture 1.3 is [2, 1.2], which shows it to be true in case \underline{N} has class two.

COROLLARY 1.5. *The soluble jnC varieties which are not abelian-by-nilpotent are the $\underline{A}_{p=q}^A A_r$, where p, q and r are any three distinct primes.*

COROLLARY 1.6. *The nonmetabelian jnC varieties which are abelian-by-nilpotent-of-class-two are the $\underline{A}_{p=q}^T$, where p and q are distinct primes.*

Theorem 1.2 is a simple consequence of [1, 3.6 (ii)], [2, 1.2], the "internal" result of John Cossey's thesis [3] (which states that $\underline{A}_{p=q}^A A_r$ is jnC), and the following two results.

THEOREM 1.7. *If \underline{V} is a soluble jnC variety which is not locally nilpotent, there are primes p, q and r , with p not equal to q or r , and a nilpotent subvariety \underline{N} of \underline{V} of exponent a power of r , such that $\underline{V} \leq \underline{A}_{p=q}^A \underline{N}$ and $p = r$ if and only if \underline{V} is abelian-by-nilpotent.*

THEOREM 1.8. *Let p, q and r be pairwise distinct primes, and suppose that \underline{V} is a jnC subvariety of $\underline{A}_{p=q}^A \underline{N}$, where \underline{N} is a*

the class of P is at least $c + 1$. //

We shall find the following lemma quite useful. We refer to [8, 53.11] for the definition of similarity of normal subgroups.

LEMMA 2.2. *Let \underline{V} be a variety, G a group in \underline{V} , and N an abelian normal subgroup of G . Then \underline{V} also contains a group G^* , which has a complemented, self-centralising (abelian) normal subgroup N^* , with N^* similar to N .*

Proof. (L.G. Kovács) Denote the direct square $G \times G$ of G by K , and the normal subgroup $N \times E$ of K by M . Let L be the diagonal in K , and denote $L \cap (C \times C)$ by J , where C is $C_G(N)$. Put ML/J equal to G^* , MJ/J equal to N^* , and L/J equal to H^* . Then $G^* \in \underline{V}$, and N^* is a self-centralising normal subgroup of G^* which is complemented in G^* by H^* . Define $\mu : N \rightarrow N^*$ by $n\mu = (n, e)J$, and $\nu : G/C \rightarrow G^*/N^*$ by $gC\nu = ((g, g)J)N^*$; it is easy to check that μ and ν are in fact isomorphisms. Finally,

$$\begin{aligned} (n^{gC})\mu &= n^g\mu = (n^g, e)J \\ &= (n, e)^{(g, g)}J \\ &= (n, e)J((g, g)J)N^* \\ &= n\mu^{gC\nu}. \quad // \end{aligned}$$

We conclude this section by recalling Kovács and Newman's version of the Oates–Powell Theorem, as it is the more convenient for our purposes. For positive integers e, m and c denote by $\underline{C}(e, m, c)$ the class of all (not necessarily finite) groups of exponent dividing e whose chief-sections have order (at most) m , and whose nilpotent sections have class (at most) c . They prove [5]:

THEOREM 2.3. *For all positive integers e, m and c , $\underline{C}(e, m, c)$ is a Cross variety. Furthermore, a variety \underline{V} is Cross if and only if there exist positive integers e, m and c such that \underline{V} is a subclass of $\underline{C}(e, m, c)$. //*

3. The proof of Theorem 1.7

As \underline{V} is not abelian, it has finite exponent, say n , and hence it

is locally finite. Since \underline{V} is not even locally nilpotent by [1, 3.7] there is a prime, say p , and an infinite set, say Γ , of monolithic groups in \underline{V} such that:

- (i) $\underline{V} = \text{var}\Gamma$;
- (ii) the monolith of each group in Γ is complemented, self-centralising, and has exponent p ;
- (iii) $\{|M(G)| : G \in \Gamma\}$ is an infinite set.

We denote by l the least positive integer such that \underline{V} is a subvariety of \underline{A}^{l+1} ; then $\underline{V} \wedge \underline{A}^l$ is Cross. (Such an l exists since \underline{V} is assumed soluble.) If $\Delta = \{G \in \Gamma : G \in \underline{A}^l\}$, then $\text{var}\Delta \leq \underline{V} \wedge \underline{A}^l$, and so it is Cross. It follows that $\text{var}(\Gamma - \Delta)$ is non-Cross, and hence equals \underline{V} , and so we may as well suppose that Δ is empty; that is, we suppose that every group in Γ has solubility length $l + 1$ precisely.

Let $G \in \Gamma$, and denote $M(G)$ by V .

LEMMA 3.1. $G^{(l)} = V$, and if $k < l$, $G^{(k)} > V$.

Proof. Since G has solubility length precisely $l + 1$, $G^{(k)}$ is trivial if and only if $k \geq l + 1$. Since G is also monolithic, $G^{(k)} \geq V$ for $k \leq l$. As $G^{(l)}$ is abelian, and V is a self-centralising subgroup of $G^{(l)}$, $V = G^{(l)}$. If $k < l$, $G^{(k)} > G^{(l)} = V$. //

Let H be a complement for V in G , and observe that by Lemma 3.1, $H \in \underline{A}^l$. Thus if $\underline{W} = \text{var}\{H : G \in \Gamma\}$, \underline{W} is a Cross subvariety of \underline{V} , and \underline{V} is a subvariety of $\underline{A}_p \underline{W}$. By properties (ii) and (iii) of Γ , and [8, 52.24], $\{|H| : G \in \Gamma\}$ is an infinite set, and hence there is a positive integer, say m , $0 < m \leq l$, which is least such that $\{|H^{(m)}| : G \in \Gamma\}$ is finite, say $|H^{(m)}| < d$ for all $G \in \Gamma$. From now on we think of V as a faithful irreducible $F(p)H$ -module.

LEMMA 3.2. $m \geq l - 1$.

Proof. Denote $C_H(H^{(m)})$ by C ; then $\{|H : C| : G \in \Gamma\}$ is

bounded (by $d!$). If T is a (right) transversal of C in H , and W is an irreducible submodule of V_C , it follows from Clifford's Theorem that $V_C = \sum\{Wt : t \in T\}$, and hence that $\{|W| : G \in \Gamma\}$ is infinite. Since \underline{V} is $\text{In}C$, we have by Theorem 2.3 that $\underline{V} = \text{var}\{VC : G \in \Gamma\}$. But $C^{(m+1)}$ is trivial, and so $\underline{V} \leq \underline{A}^{m+2}$; that is, $m \geq l - 1$. //

LEMMA 3.3. *There is a prime q , unequal to p , such that either \underline{V} is $\underline{A}_{p=Q}$, or H has an abelian normal subgroup Q of exponent q with $\{|Q| : G \in \Gamma\}$ infinite.*

Proof. By Clifford's Theorem, $V_{H^{(l-1)}}$ is completely reducible.

Since V is faithful and $H^{(l-1)}$ is abelian, it follows ([4, V, 5.17]) that $H^{(l-1)}$ is a p' -group. Thus in case $m = l$ (so that $\{|H^{(l-1)}| : G \in \Gamma\}$ is an infinite set), there is a prime, say q , such that $\{|Q| : G \in \Gamma\}$ is infinite, where Q is the socle of the Sylow q -subgroup of $H^{(l-1)}$.

Suppose, on the other hand (using Lemma 3.2), that $m = l - 1$. We replace H by $C_H(H^{(l-1)})$ (as in the proof of Lemma 3.2) and apply Lemma 2.2 and Theorem 2.3. In this way, we may suppose that $H^{(l-1)}$ is central in H , and in particular that it is cyclic. It follows that $H^{(l-2)}$ is nilpotent of class two, and so, by the argument used for $H^{(l-1)}$ above, it is a p' -group.

In case $l = 2$, we use Theorem 2.3 to reduce to the case that H is a q -group, for some prime q . It then follows from [2, 1.2] that \underline{V} is $\underline{A}_{p=Q}$.

Finally, if $l > 2$, then since $\underline{V} \wedge \underline{A}^l$ is Cross, it follows from Theorem 2.3 that the orders of the irreducible components of $V_{H^{(l-2)}}$ form a finite set as G ranges through Γ . By Lemma 3.2, $\{|H^{(l-2)}| : G \in \Gamma\}$ is an infinite set, and so there is a prime, say q , such that

$\{|A| : G \in \Gamma\}$ is infinite, where A is the Sylow q -subgroup of $H^{(l-2)}$. Since $H^{(l-1)}$ is cyclic, we have that $A^{(1)}$ is cyclic. Now V_A is completely reducible by Maschke's Theorem, and so since V is faithful and $A^{(1)}$ is cyclic, there is an irreducible direct summand, say U , of V_A such that the kernel K of U avoids $A^{(1)}$. Now $V_A \in \underline{V} \wedge \underline{A}^l$, which is Cross, and so $\{|U| : G \in \Gamma\}$ is a finite set, say $|U| < f$, for all $G \in \Gamma$. Hence $\{|A : K| : G \in \Gamma\}$ is bounded by $f!$. Since $\{|A| : G \in \Gamma\}$ is infinite, it follows that $\{|K| : G \in \Gamma\}$ is infinite. But $K \cap A^{(1)} = E$, and so $K \leq Z(A)$; hence $\{|Z(A)| : G \in \Gamma\}$ is an infinite set, and we can choose $Q = M(Z(A))$. //

If \underline{V} is $\frac{A \cdot T}{p=Q}$, Theorem 1.7 is proved (with $r = q$, $\underline{N} = \frac{A}{Q}$), and so we shall assume that H has an abelian normal subgroup Q of exponent q such that $\{|Q| : G \in \Gamma\}$ is infinite. Suppose (using Clifford's Theorem) that V_Q has homogeneous components $U_1, \dots, U_{a(G)}$, and denote the kernel of U_i by K_i . Now V is faithful, and so $\bigcap_{i=1}^{a(G)} K_i$ is trivial. Since $Q \in \frac{A}{p=Q}$, Q/K_i is (cyclic) of order q ; hence $\{a(G) : G \in \Gamma\}$ is infinite. Now $a(G)$ is the index of the inertia subgroup of U_1 in G , so that every prime dividing $a(G)$ divides n . But n is finite, and so there is a prime, say r (which for all we know at this stage could be p), such that $\{a_r(G) : G \in \Gamma\}$ is infinite, where $a_r(G)$ is the r -share of $a(G)$. If R is a Sylow r -subgroup of G and W is an irreducible submodule of V_{QR} , it follows from [1, 2.8] that if W_Q has $d(G)$ homogeneous components, then $d(G) \geq a_r(G)$. In particular, since \underline{V} is jnC, it follows from Theorem 2.3 that $\underline{V} = \text{var}\{W_{QR} : G \in \Gamma\}$, and hence $\underline{V} \leq \frac{A \cdot A}{p=Q} \text{var}\{R : G \in \Gamma\}$. Recall that since \underline{V} is not locally nilpotent, there is a bound, say c , on the nilpotency class of the nilpotent groups in \underline{V} . Hence $\text{var}\{R : G \in \Gamma\}$ is a nilpotent subvariety \underline{N} of \underline{V} of class at most c and exponent a power of r .

Since $\{d(G) : G \in \Gamma\}$ is infinite, there is a group, say G , in Γ such that $d(G) > p^c$. Lemma 2.1 now shows that r is not equal to p . Moreover, even $d(G) > 1$ implies that Q is not centralized by R , so if $q \neq r$ then QR is not nilpotent and hence neither is H . In this case, as $M(G)$ is self-centralizing, G is not abelian-by-nilpotent. Thus if $q \neq r$ then \underline{V} is not abelian-by-nilpotent. On the other hand, suppose $q = r$. Then $\underline{A}_q \underline{N}$ is locally nilpotent while \underline{V} is not, so $\underline{V} \wedge \underline{A}_q \underline{N}$ is nilpotent; and of course $\underline{V} \leq \underline{A}_p \underline{A}_q \underline{N}$ implies $\underline{V} \leq \underline{A}_p (\underline{V} \wedge \underline{A}_q \underline{N})$ so now \underline{V} is abelian-by-nilpotent. The proof of Theorem 1.7 is complete.

4. The proof of Theorem 1.8

Suppose that \underline{V} is not abelian-by-nilpotent. Observe that \underline{V} is locally finite, being soluble of finite exponent. If c denotes the class of \underline{N} , then all the nilpotent groups in \underline{V} have class at most c . It follows that \underline{V} is not locally nilpotent, and so by [1, 3.7], there is a prime, say t , and an infinite set, say Γ , of monolithic groups in \underline{V} such that:

- (i) $\underline{V} = \text{var}\Gamma$;
- (ii) the monolith of each group in Γ is complemented, self-centralising, and has exponent t ;
- (iii) $\{|M(G)| : G \in \Gamma\}$ is an infinite set.

In every group in \underline{V} , the Sylow p -subgroup is normal, and its quotient group lies in $\underline{A}_q \underline{N}$. Since $\underline{V} \wedge \underline{A}_q \underline{N}$ is Cross, it follows from Theorem 2.3 that

$$t = p .$$

Suppose $G \in \Gamma$, and denote the monolith $M(G)$ of G by V . Let H be a Hall p' -subgroup of G , and let Q be the Sylow q -subgroup and R a Sylow r -subgroup of H . Denote $M(Z(R))$ by Y . Then V can be thought of as a vector space over $F(p)$, in which case H is faithfully and irreducibly represented (by conjugation) on V . Suppose that $U_1, \dots, U_{a(G)}$ are the homogeneous components of V_Q , and let K_i be the kernel of U_i . We need to prove three preparatory lemmas.

LEMMA 4.1. *The set $T = \{|Q| : G \in \Gamma\}$ is infinite.*

Proof. By property (iii) of Γ , $\{|V| : G \in \Gamma\}$ is infinite. As V is a self-centralising chief section of G (property (ii) of Γ), it follows from [8, 52.24] that $\{|H| : G \in \Gamma\}$ is infinite. Suppose contrary to the lemma that T is finite, say $|Q| < u$ for all G in Γ , and denote $C_H(Q)$ by C . Then $\{|H : C| : G \in \Gamma\}$ is bounded by $u!$, and so by Clifford's Theorem, if W is an irreducible submodule of V_C , $\{|W| : G \in \Gamma\}$ is infinite. Using Lemma 2.2, we replace WC by $(WC)^*$; by Theorem 2.3, $\underline{V} = \text{var}\{(WC)^* : G \in \Gamma\}$. Thus we may suppose that $C = H$, for all G in Γ . In this case of course $Q \leq Z(H)$, and so Q is cyclic of order at most q . It follows that R is normal in H and $|H : R| \leq q$. Then again by Clifford's Theorem, V_R is a (homogeneous) direct sum of at most q irreducible submodules of common dimension. Thus there is no bound on the orders of the irreducible submodules of V_R , as G ranges through Γ . It follows from Theorem 2.3 that $\underline{V} = \text{var}\{VR : G \in \Gamma\}$, and hence that \underline{V} is abelian-by-nilpotent, a contradiction. //

COROLLARY 4.2. *The set $\Phi = \{|R| : G \in \Gamma\}$ is infinite.*

Proof. As Q/K_i is faithfully and irreducibly represented on an irreducible component of U_i , it is cyclic of order q . But V is faithful, and so $\cap\{K_i : 1 \leq i \leq a(G)\}$ is trivial. As T is infinite (Lemma 4.1), it follows that $\{a(G) : G \in \Gamma\}$ is infinite. By Clifford's Theorem, $a(G) \leq |H : Q| = |R|$, and so Φ is infinite. //

LEMMA 4.3. *The set $\Psi = \{|Y| : G \in \Gamma\}$ is infinite.*

Proof. By Maschke's Theorem, V_R is completely reducible; let

$$V_R = V_1 \oplus \dots \oplus V_{\ell(G)}$$

be a direct decomposition of V_R into irreducible submodules. Since

$\underline{V} \wedge \frac{\underline{A} \underline{N}}{\underline{D}}$ is Cross, it follows from Theorem 2.3 that

$\{|V_i| : 1 \leq i \leq \ell(G), G \in \Gamma\}$ is finite, say $|V_i| < m$ for all i, G .

If the kernel of V_i is R_i , then

$$\{|R : R_i| : 1 \leq i \leq \ell(G), G \in \Gamma\}$$

is bounded by $m!$. Since V is faithful, $\cap\{R_i : 1 \leq i \leq \ell(G)\}$ is trivial. Choose a subset, say $\Lambda(G)$, of $\{1, \dots, \ell(G)\}$ which is minimal with respect to $\cap\{R_i : i \in \Lambda(G)\}$ being trivial. Since Φ is an infinite set (Corollary 4.2), it follows that $\{|\Lambda(G)| : G \in \Gamma\}$ is infinite. We suppose that the components of V_R have been numbered so that $\Lambda(G) = \{1, \dots, k(G)\}$. Put $S_i = \cap\{R_j : 1 \leq j \leq k(G), j \neq i\}$, $1 \leq i \leq k(G)$. By the minimality of $\Lambda(G)$, each S_i is a nontrivial normal subgroup of R , and if $i \neq j$, $S_i \cap S_j = E$. Hence

$$|Y| \geq 1 + \sum_{i=1}^{k(G)} (|S_i \cap Y| - 1).$$

But for each i , $S_i \cap Y$ is nontrivial, so $|Y| > k(G)$. Since $k(G) = |\Lambda(G)|$, and $\{|\Lambda(G)| : G \in \Gamma\}$ is infinite, the proof is complete. //

We are now in a position to prove Theorem 1.8. Observe that QY is a normal subgroup of H which is supplemented in H by R . Thus if W is an irreducible submodule of V_{QY} , it follows from Clifford's Theorem that

$$V_{QY} = \sum\{Wx : x \in R\}.$$

Suppose that M is the kernel of W ; then the kernel of Wx is M^x . Since V is faithful, it follows that $\cap\{M^x : x \in R\}$ is trivial. But if $x \in R$,

$$M^x \cap Y = (M \cap Y)^x = M \cap Y,$$

and so $M \cap Y$ is trivial. Thus Y acts faithfully on W ; that is, Y is isomorphic to a subgroup of $\text{Aut}W$. In particular, $|W|! > |Y|$. It follows from Lemma 4.3 that $\{|W| : G \in \Gamma\}$ is an infinite set, and so by Theorem 2.3,

$$\underline{V} = \text{var}\{WQY : G \in \Gamma\}.$$

Hence \underline{V} is a subvariety of $\frac{A A A}{\underline{pqr}}$.

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