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DEGREE OF THE W-OPERATOR AND NONCROSSING PARTITIONS

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Abstract

The W-operator, $W([n])$, generalises the cut-and-join operator. We prove that $W([n])$ can be written as the sum of $n!$ terms, each term corresponding uniquely to a permutation in S_n. We also prove that there is a correspondence between the terms of $W([n])$ with maximal degree and noncrossing partitions.

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1. Introduction

The cut-and-join operator Δ,

$$\Delta = \frac{1}{2}\sum_{i,j\geq 1}\Big((i+j)p_ip_j\frac{\partial}{\partial p_{i+j}} + ijp_{i+j}\frac{\partial^2}{\partial p_i\partial p_j}\Big),$$

introduced by Goulden and Jackson [1, 2], is an infinite sum of differential operators in variables p_i, $i \geq 1$. It plays an important role in calculating the simple Hurwitz number [2–4, 6] and in many other enumerative geometry problems [7, 8, 12].

Mironov, Morosov and Natanzon [9, 10] constructed the W-operators $W([n])$, where $[n] = (1^{i_1}2^{i_2}\dots n^{i_n})$ is a partition of a positive integer n. The W-operators are differential operators acting on the formal power series $\mathbb{C}[[X_{ij}]]_{i,j\geq 1}$, where the X_{ij} are coordinate functions on the positive-half-infinite matrix. A subring of $\mathbb{C}[[X_{ij}]]_{i,j\geq 1}$ is $\mathbb{C}[p_1, p_2, \dots]$, where $p_k = \mathrm{Tr}(X^k)$ and $X = (X_{ij})_{i,j\geq 1}$. A direct calculation shows that $W([2])$ is the cut-and-join operator Δ on the ring $\mathbb{C}[p_1, p_2, \dots]$. We study the structure of the operators $W([n])$, $n \geq 1$, as operators on the ring $\mathbb{C}[p_1, p_2, \dots]$.

In Section 2, we review the natural quiver structure of permutations and show how all permutations in S_{n+1} can be constructed from permutations in S_n. The key element is Construction 2.2. In Section 3, we review the properties of $W([n])$ and prove the following theorem about the structure of $W([n])$.

Theorem 1.1. *The W-operator, $W([n])$, is a well-defined operator on $\mathbb{C}[p_1, p_2, \ldots]$. It can be written as the sum of $n!$ summands FS_α, each of which corresponds to a unique quiver $\hat{Q}_\alpha$ or, equivalently, a unique permutation $\alpha \in S_n$.*

In Section 4, we define the degree of each summand (term) of $W([n])$ and show that the maximal degree is $n + 1$. The summands with maximal degree can be used to study certain Hurwitz numbers (see [5]). In Section 5, we count the number of summands of $W([n])$ with maximal degree by showing that there is a one-to-one correspondence between the summands with maximal degree and the noncrossing permutations of $[n] = \{1, \ldots, n\}$ (Theorem 5.4).

2. Permutation group and quivers

We first review the quiver structure of permutations. Then, we give a new inductive construction of the permutations in S_n by constructing n distinct permutations in S_n from a given permutation in S_{n-1}. The idea of this construction comes from the structure of the W-operator (Remark 3.6).

A *quiver* is a directed graph. A *quiver* $Q = (V, A, s, t)$ is a quadruple, where V is the set of vertices, A is the set of arrows and s and t are two maps $A \to V$. For $a \in A$, $s(a)$ is the source of this arrow and $t(a)$ is the target. We assume that V and A are finite sets. If B is a subset of A and $V_B = \{s(a), t(a) : a \in B\}$, we call (V_B, B, s', t') the *subquiver* of Q, where $s' = s|_B$, $t' = t|_B$. A quiver $Q = (V, A, s, t)$ is *connected* if the underlying undirected graph of Q is connected. A connected quiver $Q = (V, A, s, t)$ is a *loop* if, for any vertex $v \in V$, there is a unique arrow $a \in A$ such that $s(a) = v$ and a unique arrow $b \in A$ such that $t(b) = v$. A *chain* is obtained by omitting a single arrow in a loop.

Any permutation $\alpha \in S_n$ has a natural quiver structure $Q_\alpha = (V_\alpha, A_\alpha, s, t)$, where $V_\alpha = \{1, \ldots, n\}$ and $A_\alpha = \{i \to \alpha(i) : 1 \leq i \leq n\}$. The quiver Q_α only contains loops. Since we want to use induction, we construct another quiver $\hat{Q}_\alpha$ from Q_α.

Construction 2.1. Given $\alpha \in S_n$, let Q_α be the corresponding quiver. There is a unique arrow a in Q_α such that $s(a) = 1$. We substitute this arrow by a new one $\hat{a}$, where $s(\hat{a}) = n + 1$ and $t(\hat{a}) = t(a)$. Denote the new set of arrows by $\hat{A}_\alpha$, the new vertex set by $\hat{V}_\alpha = \{1, \ldots, n, n + 1\}$ and the new quiver by $\hat{Q}_\alpha = (\hat{V}_\alpha, \hat{A}_\alpha, s, t)$.

For example, if $\alpha = (321) \in S_3$, then Q_α and $\hat{Q}_\alpha$ are

$$\begin{tikzcd} Q_\alpha : & 3 \arrow[r] & 2 \arrow[r] & 1 \arrow[ll, bend right] & \text{and} & \hat{Q}_\alpha : & 4 \arrow[r] & 3 \arrow[r] & 2 \arrow[r] & 1. \end{tikzcd}$$

Here, Q_α is a loop, while $\hat{Q}_\alpha$ is a chain. In general, $\hat{Q}_\alpha$ consists of one chain and possibly a number of loops. The chain in $\hat{Q}_\alpha$ always starts from $n + 1$ and stops at 1. Since we can construct Q_α uniquely from $\hat{Q}_\alpha$, we have a one-to-one correspondence between permutations α and quivers $\hat{Q}_\alpha$.

Let $\alpha \in S_n$. From the quiver $\hat{Q}_\alpha$, we construct $n+1$ quivers corresponding to permutations in S_{n+1}. To construct a new quiver $\hat{Q}_\beta$ representing an element $\beta \in S_{n+1}$, we add one more vertex $n+2$ into $\hat{V}_\alpha$ and add arrows a_1, a_2 in $\hat{A}_\alpha$ such that

$$s(a_1) = n+2, \quad t(a_2) = n+1,$$

where a_1, a_2 can be the same arrow. Here is the construction.

Construction 2.2. Given $\alpha \in S_n$, write $\alpha = \alpha_1\alpha_2 \ldots \alpha_k$ as the product of disjoint cycles. We assume that $1 \in \alpha_1$. The corresponding subquiver for α_1 in $\hat{Q}_\alpha$ is the chain

$$\hat{Q}_{\alpha_1}: \quad n+1 \longrightarrow \cdots \longrightarrow 1.$$

Case 0. We extend the quiver for α_1 directly to

$$\hat{Q}_{\beta_1}: \quad n+2 \longrightarrow n+1 \longrightarrow \cdots \longrightarrow 1.$$

This subquiver represents a well-defined cycle β_1 (by replacing $n+2$ by 1), leading to a permutation $\beta \in S_{n+1}$, where $\beta = \beta_1\alpha_2 \ldots \alpha_k$. In this case, a_1, a_2 are the same arrow

$$a_1 = a_2: \quad n+2 \longrightarrow n+1.$$

Next, we consider the general case. Choose an arbitrary arrow $a : i \to j$ in $\hat{Q}_\alpha$. The idea is to cut this arrow and reconnect the chain and loops in $\hat{Q}_\alpha$. Since there are n choices of the arrow in $\hat{Q}_\alpha$, we can construct n permutations.

Case 1: Cut case, $a \in \hat{Q}_{\alpha_1}$. In this case, $\hat{Q}_{\alpha_1}$ is

$$\hat{Q}_{\alpha_1}: \quad n+1 \longrightarrow \cdots \longrightarrow i \longrightarrow j \longrightarrow \cdots \longrightarrow 1.$$

First, cut the arrow $i \to j$, giving

$$n+1 \longrightarrow \cdots \longrightarrow i, \qquad j \longrightarrow \cdots \longrightarrow 1.$$

Then, add the two arrows

$$a_1 : n+2 \longrightarrow j \qquad a_2 : i \longrightarrow n+1$$

to get the quivers

$$\begin{tikzcd} \hat{Q}_{\beta_1\beta_2}: n+2 \arrow[r] & j \arrow[r] & \cdots \arrow[r] & 1, & i \arrow[r] & n+1 \arrow[r] & \cdots \arrow[r] & {} \arrow[llll, bend right] & . \end{tikzcd}$$

They represent two disjoint cycles in S_{n+1} by replacing $n+2$ by 1. Let β_1 and β_2 be the permutations corresponding to these two quivers, where $1 \in \beta_1$, and let $\beta = \beta_1\beta_2\alpha_2 \ldots \alpha_k$ be the permutation in S_{n+1} obtained by cutting the arrow a.

Case 2: Join case, $a \notin \hat{Q}_{\alpha_1}$. Without loss of generality, assume that $a \in \hat{Q}_{\alpha_2}$. The corresponding quivers for α_1 and α_2 are

$$\begin{tikzcd} \hat{Q}_{\alpha_1\alpha_2}: & n+1 \arrow[r] & \cdots \arrow[r] & 1, & i \arrow[r] & j \arrow[r] & \cdots \arrow[r] & {} \arrow[llll, bend right] \end{tikzcd}.$$

As in Case 1, we cut the arrow $i \to j$ to get

$$n+1 \longrightarrow \cdots \longrightarrow 1, \qquad j \longrightarrow \cdots \longrightarrow i.$$

With the same process as in Case 1, we add the two arrows

$$a_1 : n+2 \longrightarrow j \qquad a_2 : i \longrightarrow n+1,$$

giving the chain

$$\hat{Q}_{\beta_1}: \quad n+2 \longrightarrow j \longrightarrow \cdots \longrightarrow i \longrightarrow n+1 \longrightarrow \cdots \longrightarrow 1.$$

The quiver $\hat{Q}_{\beta_1}$ represents a cycle in S_{n+1} by replacing $n+2$ by 1. Let β_1 be the corresponding permutation of $\hat{Q}_{\beta_1}$ and let $\beta = \beta_1\alpha_3 \ldots \alpha_k$ be the permutation in S_{n+1} constructed in this case.

The following theorem is a direct result from Construction 2.2.

THEOREM 2.3. *For any $\alpha \in S_n$, Construction 2.2 gives $n+1$ distinct permutations in S_{n+1}. Applying the construction to all $\alpha \in S_n$ gives all $(n+1)!$ permutations of S_{n+1}.*

DEFINITION 2.4. Let α be a permutation in S_n and let j be an integer such that $0 \le j \le n$. Denote by $[\alpha, j]$ the permutation in S_{n+1} obtained from α in Construction 2.2 as follows:

(1) $[\alpha, 0]$ corresponds to Case 0; and
(2) $[\alpha, j]$ for $j > 0$ corresponds to Case 1 and Case 2 by cutting arrow a with $t(a) = j$.

3. *W*-operator

First, we review the properties of the W-operator $W([n])$ (details can be found in [9, 10]). Then, we calculate $W([n+1])$ from $W([n])$ and relate the structure of the W-operator $W([n])$ to the permutation group S_n and the quivers in Section 2.

Let $X := (X_{ab})_{a\ge1,b\ge1}$ be an infinite matrix. Given a positive integer k, let

$$p_k = \sum_{a_1,\ldots,a_k\ge1} X_{a_1a_k}X_{a_ka_{k-1}}\ldots X_{a_2a_1}$$

denote the trace of X^k. Clearly, p_k is a formal power series in $\mathbb{C}[[X_{ab}]]_{a,b\ge1}$.

The operator matrix $D = (D_{ab})_{a\ge1,b\ge1}$ is the infinite matrix whose (a,b)-entry is

$$D_{ab} = \sum_{c=1}^{\infty} X_{ac}\frac{\partial}{\partial X_{bc}}.$$

In the rest of the paper, we prefer to write $D_{ab} = X_{ac}\partial/\partial X_{bc}$ with the sum over c implied. As differential operators, the normal ordered product of D_{ab} and D_{cd} is

$$: D_{ab}D_{cd} := X_{ae_1}X_{ce_2}\frac{\partial}{\partial X_{be_1}}\frac{\partial}{\partial X_{de_2}},$$

meaning that we always calculate the differentiation first. The normal ordered product $: D_{a_{n+1}a_n}D_{a_na_{n-1}}\dots D_{a_2a_1} :$ is defined similarly.

Definition 3.1. For any positive integer n, the W-operator $W([n])$ is defined by

$$W([n]) := \frac{1}{n} : tr(D^n) := \frac{1}{n}\sum_{a_1,\dots,a_n\geq 1} : D_{a_1a_n}D_{a_na_{n-1}}\dots D_{a_2a_1} : .$$

Next, we review some important formulas.

Lemma 3.2 [9]. *Let $F(p)$ be an element in $\mathbb{C}[p_1, p_2, \dots]$. Then*

$$D_{ab}F(p) = \sum_{k=1}^{\infty} k(X^k)_{ab}\frac{\partial F(p)}{\partial p_k}.$$

Lemma 3.3 [9]. *We have*

$$D_{cd}(X^k)_{ab} = \sum_{j=0}^{k-1}(X^j)_{ad}(X^{k-j})_{cb}.$$

In particular, by setting $a = a_i, b = a_j, c = a_{n+1}, d = a_n$, Lemma 3.3 gives

$$\sum_{k_j=1}^{\infty} D_{a_{n+1}a_n}(X^{k_j})_{a_ia_j} = \sum_{k_j=1}^{\infty}\sum_{k_n=0}^{k_j-1}(X^{k_n})_{a_ia_n}(X^{k_j-k_n})_{a_{n+1}a_j} = \sum_{k_j=1}^{\infty}\sum_{k_n=1}^{\infty}(X^{k_n})_{a_ia_n}(X^{k_j})_{a_{n+1}a_j}.$$

Lemma 3.4 [9, 10]. *We have*

$$\begin{aligned} D_{a_{n+2}a_{n+1}}D_{a_ia_j} &= \sum_{k\geq 1, j\geq 0}\bigg((k+j)(X^j)_{a_ia_{n+1}}(X^k)_{a_{n+2}a_j}\frac{\partial}{\partial p_{k+j}}\bigg) \\ &\quad + \sum_{k,j\geq 1}\bigg(kj(X^k)_{a_{n+2}a_{n+1}}(X^j)_{a_ia_j}\frac{\partial^2}{\partial p_k\partial p_j}\bigg), \\ : D_{a_{n+2}a_{n+1}}D_{a_ia_j} : &= \sum_{k,j\geq 1}\bigg((k+j)(X^j)_{a_ia_{n+1}}(X^k)_{a_{n+2}a_j}\frac{\partial}{\partial p_{k+j}}\bigg) \\ &\quad + \sum_{k,j\geq 1}\bigg(kj(X^k)_{a_{n+2}a_{n+1}}(X^j)_{a_ia_j}\frac{\partial^2}{\partial p_k\partial p_j}\bigg). \end{aligned}$$

Remark 3.5. The calculation of the differential operator $D_{a_{n+2}a_{n+1}}D_{a_ia_j}$ in Lemma 3.4 is an application of the chain rule; the first line comes from the action on the polynomial part (Lemma 3.3) and the second line comes from the action on the differential part (Lemma 3.2).

Notice that the only difference between the two formulas in Lemma 3.4 is that the subscript j in the first sum starts from 0 in the first formula and from 1 in the second. This arises because the normal ordered products $:D_{a_{n+2}a_{n+1}}D_{a_ia_j}:$ and $D_{a_{n+2}a_{n+1}}D_{a_ia_j}$ differ by one term: that is,

$$D_{a_{n+2}a_{n+1}}D_{a_{n+1}a_{n-1}} =: D_{a_{n+2}a_{n+1}}D_{a_{n+1}a_{n-1}} : +X_{a_{n+2}e_1}\left[\frac{\partial}{\partial X_{a_{n+1}e_1}}, X_{a_{n+1}e_2}\right]\frac{\partial}{\partial X_{a_ne_2}}.$$

The same approach can be used to calculate $:D_{a_{n+2}a_{n+1}}\dots D_{a_2a_1}:$ from the product $D_{a_{n+2}a_{n+1}}\dots D_{a_2a_1}$. In the formula for the normal ordered product $:D_{a_{n+2}a_{n+1}}\dots D_{a_2a_1}:$, the sum always goes from one to infinity, while some subscripts start from zero in the formula for $D_{a_{n+2}a_{n+1}}\dots D_{a_2a_1}$.

REMARK 3.6. We now explain the connection with Construction 2.2. Fix a permutation $\alpha \in S_n$ and positive integers k, k_j and a_j $(1 \le j \le n)$. Let $\hat{Q}_\alpha = (\hat{V}_\alpha, \hat{A}_\alpha)$ be the quiver from Construction 2.1. We consider a special differential operator

$$\prod_{b\in\hat{A}_\alpha}(X^{k_j})_{a_{s(b)}a_{t(b)}}\frac{\partial}{\partial p_k},$$

where the polynomial part $\prod_{b\in\hat{A}_\alpha}(X^{k_j})_{a_{s(b)}a_{t(b)}}$ corresponds to the quiver $\hat{Q}_\alpha$. We calculate $D_{a_{n+2}a_{n+1}}(\prod^n_{b\in\hat{A}_\alpha}(X^{k_j})_{a_{s(b)}a_{t(b)}}\partial/\partial p_k)$ by the chain rule.

(1) When $D_{a_{n+2}a_{n+1}}$ acts on the differential part, we use the formula in Lemma 3.2. In the language of quivers, we add one more arrow $a_{n+2} \to a_{n+1}$ to the quiver $\hat{Q}_\alpha$, which corresponds to *Case 0* in Construction 2.2.

(2) When $D_{a_{n+2}a_{n+1}}$ acts on the polynomial part, without loss of generality, we assume that it acts on $(X^{k_j})_{a_ia_j}$ and use Lemma 3.3. In the language of quivers, we cut the arrow $i \to j$ and add two arrows $i \to n$ and $n+1 \to j$, which corresponds to *Case 1* and *Case 2* in Construction 2.2.

Now we are ready to prove Theorem 1.1 and calculate the W-operator $W([n])$ by induction. We restate the theorem here for convenience.

THEOREM 3.7. *$W([n])$ is a well-defined operator on $\mathbb{C}[p_1, p_2, \dots]$. It can be written as the sum of $n!$ summands FS_α, each of which corresponds to a unique quiver $\hat{Q}_\alpha$ or, equivalently, a unique permutation $\alpha \in S_n$.*

PROOF. To calculate $W([n])$, we need the formula for $:D_{a_1a_n}D_{a_na_{n-1}}\dots D_{a_2a_1}:$ for any positive integers a_i $(1 \le i \le n)$. By Lemma 3.4 and Remark 3.5, it is equivalent to calculating the product $D_{a_1a_n}D_{a_na_{n-1}}\dots D_{a_2a_1}$. To facilitate the induction, we replace $D_{a_1a_n}$ by $D_{a_{n+1}a_n}$.

For the base step, $n = 1$, by Lemma 3.2,

$$D_{a_2a_1} = \sum_{k_1=1}^{\infty} k_1(X^{k_1})_{a_2a_1}\frac{\partial}{\partial p_{k_1}}.$$

We associate this summand to the quiver

$$\hat{Q}_{(1)}: \quad 2 \longrightarrow 1,$$

which corresponds to the subscript of $(X^{k_1})_{a_2a_1}$. Note that there is only one summand. Thus we define

$$FS'_{(1)} = \sum_{k_1=1}^{\infty} k_1 (X^{k_1})_{a_2a_1} \frac{\partial}{\partial p_{k_1}}.$$

Replacing a_2 by a_1 and taking the sum over a_1,

$$W([1]) = \underbrace{\sum_{k_1 \geq 1} k_1 p_{k_1} \frac{\partial}{\partial p_{k_1}}}_{FS_{(1)}}.$$

Denote by $FS_{(1)}$ the summand in $W([1])$ corresponding to $FS'_{(1)} = D_{a_2a_1}$.

When $n = 2$, we have to calculate $D_{a_3a_2}D_{a_2a_1}$.

$$D_{a_3a_2}D_{a_2a_1} = \sum_{k_1=1}^{\infty} (D_{a_3a_2}(k_1(X^{k_1})_{a_2a_1}))\frac{\partial}{\partial p_{k_1}} + \sum_{k_1=1}^{\infty} k_1 (X^{k_1})_{a_2a_1}\Big(D_{a_3a_2} \circ \frac{\partial}{\partial p_{k_1}}\Big).$$

By Lemma 3.4,

$$\begin{aligned} D_{a_3a_2}D_{a_2a_1} = & \sum_{k_1 \geq 1, k_2 \geq 0} \Big((k_1 + k_2)(X^{k_2})_{a_2a_2}(X^{k_1})_{a_3a_1} \frac{\partial}{\partial p_{k_1+k_2}}\Big) \\ & + \sum_{k_1, k_2 \geq 1} \Big(k_1 k_2 (X^{k_2})_{a_3a_2}(X^{k_1})_{a_2a_1} \frac{\partial^2}{\partial p_{k_1} \partial p_{k_2}}\Big). \end{aligned}$$

We associate the first summand with the quiver $\hat{Q}_{(1)(2)}$

$$\hat{Q}_{(1)(2)}: \begin{tikzcd} 2 \arrow[loop right] \end{tikzcd}, \qquad 3 \longrightarrow 1,$$

which comes from the subscripts of the polynomial part $(X^{k_2})_{a_2a_2}(X^{k_1})_{a_3a_1}$. Similarly, the second summand corresponds to the quiver $\hat{Q}_{(12)}$

$$\hat{Q}_{(12)}: \quad 3 \longrightarrow 2 \longrightarrow 1.$$

Now $D_{a_3a_2}$ acting on $(X^{k_1})_{a_2a_1}$ gives the first summand, which corresponds to *Case 1* cutting the arrow $2 \to 1$ in $\hat{Q}_{(1)}$ in Construction 2.2. The same argument holds for the second summand, where $D_{a_3a_2}$ acts on $\partial/\partial p_{k_1}$, and this action corresponds to *Case 0* in Construction 2.2.

By Lemma 3.4 and Remark 3.5, $: D_{a_3a_2}D_{a_2a_1} :$ and $D_{a_3a_2}D_{a_2a_1}$ only differ by the term with subscript $j = 0$ in the first summand. Therefore we can use quivers to describe the summands of $: D_{a_3a_2}D_{a_2a_1} :$ in the same way as $D_{a_3a_2}D_{a_2a_1}$. Using the notation FS'_{α} for the summand corresponding to $\alpha \in S_2$,

$$: D_{a_3a_2}D_{a_2a_1} := \sum_{\alpha \in S_2} FS'_{\alpha}.$$

In conclusion, $:D_{a_3a_2}D_{a_2a_1}:$ is the sum of two summands, which correspond to the quivers $\hat{Q}_\alpha$, $\alpha \in S_2$, that is,

$$:D_{a_3a_2}D_{a_2a_1}: = \underbrace{\sum_{k_1,k_2\geq 1}\Big((k_1+k_2)(X^{k_2})_{a_2a_2}(X^{k_1})_{a_3a_1}\frac{\partial}{\partial p_{k_1+k_2}}\Big)}_{FS'_{(1)(2)}} + \underbrace{\sum_{k_1,k_2\geq 1}\Big(k_1k_2(X^{k_2})_{a_3a_2}(X^{k_1})_{a_2a_1}\frac{\partial^2}{\partial p_{k_1}\partial p_{k_2}}\Big)}_{FS'_{(12)}}.$$

Replacing a_3 by a_1 and taking the sum over a_1, a_2,

$$W([2]) = \underbrace{\frac{1}{2}\sum_{k_1,k_2\geq 1}(k_1+k_2)p_{k_1}p_{k_2}\frac{\partial}{\partial p_{k_1+k_2}}}_{FS_{(1)(2)}} + \underbrace{\frac{1}{2}\sum_{k_1,k_2\geq 1}k_1k_2p_{k_1+k_2}\frac{\partial^2}{\partial p_{k_1}\partial p_{k_2}}}_{FS_{(12)}}.$$

We define FS_α to be summand of $W([2])$ which corresponds to FS'_α in the formula for $:D_{a_3a_2}D_{a_2a_1}:$.

Similarly, when $n = 3$, we can calculate $:D_{a_4a_3}D_{a_3a_2}D_{a_2a_1}:$ from the product $D_{a_4a_3}D_{a_3a_2}D_{a_2a_1}$. Consider the operator $D_{a_4a_3}$ acting on $D_{a_3a_2}D_{a_2a_1}$. Since the polynomial part of each summand is a product of two terms (the corresponding quiver has two arrows), we get three new summands by the chain rule: two come from the polynomial part and one from the differential part. By Theorem 2.3 and Remark 3.6, each of the new summands corresponds to a unique permutation in S_3. Thus $:D_{a_4a_3}D_{a_3a_2}D_{a_2a_1}:$ can be written as the sum of summands FS'_α labelled by permutations α in S_3. Replacing a_4 by a_1 and taking the sum over a_1, a_2, a_3, we get the formula for $W([3])$. We define FS_α to be the summand of $W([3])$, which corresponds to the summand FS'_α. In this way, the operator $W([n])$ can be written as the sum of $n!$ summands by induction, and each summand corresponds to a unique permutation in S_n.

We give two examples to show that $W([n])$ is a well-defined operator on $\mathbb{C}[p_1, p_2, \dots]$. When $n = 1$, consider $D_{a_2a_1}$. Let $a_2 = a_1$. Taking the sum over a_1,

$$W([1]) = \sum_{k_1} k_1p_{k_1}\frac{\partial}{\partial p_{k_1}}.$$

Now consider $:D_{a_3a_2}D_{a_2a_1}:$. Let $a_3 = a_1$. Taking the sum over a_1, a_2,

$$\begin{aligned} W([2]) &= \frac{1}{2}\sum_{a_1,a_2\geq 1} :D_{a_1a_2}D_{a_2a_1}: \\ &= \frac{1}{2}\sum_{k_1,k_2\geq 1}\Big((k_1+k_2)p_{k_1}p_{k_2}\frac{\partial}{\partial p_{k_1+k_2}} + k_1k_2p_{k_1+k_2}\frac{\partial^2}{\partial p_{k_1}\partial p_{k_2}}\Big). \end{aligned}$$

Clearly, $W([1])$ and $W([2])$ are well-defined operators on $\mathbb{C}[p_1, p_2, \dots]$. The operator $W([n])$ can be proved to be a well-defined operator on $\mathbb{C}[p_1, p_2, \dots]$ by induction. □

EXAMPLE 3.8. We write $W([3])$ as a sum of 3! summands, each corresponding to a unique permutation in S_3.

$$
\begin{aligned}
W([3]) = \frac{1}{3} \sum_{k_1,k_2,k_3 \geq 1} \biggl(& k_1 k_2 k_3 p_{k_1+k_2+k_3} \frac{\partial^3}{\partial p_{k_1} \partial p_{k_2} \partial p_{k_3}} && (321)\\
& + k_1(k_2 + k_3) p_{k_1+k_3} p_{k_2} \frac{\partial^2}{\partial p_{k_1} \partial p_{k_2+k_3}} && (13)(2)\\
& + k_2(k_1 + k_3) p_{k_1+k_2} p_{k_3} \frac{\partial^2}{\partial p_{k_2} \partial p_{k_1+k_3}} && (12)(3)\\
& + k_3(k_1 + k_2) p_{k_3+k_2} p_{k_1} \frac{\partial^2}{\partial p_{k_3} \partial p_{k_1+k_2}} && (1)(23)\\
& + (k_1 + k_2 + k_3) p_{k_1} p_{k_2} p_{k_3} \frac{\partial}{\partial p_{k_1+k_2+k_3}} && (1)(2)(3)\\
& + (k_1 + k_2 + k_3) p_{k_1+k_2+k_3} \frac{\partial}{\partial p_{k_1+k_2+k_3}} \biggr) && (123).
\end{aligned}
$$

4. Degree of the summand FS_α

DEFINITION 4.1. Given any summand FS_α of $W([n])$, define $dP(FS_\alpha)$ to be the degree of its polynomial part and $dD(FS_\alpha)$ to be the order of its differential part. The degree $d(FS_\alpha)$ of the summand FS_α is $d(FS_\alpha) = dP(FS_\alpha) + dD(FS_\alpha)$.

We give two easy examples to explain this definition. Consider the summand

$$FS_{(1)(2)} = \frac{1}{2} \sum_{k_1,k_2 \geq 1} (k_1 + k_2) p_{k_1} p_{k_2} \frac{\partial}{\partial p_{k_1+k_2}}.$$

Then

$$dP(FS_\alpha) = 2, \quad dD(FS_\alpha) = 1, \quad d(FS_\alpha) = 3.$$

Similarly, the degree data of $FS_{(12)}$ are

$$dP(FS_\alpha) = 1, \quad dD(FS_\alpha) = 2, \quad d(FS_\alpha) = 3.$$

The following lemma describes the relationship between the degrees of FS_β and FS_α, where $\beta = [\alpha, j]$ (see Definition 2.4).

LEMMA 4.2. *Let* $\alpha \in S_n$.

(1) *If* $[\beta] = [\alpha, 0]$ *(Case 0), then*

$$dP(FS_\beta) = dP(FS_\alpha), \quad dD(FS_\beta) = dD(FS_\alpha) + 1, \quad d(FS_\beta) = d(FS_\alpha) + 1.$$

(2) *If* $[\beta] = [\alpha, j]$ *and* j *is a vertex in the chain of* $\hat{Q}_\alpha$ *(Case 1), then*

$$dP(FS_\beta) = dP(FS_\alpha) + 1, \quad dD(FS_\beta) = dD(FS_\alpha), \quad d(FS_\beta) = d(FS_\alpha) + 1.$$

(3) *If* $[\beta] = [\alpha, j]$ *and* j *is not a vertex in the chain of* $\hat{Q}_\alpha$ *(Case 2), then*

$$dP(FS_\beta) = dP(FS_\alpha) - 1, \quad dD(FS_\beta) = dD(FS_\alpha), \quad d(FS_\beta) = d(FS_\alpha) - 1.$$

Proof. Notice that $dP(FS_\alpha)$ is exactly the number of disjoint cycles of α.

When $j = 0$, the differential degree of FS'_β increases by one by Lemma 3.2. The disjoint cycle of $\beta = [\alpha, 0]$ is the same as α, so $dP(FS_\beta) = dP(FS_\alpha)$.

When $j \geq 1$, Lemma 3.3 and Remark 3.6 imply that the operator $D_{a_{n+n}a_{n+1}}$ fixes the differential degree. Now we consider the polynomial degree. If j is in the chain of $\hat{Q}_\alpha$, *Case 1* in Construction 2.2 shows that β has one more disjoint cycle than α. When j is not in the chain of $\hat{Q}_\alpha$, β corresponds to *Case* 2 and $dP(FS_\beta) = dP(FS_\alpha) - 1$.

This proves the lemma. □

From Lemma 4.2, the maximal degree of summands in $W([n])$ is $n + 1$ and the other possible degrees are $n - 1, n - 3, \ldots$.

Definition 4.3 (Ordinary summand). Let α be a permutation in S_n. We say that FS_α is an *ordinary summand* (OS) if $d(FS_\alpha) = n + 1$. An ordinary summand FS_α is of *type* $(r, n - r + 1)$ if $dP(FS_\alpha) = r$ and $dD(FS_\alpha) = n - r + 1$.

5. Noncrossing permutations

In this section, we prove that the permutation α is a noncrossing permutation if and only if FS_α is of maximal degree. As a corollary of this correspondence, we show that the number of ordinary summands with maximal degree is the Catalan number.

Noncrossing permutations come from noncrossing partitions with respect to a fixed order of objects. A partition of $[n] = \{1, \ldots, n\}$ is *noncrossing* if whenever four elements, $1 \leq a < b < c < d \leq n$, are such that a, c are in the same block and b, d are in the same block, then the two blocks coincide. With respect to the natural order of integers, each noncrossing partition corresponds to a unique permutation, where each block corresponds to a disjoint cycle and the order $i > j$ implies an arrow $i \to j$ in the disjoint cycle.

Definition 5.1 (Noncrossing permutation). Let α be a permutation in S_n and suppose that $\alpha = \alpha_1 \ldots \alpha_r$ is its decomposition into disjoint cycles. The permutation α is a *noncrossing permutation* if it satisfies the following conditions.

$(*_1)$ For each arrow a in the unique chain of $\hat{Q}_\alpha$, we have $t(a) < s(a)$ and there is only one arrow b in each loop of $\hat{Q}_\alpha$ such that $s(b) < t(b)$.

$(*_2)$ Any two distinct cycles α_i and α_j, satisfy at least one of the following conditions:

(a) for any m in α_i, either $m > n$ for any n in α_j or $m < n$ for any n in α_j;
(b) for any m in α_j, either $m > n$ for any n in α_i or $m < n$ for any n in α_i.

Condition $(*_1)$ means that we have an order on the finite set which determines the permutation. The definition of the order depends on the order of the elements in the set $\{1, \ldots, n\}$ and we choose the standard order for positive integers. Condition

$(*_2)$ corresponds to the *noncrossing* condition. If α_i and α_j satisfy only one of the conditions $(a), (b)$, then one is contained in the other. If α_i and α_j satisfy both conditions, they are disjoint. For instance, consider the permutations

$$\tau_1 = (123)(45), \quad \tau_2 = (125)(34), \quad \tau_3 = (124)(35).$$

The two disjoint cycles in τ_1 satisfy both conditions, while the cycles (125) and (34) in the permutation τ_2 satisfy only one of the conditions $(a), (b)$.

Note that there is a one-to-one correspondence between noncrossing partitions and noncrossing permutations [11]. Therefore, although we are working with noncrossing permutations, everything can be considered in terms of noncrossing partitions.

Next, we show that FS_α is an ordinary summand if and only if α is a noncrossing permutation.

Lemma 5.2. *Given $\alpha \in S_n$, if FS_α is an OS, then α satisfies condition* $(*_1)$.

Proof. We prove this lemma by induction on n. For the base step $n = 1$, $\hat{Q}_{(1)}$ is the only quiver and $FS_{(1)}$ is an OS. There is only one arrow $2 \to 1$ in the quiver $\hat{Q}_{(1)}$. Clearly, (1) satisfies condition $(*_1)$.

By induction, we assume that, for all $\alpha \in S_{k-1}$, if FS_α is an OS, then α satisfies $(*_1)$. Let $\beta = [\alpha, j]$ be a permutation in S_k such that FS_β is an OS. This implies that FS_α is also an OS. Indeed, if FS_α is not an OS, then $d(FS_\alpha) < k$. By Lemma 4.2, $d(FS_\beta) < k + 1$, which violates the fact that β is an OS.

Let $\alpha = \alpha_1 \dots \alpha_r$ be the decomposition of α into disjoint cycles with $1 \in \alpha_1$. By Lemma 4.2, the integer j is either zero or the target of some arrow in the chain of $\hat{Q}_\alpha$.

Case A. If $j = 0$, then $\beta = \beta_1\alpha_2 \dots \alpha_r$, where $\hat{Q}_{\beta_1}$ is constructed from $\hat{Q}_{\alpha_1}$ by adding another arrow $k + 1 \to k$. By induction, the statement is true for β.

Case B. If $j \neq 1$, then β is constructed from α by cutting the arrow $a : i \to j$, which is an arrow a in the chain of $\hat{Q}_\alpha$. We use the notation of *Case 1* in Construction 2.2. Let $\beta = \beta_1\beta_2\alpha_2 \dots \alpha_r$. The quiver $\hat{Q}_{\beta_1}$ of the cycle β_1 is

$$\begin{tikzcd} \hat{Q}_{\beta_1}: & k+2 \arrow[r] & j \arrow[r] & \cdots \arrow[r] & 1, \end{tikzcd}$$

where $j \to \cdots \to 1$ is a subquiver of α_1. For all arrows in this chain, the source is larger than the target. The quiver $\hat{Q}_{\beta_2}$ is

$$\begin{tikzcd} \hat{Q}_{\beta_2}: & i \arrow[r] & k+1 \arrow[r] & \cdots \arrow[r] & {} \arrow[lll, bend right] , \end{tikzcd}$$

where $k + 1 \to \cdots \to i$ is a subquiver of α_1 by construction. So the only arrow a in the cycle $\hat{Q}_{\beta_2}$ satisfying $s(a) < t(a)$ is $i \to k + 1$. Hence, the statement is true for $n = k$. □

Lemma 5.3. *Given $\alpha \in S_n$, if FS_α is an OS, then α satisfies condition* $(*_2)$.

Proof. We prove the lemma by induction on n. When $n = 1$, it is clear that the unique permutation (1) in S_1 satisfies condition $(*_2)$.

By induction, we assume that, for all $\alpha \in S_{k-1}$, if FS_α is an OS, then α satisfies $(*_2)$. Let $\beta = [\alpha, j]$ be a permutation in S_k such that FS_β is an OS. Let $\alpha = \alpha_1 \ldots \alpha_r$ be the decomposition of α into disjoint cycles. We will prove that if FS_β is an OS, then β satisfies condition $(*_2)$. Note that if $[\beta] = [\alpha, j]$ and FS_β is an OS, then FS_α is also an OS. This property comes from the proof of Lemma 5.2.

If $j = 0$, then $\beta = \beta_1\alpha_2 \ldots \alpha_r$, where $\hat{Q}_{\beta_1}$ is constructed from $\hat{Q}_{\alpha_1}$ by adding another arrow $k + 1 \rightarrow k$. In other words, we put another element k into the cycle α_1 (see Construction 2.2). By assumption, any two disjoint cycles of $\alpha \in S_{k-1}$ satisfy at least one of the conditions, so we only have to check whether the pair (β_1, α_i) satisfies condition $(*_2)$ for $2 \leq i \leq r$. Since α_1 contains the smallest element 1, if α_1 and α_i are disjoint, then any element in α_1 is smaller than any element in α_i. Since k is the largest element, the statement is true for β_1 and α_i. Next, suppose that α_1 and α_i are not disjoint. Since 1 is contained in α_1, it follows that α_i is contained in α_1. Clearly, this still holds for β_1 and α_i. So (β_1, α_i) satisfies condition $(*_2)$.

Now suppose that β is constructed from α by cutting the arrow $a : i \rightarrow j$ lying in the chain of α. We use the notation of *Case 1* in Construction 2.2. Let $\beta = \beta_1\beta_2\alpha_2 \ldots \alpha_r$. For $2 \leq i \leq r$, we check whether the following three types of pairs satisfy the condition: the pairs are

$$(\beta_1, \beta_2), \quad (\beta_1, \alpha_i), \quad (\beta_2, \alpha_i).$$

(a) (β_1, β_2). Since FS_α is OS, all arrows a in $\hat{Q}_{\alpha_1}$ satisfy $t(a) < s(a)$. Hence, when cutting the arrow $i \rightarrow j$, any elements in β_2 are larger than any elements in β_1. The condition is true in this case.

(b) (β_1, α_i). By induction, we know that the lemma is true for (α_1, α_i), $2 \leq i \leq r$. Since the elements of β_1 form a subset of the elements of α_1, it is also true for (β_1, α_i) for $2 \leq i \leq r$.

(c) (β_2, α_i). If β_2 is a single disjoint 'one cycle' (k), the statement is true. If $\beta_2 \neq (k)$, assume that the largest element in β_2 other than k is ϕ. If ϕ is smaller than the smallest element in α_i, then any element u other than k in β_2 is smaller than any element in α_i. Also, k is larger than any element in α_i. Hence, the statement is true in this case. Next, suppose that ϕ is larger than the smallest element in α_i. By construction, ϕ is an element in α_1, which contains 1. Hence, ϕ is larger than any element in α_i by induction. Similarly, any other element in β_2 is larger or smaller than all elements in α_i by induction. So, the statement is true. This finishes the proof of this lemma. □

Theorem 5.4. *The summand FS_α is an OS if and only if α is a noncrossing permutation.*

Proof. The 'only if' part follows from Lemmas 5.2 and 5.3. So, we only have to prove the 'if' part. We do so by induction on n.

When $n = 1$ it is clear, since (1) is the only permutation.

By induction, we assume that if $\alpha \in S_{k-1}$ satisfies condition $(*)$, then FS_α is an OS. We will prove that if $\beta \in S_k$ satisfies condition $(*)$, then FS_β is an OS. Assume that $[\beta] = [\alpha, j]$ for some α in S_{n-1} and some nonnegative integer j. We make two claims.

Claim 1: j is 0 or in the chain of $\hat{Q}_\alpha$.

Claim 2: α is a noncrossing permutation.

By *Claim 1*, j is 0 or in the chain of $\hat{Q}_\alpha$. By *Claim 2*, FS_α is an ordinary summand. By Construction 2.2 and Lemma 4.2, FS_β is also an OS. Therefore, if the above two claims are correct, the theorem is proved. Now we prove these two claims. □

PROOF OF CLAIM 1. If not, β is constructed from α by cutting an arrow $a : i \to j$ which is not in the chain of $\hat{Q}_\alpha$. By *Case 2* in Construction 2.2, we get a long chain

$$k+1 \longrightarrow j \longrightarrow \cdots \longrightarrow i \longrightarrow k \longrightarrow \cdots \longrightarrow 1.$$

In this chain, $i < k$, which contradicts the assumption that β satisfies condition $(*_1)$. So, j must be in the chain of $\hat{Q}_\alpha$ or $j = 0$. □

PROOF OF CLAIM 2. By *Claim 1*, $j = 0$ or j is in the chain of $\hat{Q}_\alpha$. If $j = 0$, it is easy to prove that α is a noncrossing permutation. Next, we assume that j is in the chain of $\hat{Q}_\alpha$. With the same notation as in Construction 2.2, let $\beta = \beta_1\beta_2\alpha_2 \ldots \alpha_r$ with $1 \in \beta_1$.

First, we check that α satisfies condition $(*_1)$. By the assumption on β, there is exactly one arrow a in the quiver of α_i such that $t(a) > s(a)$, where $2 \le i \le r$. So we have to show that all arrows a in the chain of $\hat{Q}_\alpha$ satisfy $t(a) < s(a)$. Suppose that there is an arrow a in the chain of $\hat{Q}_\alpha$ with $s(a) < t(a)$. If $t(a) \neq j$, then this arrow will be in either β_1 or β_2, which contradicts the assumption on β. If $t(a) = j$, then

$$\hat{Q}_{\beta_1} : \quad k+1 \longrightarrow j \longrightarrow \cdots \longrightarrow 1$$

and

$$\begin{tikzcd} \hat{Q}_{\beta_2} : & i \arrow[r] & k \arrow[r] & \cdots \arrow[r] & {} \arrow[lll, bend right] . \end{tikzcd}$$

Since $k > j > i$, it follows that (β_1, β_2) does not satisfy condition $(*_2)$. So $t(a) < s(a)$ for each arrow a in the chain of $\hat{Q}_\alpha$ and there is exactly one arrow b in each loop of $\hat{Q}_\alpha$ such that $s(b) < t(b)$.

Next, we prove that α satisfies condition $(*_2)$. The problem pair is (α_1, α_i) for $2 \le i \le r$. By assumption, β_1 contains the smallest element 1 and β_2 contains the element k. Hence, by Construction 2.2 and Lemma 5.3, any element in β_1 is smaller than any element in β_2. Since β is a noncrossing permutation, for any cycle α_i with $2 \le i \le r$, there are three possible cases.

(a) α_i is contained in β_1. If we pick an arbitrary element m in β_1, then either $m > n$ for any n in α_i or $m < n$ for any n in α_i.

TABLE 1. Ordinary summands in $W([3])$ and noncrossing permutations in S_3.

Ordinary summand	Noncrossing permutation
$p_{k_1+k_2+k_3}\,\partial^3/\partial p_{k_1}\partial p_{k_2}\partial p_{k_3}$	(321)
$p_{k_1+k_3}p_{k_2}\,\partial^2/\partial p_{k_1}\partial p_{k_2+k_3}$	(13)(2)
$p_{k_1+k_2}p_{k_3}\,\partial^2/\partial p_{k_2}\partial p_{k_1+k_3}$	(12)(3)
$p_{k_3+k_2}p_{k_1}\,\partial^2/\partial p_{k_3}\partial p_{k_1+k_2}$	(1)(23)
$p_{k_1}p_{k_2}p_{k_3}\,\partial/\partial p_{k_1+k_2+k_3}$	(1)(2)(3)

(b) α_i is contained in β_2. If we pick an arbitrary element m in β_2, then either $m > n$ for any n in α_i or $m < n$ for any n in α_i.

(c) α_i is disjoint with β_1 and β_2. Any element in α_i is larger than any element in β_1 and smaller than any element in β_2.

In the first case, if α_i is 'contained' in β_1, then any element in β_2 is larger than any element in α_i, because the elements in β_2 are always larger than the elements in β_1. By the construction of α_1, the condition is true for (α_1, α_i). The same argument holds for the second case. For the third case, β_1 and β_2 are constructed from α_1 by cutting the arrow with target j and adding another element k. Hence α_i is 'contained' in α_1. Therefore α satisfies condition $(*_2)$. □

EXAMPLE 5.5. In this example, we give the correspondence between the ordinary summands of $W([3])$ and noncrossing permutations in S_3 based on Theorem 5.4.

By Theorem 3.7, we know that $W([3])$ has 3! summands and that each corresponds to a unique permutation in S_3, as given in Example 3.8. Note that the first five summands in Example 3.8 are ordinary summands of maximal degree four, while the last one is of degree two. The correspondence between ordinary summands in $W([3])$ and noncrossing permutations in S_3 is given in Table 1, where we omit the symbol $\sum$ and the coefficients.

COROLLARY 5.6. *The number of $(r, n-r+1)$-type OS in $W([n])$ is the Narayana number*

$$\frac{1}{n+1}\binom{n+1}{r}\binom{n-1}{r-1}.$$

The number of all ordinary summands in $W([n])$ is the Catalan number

$$\sum_{r=1}^{n}\frac{1}{n+1}\binom{n+1}{r}\binom{n-1}{r-1} = \frac{1}{n+1}\binom{2n}{n}.$$

PROOF. By Theorem 5.4, there is a one-to-one correspondence between the ordinary summands and noncrossing permutations (also noncrossing partitions). The number of $(r, n-r+1)$-type OS is exactly the number of noncrossing partitions with r blocks, which is the Narayana number [11]. The number of all ordinary summands in $W([n])$ is the sum of Narayana numbers, which is the Catalan number. □

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