Canad. Math. Bull. Vol. **51** (1), 2008 pp. 100–113

Dynamical Zeta Function for Several Strictly Convex Obstacles

Vesselin Petkov

Abstract. The behavior of the dynamical zeta function $Z_D(s)$ related to several strictly convex disjoint obstacles is similar to that of the inverse $Q(s) = \frac{1}{\zeta(s)}$ of the Riemann zeta function $\zeta(s)$. Let $\Pi(s)$ be the series obtained from $Z_D(s)$ summing only over primitive periodic rays. In this paper we examine the analytic singularities of $Z_D(s)$ and $\Pi(s)$ close to the line $\Re s = s_2$, where s_2 is the abscissa of absolute convergence of the series obtained by the second iterations of the primitive periodic rays. We show that at least one of the functions $Z_D(s), \Pi(s)$ has a singularity at $s = s_2$.

1 Introduction

Let $\Omega \subset \mathbb{R}^n, n = 2, 3$, be an open and connected domain with C^∞ boundary $\partial\Omega$ having the form $\Omega = \mathbb{R}^n \setminus K$, where

$$K = \bigcup_{j=1}^{Q} K_j, \quad K_i \cap K_j = \varnothing, \text{ for } i \neq j$$

and K_j are strictly convex compact obstacles for $j = 1, \dots, Q$, $Q \geq 3$. Throughout this paper we suppose that K satisfies the following condition introduced by Ikawa [6]:

(H) The convex hull of every two connected components of K does not have common points with any other connected component of K.

Consider the reflecting rays in $\overline{\Omega}$ (see [6] and [19, Ch. 2] for a precise definition). Under condition (H) every periodic ray is ordinary reflecting, that is, γ has no tangent segments. Given a *periodic reflecting ray* γ in $\overline{\Omega}$ with m_γ reflections, we denote by T_γ the primitive period (length) of γ, by $d_\gamma = lT_\gamma,\ l \in \mathbb{N}$, the period of γ and by P_γ the linear Poincaré map related to γ. Setting $|\det(I - P_\gamma)| = |I - P_\gamma|$, it is easy to prove (see [18, Appendix]) that there exist constants $b_1 > 0$, $b_2 > 0$, $B_0 > 0$ so that

$$B_0 e^{2b_1 d_\gamma} \leq |I - P_\gamma| \leq e^{2b_2 d_\gamma}. \tag{1.1}$$

Denote by Ξ the set of all reflecting periodic rays in $\overline{\Omega}$ and set

$$d_0 = \min \operatorname{dist}_{i\neq j}(K_i, K_j), \quad D_0 = \max \operatorname{dist}_{i\neq j}(K_i, K_j).$$

Received by the editors October 21, 2005; revised December 29, 2005.
AMS subject classification: Primary: 11M36; secondary: 58J50.
Keywords: dynamical zeta function, periodic rays.

https://doi.org/10.4153/CMB-2008-012-8 Published online by Cambridge University Press

For the counting function of the lengths of periodic rays, there exists a constant $a_0 > 0$ such that

$$\sharp\{\gamma \in \Xi : d_\gamma \leq q\} \leq e^{a_0 q} \tag{1.2}$$

(see [6, 22] and [19, Ch. 2]). In this note we examine the dynamical zeta function

$$Z_D(s) = \sum_{\gamma \in \Xi} (-1)^{m_\gamma} T_\gamma |I - P_\gamma|^{-1/2} e^{-s d_\gamma}, \quad s \in \mathbb{C}, \tag{1.3}$$

where the summation is over all periodic rays $\gamma \in \Xi$. This zeta function is related to the trace formula for the unitary group associated with the Dirichlet problem for the wave equation

$$\begin{gathered} (\partial_t^2 - \Delta_x) u = 0 \text{ in } \mathbb{R} \times \Omega, \\ u = 0 \text{ on } \mathbb{R} \times \partial\Omega, \\ u(0, x) = f_1(x), \quad \partial_t u(0, x) = f_2(x). \end{gathered} \tag{1.4}$$

The form of $Z_D(s)$ is obtained by the Laplace transformation of the distribution

$$\sum_{\gamma \in \Xi} (-1)^{m_\gamma} T_\gamma |I - P_\gamma|^{-1/2} \delta(t - d_\gamma) \tag{1.5}$$

which in turn is the sum of the principal singularities of $u(t) \in \mathcal{D}'(\mathbb{R}^+)$ given by

$$u(t) = \sum_{\lambda_j} e^{it\lambda_j}, \quad t > 0.$$

Here $\lambda_j \in \mathbb{C}$ are the poles of the scattering matrix $S(z)$ related to the problem (1.4) and the summation is over all poles counted with their multiplicities. We refer to [7, 8, 18, 21] for a more detailed description of this link and to [1, 5, 13, 19, 21, 28] for the trace formulas leading to (1.5).

Following a result of Ikawa [7, 8], the existence of an analytic singularity of $Z_D(s)$ implies the existence of $\delta > 0$ such that there are an infinite number of poles $\{z_j\}_{j \in \mathbb{N}}$ of the scattering matrix $S(z)$ satisfying

$$0 < \Im z_j \leq \delta, \quad \forall j \in \mathbb{N},$$

and the last property is known as the modified Lax–Phillips conjecture. Another motivation for the analysis of $Z_D(s)$ is the folklore conjecture that the singularities of $Z_D(s)$ should determine approximatively the scattering poles.

By using (1.1) and (1.2), it is easy to see that there exists $s_1 \in \mathbb{R}$ called abscissa of absolute convergence such that for $\Re s > s_1$ the series (1.3) is absolutely convergent. Despite an extensive search in the physics and numerical analysis literature concerning n-disk problems (see [3, 12, 28, 29] and the references cited therein), to the best of our knowledge, in the general case the problem of the existence of *at least one singularity* of $Z_D(s)$ is still open. The existence of an analytic non-real singularity has been proved by Ikawa [9] in the case when K is the union of several balls with radius $r \leq r_0$, provided $r_0 > 0$ is sufficiently small. Recently, Stoyanov [25] generalized the result of Ikawa for several obstacles satisfying some geometrical conditions and having diameters less than r_0. It was proved in [18] that $Z_D(s)$ has no singularities on the line $\Re s = s_1$. In fact, we have a stronger result and following the recent works of Stoyanov [23, 26], we know that there exists $\delta_0 > 0$ such that $Z_D(s)$ is analytic for $\Re s > s_1 - \delta_0$ (see also [10] for the special case $s_1 > 0$). This means that $Z_D(s)$ is analytic in a domain around $\Re s = s_1$ and this phenomenon of cancellations is typical for dynamical zeta functions (see [4, 16, 23, 24, 26]). On the other hand, since $Z_D(s)$ is a Dirichlet series with real coefficients changing their signs, the situation is very similar to that for the inverse $Q(s) = \frac{1}{\zeta(s)}$ of the classical Riemann zeta function $\zeta(s)$. It is well known that $Q(s)$ is analytic on the line $\Re s = 1$ and $Q(s)$ has non-real singularities on the *critical line* $\Re s = 1/2$. Moreover, we have the representation

$$\log \zeta(s) = \sum_{m=1}^{\infty} \sum_{p \in \mathbf{P}} \frac{1}{m} \frac{1}{p^{ms}}, \quad \Re s > 1, \tag{1.6}$$

where $\mathbf{P}$ denotes the set of prime numbers. Consequently, the analytic behavior of $\log \zeta(s)$ for $1/2 < \Re s \leq 1$ is characterized by the continuation of the function

$$\pi(s) = \sum_{p \in \mathbf{P}} \frac{1}{p^s}, \quad \Re s > 1,$$

and the critical line $\Re s = 1/2$ is related to $m = 2$ in the representation (1.6).

Denote by $\mathcal{P}$ the set of all primitive periodic rays. In this note we examine the analytic singularities of $Z_D(s)$ close to the line $\Re s = s_2$, where $s_2 < s_1$ is the abscissa of the absolute convergence of the series $\Pi_2(s)$ obtained from $Z_D(s)$ when we sum only over the rays $2\gamma,\ \gamma \in \mathcal{P}$, that is, over the second iteration of primitive rays (see Section 4 for a precise definition). We show that the line $\Re s = s_2$ plays a role in the investigation of the singularities of $Z_D(s)$. Similarly to $\pi(s)$, introduce the function

$$\Pi(s) = \sum_{\gamma \in \mathcal{P}} (-1)^{m_\gamma} T_\gamma |I - P_\gamma|^{-1/2} e^{-sT_\gamma}, \quad \Re s > s_1,$$

where the summation is over the primitive rays $\gamma \in \mathcal{P}$. Next let $h_\Pi < s_1$ be the abscissa of holomorphy of $\Pi(s)$ given by

$$h_\Pi = \inf\{t \in \mathbb{R} : \Pi(s) \text{ is analytic for } \Re s > t\}.$$

Our main result is the following.

Theorem 1.1 *At least one of the functions $Z_D(s), \Pi(s)$ has a singularity at $s = s_2$ and the difference $Z_D(s) - \Pi(s)$ is analytic for $s \in \{z \in \mathbb{C} : \Re z > s_2\}$. Moreover, if $s_2 \neq h_\Pi$, then $Z_D(s)$ has a singularity at z with $\Re z > \max\{s_2, h_\Pi\} - \epsilon_1$, where $\epsilon_1 > 0$ is sufficiently small.*

In the same way, we may show that if we consider the series obtained by summing over all iterations of the primitive rays of order $(2m-1)$, the corresponding function will be singular at $s = s_{2m}$ if $Z_D(s)$ is analytic at $s = s_{2m}$. Here s_k is the abscissa of absolute convergence of the series obtained by summing over all iterations of order $k \geq 2$, and we show that $s_1 - h_t < s_k < s_{k-1}$, $h_t > 0$ being the topological entropy of the billiard flow (see Proposition 3.3). Thus if $Z_D(s)$ is analytic for $\Re s > s_1 - h_t$, for any fixed $M \geq 2$ one obtains a singularity of the sum of series related to the iterations $m \leq M$. This corollary yields some information for the numerical analysis, since in the numerical experiences one treats series with finite number iterations.

The existence of a singularity z_0 of $\Pi(s)$ such that $\Re z_0 > s_2 - \epsilon_0$, $\epsilon_0 > 0$, $\Im z_0 \neq 0$, is an interesting open problem, but it seems that the difficulty of this problem could be compared with that of the existence or the absence of singularities of $\pi(s)$ for $1/2 < \Re s < 1$. If fact, the dynamics of the periodic orbits is chaotic and the random change of signs of the coefficients in (1.3) plays some essential role. We *conjecture* that in general $Z_D(s)$ is not singular at s_2 and Theorem 1.1 shows that in this case $\Pi(s)$ must be singular at s_2. It is expected that there exist non-real singularities z of $\Pi(s)$ with $\Re z$ arbitrary close to line of holomorphy $\Re s = h_\Pi$ of $\Pi(s)$. This will lead to singularities of $Z_D(s)$. In fact we have two possibilities:

$$\text{(i) } s_2 \neq h_\Pi, \qquad\qquad \text{(ii) } s_2 = h_\Pi.$$

Our analysis in Section 4 implies that in case (i) the function $Z_D(s)$ must be singular either at $s = s_2$ $(s_2 > h_\Pi)$ or at a point z close to the line $\Re s = h_\Pi$ $(s_2 < h_\Pi)$ and we obtain a solution of the modified Lax–Phillips conjecture (see [8, 9, 25]). In case (ii) we have a phenomenon similar to the famous Riemann conjecture for $\zeta(s)$ and the maximal domain $\Re s > t$, where $\Pi(s)$ is analytic, is determined by the line $\Re s = s_2$. Finally, it is not clear if the singularities found in [9, 25] lie in the domain $\Re s > s_2$. We will discuss this problem in Section 4.

2 Symbolic Dynamics

We will write $Z_D(s)$ as a Selberg zeta function using the argument of [18, §5]. First assume $n = 3$ and let $\lambda_{\gamma,i}$, $i = 1, 2$, $|\lambda_{\gamma,i}| > 1$, be the eigenvalues of the Poincaré map P_γ of the ray $\gamma \in \mathcal{P}$. Set

$$\delta_\gamma = -\frac{1}{2}\log(\lambda_{\gamma,1}\lambda_{\gamma,2}), \quad \nu_\gamma = -\log\lambda_{\gamma,1}, \quad \mu_\gamma = -\log\lambda_{\gamma,2}.$$

The product $\lambda_{\gamma,1}\lambda_{\gamma,2}$ and the sum $\lambda_{\gamma,1} + \lambda_{\gamma,2}$ are positive and $\delta_\gamma < 0$. Given $\gamma \in \mathcal{P}$, introduce

$$r_\gamma = \begin{cases} 0 & \text{if } m_\gamma = 2k, \\ 1 & \text{if } m_\gamma = 2k+1. \end{cases}$$

Then for $\Re s \gg s_1$ we have

$$Z_D(s) = \sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\sum_{m=1}^{\infty}\sum_{\gamma\in\mathcal{P}} T_\gamma(-1)^{mr_\gamma} e^{m(-sT_\gamma+\delta_\gamma+k\nu_\gamma+p\mu_\gamma)}.$$

We refer to [18] for the details of the proof of this representation. For $n = 2$ we have a simpler formula since there is only one eigenvalue $\lambda_\gamma > 1$ and we get

$$Z_D(s) = \sum_{k=0}^{\infty}\sum_{m=1}^{\infty}\sum_{\gamma\in\mathcal{P}} T_\gamma(-1)^{mr_\gamma} e^{m(-sT_\gamma+\delta_\gamma+k\nu_\gamma)},$$

where $\delta_\gamma = -\frac{1}{2}\log\lambda_\gamma,\ \nu_\gamma = 2\delta_\gamma$. Consider the leading term of $Z_D(s)$ obtained for $k = p = 0$ (resp. $k = 0$ for $n = 2$) and having the form

$$Z(s) = -\frac{d}{ds}Z_0(s), \quad Z_0(s) = \sum_{m=1}^{\infty}\frac{1}{m}\sum_{\gamma\in\mathcal{P}}(-1)^{mr_\gamma} e^{m(-sT_\gamma+\delta_\gamma)}.$$

We will write $Z_0(s)$ by using a symbolic model. Let us recall some notations concerning the symbolic dynamics. Given a $Q\times Q$ matrix $A(i,j)_{i,j=1,\dots,Q}$ such that

$$A(i,j) = \begin{cases} 1 & \text{if } i \neq j,\\ 0 & \text{if } i = j,\end{cases}$$

introduce the spaces

$$\Sigma_A = \{\xi = \{\xi_i\}_{i=-\infty}^{\infty} : \xi_i \in \{1,\dots,Q\},\ A(\xi_i,\xi_{i+1}) = 1\},$$
$$\Sigma_A^+ = \{\xi = (\xi_0,\xi_1,\dots) : A(\xi_i,\xi_{i+1}) = 1, \forall i \geq 0\}.$$

Let σ_A be the shift on Σ_A, Σ_A^+ given, respectively, by

$$(\sigma_A\xi)_i = \xi_{i+1},\ \forall i \in \mathbb{Z}, \quad (\sigma_A\xi)_i = \xi_{i+1},\ \forall i \geq 0.$$

For every $\xi \in \Sigma_A$ there exists a unique ray $\gamma(\xi)$ with successive reflection points on

$$\dots,\partial K_{j-1},\partial K_j,\partial K_{j+1},\dots$$

(see [6,19]). Let $P_j(\xi)$ be the j-th reflection point of $\gamma(\xi)$ and let

$$f(\xi) = \|P_0(\xi) - P_1(\xi)\|.$$

If $\gamma = \gamma(\xi) \in \mathcal{P}$ has m reflections and primitive period T_γ, then

$$T_\gamma = f(\xi) + f(\sigma_A\xi) + \dots + f(\sigma_A^{m-1}\xi) = S_m f(\xi).$$

Also (See [8,9]) there exists a function $g(\xi)$ such that

$$\delta_\gamma = g(\xi) + g(\sigma_A\xi) + \dots + g(\sigma_A^{m-1}\xi) = S_m g(\xi).$$

For $\Re s$ large we may write $Z_0(s)$ as follows,

$$Z_0(s) = \sum_{m=1}^{\infty}\frac{(-1)^m}{m}\sum_{\sigma_A^m\xi=\xi} e^{S_m(-sf(\xi)+g(\xi))}.$$

Given a continuous function $F(\xi) \in C(\Sigma_A)$, introduce

$$\operatorname{var}_n F = \sup_{\xi,\eta \in \Sigma_A} \{|F(\xi) - F(\eta)| : \xi_i = \eta_i \text{ for } |i| \leq n\}$$

and for $0 < \theta < 1$ consider the norms

$$|F|_\theta = \sup_n \frac{\operatorname{var}_n F}{\theta^n}, \ \|F\|_\infty = \sup_{\xi \in \Sigma_A} |F(\xi)|, \|F\|_\theta = \|F\|_\infty + |F|_\theta.$$

Let $\mathcal{F}_\theta(\Sigma_A) \subset C(\Sigma_A), \mathcal{F}_\theta(\Sigma_A^+) \subset C(\Sigma_A^+)$ be Banach spaces with norm $\|\cdot\|_\theta$. It follows from the exponential instability of the billiard ball map that with some constant $0 < \theta < 1$, depending on the geometry of K, we have $f(\xi), g(\xi) \in \mathcal{F}_\theta(\Sigma_A)$ (see [8,9,18,23,25] for more details). We introduce the suspended flow σ^f over the space

$$\Sigma_A^f = \{(\xi, t) : \xi \in \Sigma_A, 0 \leq t \leq f(\xi)\}$$

with the identification $(\xi, f(\xi)) \sim (\sigma_a(\xi), 0)$ (see [17]) and notice that the topological entropy $h_t > 0$ of the suspended flow σ^f over Σ_A^f is given by

$$h_t = \sup_{\mu \in \mathcal{M}} \frac{h_\mu(\sigma_A)}{\int_{\Sigma_A} f d\mu}.$$

Finally, recall that the pressure $P(F)$ of a function $F \in C(\Sigma_A)$ is given by

$$P(F) = \sup_{\mu \in \mathcal{M}} \Big(h_\mu(\sigma_A) + \int_{\Sigma_A} F d\mu \Big),$$

where $h_\mu(\sigma_A)$ is the measure entropy of σ_A and the sup is taken over the set $\mathcal{M}$ of all probabilistic measures on Σ_A invariant with respect to σ_A.

3 Summation over the Iterated Periodic Rays

It is well known [17] that for every function $\varphi(\xi) \in \mathcal{F}_\theta(\Sigma_A)$ there exists $h, \psi \in \mathcal{F}_{\theta^{1/2}}(\Sigma_A)$ so that

$$\varphi(\xi) = h(\xi) + \psi(\sigma_A(\xi)) - \psi(\xi),$$

and the function $h(\xi)$ depends only on the coordinates $(\xi_0, \xi_1, \dots)$. In this case we will write $\varphi \sim h$. Obviously, if $F \sim \tilde{F}$, we have $P(F) = P(\tilde{F})$. Passing to functions $f \sim \tilde{f}$, $g \sim \tilde{g}$, we get

$$Z_0(s) = \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \sum_{\sigma_A^m \xi = \xi} e^{S_m(-s\tilde{f}(\xi) + \tilde{g}(\xi))}.$$

The function $\mathbb{R} \ni s \longrightarrow P(-skf + kg)$ is strictly decreasing and given an integer $k \geq 1$ we may introduce the number $s_k \in \mathbb{R}$ determined uniquely by the equality

$$P(-s_k kf + kg) = 0.$$

It follows easily from the results in [17] that s_k is the abscissa of absolute convergence of the series

$$P_k(s) = \frac{1}{k} \sum_{\gamma \in \mathcal{P}} (-1)^{km_\gamma} e^{-ksT_\gamma + k\delta_\gamma}.$$

Indeed, s_k is the abscissa of absolute convergence of the series

$$G_k(s) = \sum_{m=1}^{\infty} \frac{1}{m} \sum_{\sigma_A^m \xi = \xi} e^{S_m(-skf(\xi)+kg(\xi))}.$$

On the other hand, for $\Re s > s_k$ we have

$$G_k(s) = \sum_{\gamma \in \mathcal{P}} e^{-skT_\gamma + k\delta_\gamma} + \sum_{m=2}^{\infty} \frac{1}{m} \sum_{\gamma \in \mathcal{P}} e^{m(-skT_\gamma + k\delta_\gamma)}$$

and as in [17, Ch. 6] and [18, §4], we deduce that the series

$$\sum_{m=2}^{\infty} \frac{1}{m} \sum_{\gamma \in \mathcal{P}} e^{m(-skT_\gamma + k\delta_\gamma)}$$

is absolutely convergent for $\Re s \geq s_k - \epsilon$ for some small $\epsilon > 0$. Next we will prove the following.

Lemma 3.1 *For all $k \geq 1$ we have $s_{k+1} < s_k$.*

Proof The pressure of the function $-s_k kf + kg$ is zero, so we may find a function $h \in \mathcal{F}_{\theta^{1/2}}(\Sigma_A^+)$ so that $h \sim -s_k kf + kg, P(h) = 0$ and we may choose h (for more details, see [17]) so that

$$\sum_{\sigma_A \eta = \xi} e^{h(\eta)} = 1, \quad \forall \xi \in \Sigma_A^+.$$

This implies $h(\eta) \leq \alpha_k < 0$ for all $\eta \in \Sigma_A^+$ and $k \int_{\Sigma_A} (-s_k f + g)\, d\mu \leq \alpha_k$ for each $\mu \in \mathcal{M}$. It is clear that

$$h_\mu(\sigma) + \int_{\Sigma_A} (-s_k(k+1)f + (k+1)g)\, d\mu$$
$$\leq \sup_{\mu \in \mathcal{M}} \Big[h_\mu(\sigma) + \int_{\Sigma_A} (-s_k kf + kg)\, d\mu \Big] + \frac{\alpha_k}{k} = \frac{\alpha_k}{k} < 0, \quad \forall \mu \in \mathcal{M}.$$

This implies

$$P(-s_k(k+1)f + (k+1)g) = \sup_{\mu \in \mathcal{M}} \Big[h_\mu(\sigma) + \int_{\Sigma_A} (-s_k(k+1)f + (k+1)g)\, d\mu \Big] \leq \frac{\alpha_k}{k}.$$

On the other hand, $P(-s_{k+1}(k+1)f + (k+1)g) = 0$ and since the function

$$\mathbb{R} \ni s \longrightarrow P(-s(k+1)f + (k+1)g)$$

is strictly decreasing, we get $s_{k+1} < s_k$. ■

To study the convergence of the series over the iterated rays we need the following.

Proposition 3.2 *For every $k \geq 1$ there exists $\epsilon_o(k) > 0$, depending on k, such that the series*

$$\sum_{m=k+1}^{\infty} P_m(s) = \sum_{m=k+1}^{\infty} \sum_{\gamma \in \mathcal{P}} \frac{(-1)^{mr_\gamma}}{m} e^{m(-sT_\gamma + \delta_\gamma)}$$

is absolutely convergent for $\Re s \geq s_k - \epsilon_o(k)$.

Proof As in the proof of Lemma 3.1, we choose h so that $h \sim -ks_k f + kg$, $h(\eta) < 0$, for all $\eta \in \Sigma_A^+$. First, assume that $s_k < 0$. We choose $\epsilon = \epsilon(k) > 0$ small enough in order to arrange the inequality $\sup_{\eta \in \Sigma_A^+} h(\eta) = \alpha_k \leq (k+1)k\epsilon s_k \|f\|_\infty$. Let $\eta \in \Sigma_A^+$ correspond to a primitive periodic ray $\gamma \in \mathcal{P}$ with m reflections as explained in Section 2. We obtain $S_m(-ks_k f + kg)(\eta) = -ks_k T_\gamma + k\delta_\gamma$. On the other hand, it is clear that $T_\gamma \leq m\|f\|_\infty$ and we get

$$S_m h(\eta) \leq m(k+1)k\epsilon s_k \|f\|_\infty \leq (k+1)k\epsilon s_k T_\gamma.$$

From the equality $S_m(-ks_k f + kg)(\eta) = S_m h(\eta)$, we deduce

$$-s_k T_\gamma + \delta_\gamma \leq (k+1)\epsilon s_k T_\gamma, \forall \gamma \in \mathcal{P}.$$

Now let $0 \leq u \leq \frac{\epsilon}{k+1}$. Then

$$-s_k(1+u)T_\gamma + \delta_\gamma \leq (k+1)\epsilon s_k T_\gamma - s_k u T_\gamma \leq \Big((k+1)\epsilon - \frac{\epsilon}{k+1}\Big) s_k T_\gamma \leq \epsilon s_k T_\gamma$$

and we get the lower bound

$$1 > 1 - e^{-s_k(1+u)T_\gamma + \delta_\gamma} \geq 1 - e^{\epsilon s_k T_\gamma} \geq 1 - e^{2s_k \epsilon d_0} = \frac{1}{C_{\epsilon,k}} > 0.$$

Thus for $0 \leq u \leq \frac{\epsilon}{k+1}$, the series

$$\sum_{m=k+1}^{\infty} e^{m(-s_k(1+u)T_\gamma + \delta_\gamma)} = \frac{e^{(k+1)(-s_k(1+u)T_\gamma + \delta_\gamma)}}{1 - e^{-s_k(1+u)T_\gamma + \delta_\gamma}} \leq C_{\epsilon,k} e^{(k+1)(-s_k(1+u)T_\gamma + \delta_\gamma)}$$

is convergent.

Next we obtain

$$\begin{aligned} &-(k+1)s_k(1+u)T_\gamma + (k+1)\delta_\gamma \\ &\qquad \leq -s_k k T_\gamma + k\delta_\gamma + (k+1)\epsilon s_k T_\gamma - (k+1)u s_k T_\gamma \leq -s_k(1-\epsilon)kT_\gamma + k\delta_\gamma. \end{aligned}$$

Since s_k is the abscissa of absolute convergence of the series of k iterated rays, we deduce

$$\sum_{\gamma \in \mathcal{P}} e^{-s_k(1-\epsilon)kT_\gamma + k\delta_\gamma} < \infty.$$

Thus we conclude that

$$\sum_{m=k+1}^{\infty}\sum_{\gamma\in\mathcal{P}} e^{m(-s_k(1+u)T_\gamma+\delta_\gamma)} < \infty,$$

and the series

$$\sum_{m=k+1}^{\infty}\sum_{\gamma\in\mathcal{P}} \frac{(-1)^{mr_\gamma}}{m} e^{m(-sT_\gamma+\delta_\gamma)}$$

is absolutely convergent for $\Re s \geq s_k - \frac{\epsilon}{k+1}$. Setting $\epsilon_o(k) = \frac{\epsilon}{k+1}$, we obtain the result in this case.

Passing to the case $s_k > 0$, choose $\epsilon = \epsilon(k) > 0$ to arrange the inequalities

$$\sup_{\eta\in\Sigma_A^+} h(\eta) \leq -(k+1)k\epsilon s_k\|f\|_\infty,$$

$$-s_kT_\gamma + \delta_\gamma \leq -(k+1)\epsilon s_k T_\gamma, \quad \forall T_\gamma \in \mathcal{P}.$$

For $0 \leq u \leq \frac{\epsilon}{k+1}$ we deduce $-s_k(1-u)T_\gamma + \delta_\gamma \leq -(k+1)\epsilon s_k T_\gamma + s_k u T_\gamma \leq -\epsilon s_k T_\gamma$, which yields

$$\sum_{m=k+1}^{\infty} e^{m(-s_k(1-u)T_\gamma+\delta_\gamma)} \leq C_{\epsilon,k} e^{(k+1)(-s_k(1-u)T_\gamma+\delta_\gamma)}.$$

On the other hand,

$$-(k+1)s_k(1-u)T_\gamma + (k+1)\delta_\gamma \leq -s_k(1+\epsilon)kT_\gamma + k\delta_\gamma$$

and this leads to

$$\sum_{m=k+1}^{\infty}\sum_{\gamma\in\mathcal{P}} e^{m(-s_k(1-u)T_\gamma+\delta_\gamma)} < \infty.$$

Finally, in the case $s_k = 0$, we arrange

$$\sup_{\eta\in\Sigma_A^+} h(\eta) \leq -(k+1)k\epsilon\|f\|_\infty,$$

$$\delta_\gamma \leq -(k+1)\epsilon T_\gamma, \forall T_\gamma \in \mathcal{P}.$$

Repeating the above argument, we establish for $0 \leq u \leq \frac{\epsilon}{k+1}$ the convergence of the series

$$\sum_{m=k+1}^{\infty}\sum_{\gamma\in\mathcal{P}} e^{m(uT_\gamma+\delta_\gamma)} < \infty,$$

and this completes the proof. ■

To compare s_k and s_1, consider the measure $\nu \in \mathcal{M}$ for which we have

$$P(-s_1 f + g) = h_\nu(\sigma_A) + \int_{\Sigma_A} (-s_1 f + g)\, d\nu = 0.$$

This measure is called the *equilibrium state* of $-s_1 f + g$ (see [17]). Then we obtain

$$\begin{aligned} P\big(-k\big(s_1 - \frac{k-1}{k} h_t\big) f + kg\big) &\geq h_\nu(\sigma_A) + k \int_{\Sigma_A} (-s_1 f + g)\, d\nu + (k-1) h_t \int_{\Sigma_A} f\, d\nu \\ &= (k-1)\Big[h_t \int_{\Sigma_A} f\, d\nu - h_\nu(\sigma_A)\Big] \geq 0. \end{aligned}$$

Comparing this with $P(-k s_k f + kg) = 0$, we deduce

$$s_k \geq s_1 - \frac{k-1}{k} h_t. \tag{3.1}$$

Thus we have proved the following.

Proposition 3.3 *The sequence s_k is convergent and* $\lim_{k\to\infty} s_k \geq s_1 - h_t$.

It is interesting to note that the abscissa c_0 of simple convergence of the Dirichlet series $Z_0(s)$ satisfies the estimate $c_0 \geq s_1 - h_t$, but it is difficult to compare c_0 with s_k.

4 Singularities on the Line $\Re s = s_2$

Consider the Dirichlet series $P_2(s) = \frac{1}{2}\sum_{\gamma\in\mathcal{P}} e^{-2sT_\gamma + 2\delta_\gamma}$, with positive coefficients. According to a classical result, this series has an analytic singularity at $s = s_2$. On the other hand, Proposition 3.2 implies that the sum over all iterated rays $k\gamma$, $\gamma \in \mathcal{P}$, $k \geq 3$, given by $\sum_{k=3}^{\infty} P_k(s)$, is analytic for $\Re s \geq s_2 - \epsilon_o(2)$ for some $\epsilon_o(2) > 0$. It is clear that the singularities of $Z_0(s)$ for $\Re s > s_2$ are related to those of the series obtained by summing only over the primitive rays

$$P_1(s) = \sum_{\gamma\in\mathcal{P}} (-1)^{r_\gamma} e^{-sT_\gamma + \delta_\gamma}.$$

Let h_p be the abscissa of *holomorphy* of the Dirichlet series $P_1(s)$. More precisely, h_p is the *minimal* real number t such that $P_1(s)$ is analytic for $\Re s > t$. We have three possibilities:

$$\text{(i) } h_p > s_2, \qquad \text{(ii) } h_p = s_2, \qquad \text{(iii) } h_p < s_2.$$

In case (i), the function $P_1(s)$, and hence $Z_0(s)$, has either a singularity on the line $\Re s = h_p$ or there exists a sequence of singularities z_j with $\Re z_j \to h_p, |\Im z_j| \to \infty$. In case (iii), the function $P_2(s)$ produces a singularity of $Z_0(s)$ at $s = s_2$. In case (ii), we must examine the singularities of the sum $P_1(s) + P_2(s)$. Of course, if $P_1(s)$ is analytic at $s = s_2$, we have the same situation as in case (iii). Thus a cancellation of the singularities of $P_1(s) + P_2(s)$ at the point s_2 is possible only if $P_1(s)$ is singular at $s = s_2$. Thus we have the following.

Theorem 4.1 *At least one of the functions $Z_0(s), P_1(s)$ has a singularity at $s = s_2$. Moreover, the difference $Z_0(s) - P_1(s)$ is analytic for $s \in \{z \in \mathbb{C} : \Re z > s_2\}$.*

We may compare the functions $Z_0(s)$ and $Z_D(s)$. As was shown in [8, 18, 25] there exists $\mu_1 > 0$ such that $Z_D(s) - Z_0(s)$ is analytic for $\Re s > s_1 - \mu_1$. The number μ_1 depends on the geometry of obstacles (see [18, Appendix] and [25]). In some cases we may show that $s_2 > s_1 - \mu_1$. For example, this is true if $n = 2$ and $s_2 < 0$. Nevertheless, it is more natural to deal with the function $\Pi(s)$ introduced in Section 1. As above, let h_Π be the abscissa of the holomorphy of the Dirichlet series $\Pi(s)$ introduced in Section 1. We consider again three cases:

$$\text{(i) } h_\Pi > s_2, \qquad \text{(ii) } h_\Pi = s_2, \qquad \text{(iii) } h_\Pi < s_2.$$

For $m \geq 2$ and $n = 3$, the analysis of the series

$$\Pi_m(s) = \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \sum_{\gamma \in \mathcal{P}} \frac{1}{m} (-1)^{m r_\gamma} e^{m(-sT_\gamma + \delta_\gamma + k\nu_\gamma + p\mu_\gamma)}, \Re s > s_1$$

is completely similar to that of $P_m(s)$. In fact the abscissa of absolute convergence of $\Pi_m(s)$ coincides with that of $P_m(s)$ and we may apply Proposition 3.2 for the series

$$\sum_{m=j+1}^{\infty} \Pi_m(s) = \sum_{m=j+1}^{\infty} \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \sum_{\gamma \in \mathcal{P}} \frac{1}{m} (-1)^{m r_\gamma} e^{m(-sT_\gamma + \delta_\gamma + k\nu_\gamma + p\mu_\gamma)},$$

assuming $j \geq 1$. Case $n = 2$ is treated in a similar way and repeating the argument of the proof of Theorem 4.1, we obtain Theorem 1.1.

In the same way, we may consider the function

$$\mathbf{\Pi}_3(s) = \Pi(s) + \Pi_2(s) + \Pi_3(s) = \sum_{\gamma \in \Xi_3} (-1)^{m_\gamma} T_\gamma |I - P_\gamma|^{-1/2} e^{-sd_\gamma}, \Re s > s_1,$$

where the summation is over all rays $\gamma \in \Xi_3 \subset \Xi$, which are either primitive or are obtained by two or three iterations of primitive periodic rays. Then at least one of the functions $Z_D(s)$, $\mathbf{\Pi}_3(s)$ has a singularity at $s = s_4$ and it is possible to iterate this argument.

Let us mention that from our results it is not clear if the analytic singularity z of $\Pi(s)$ or $Z_D(s)$ given by Theorem 1.1 is a pole. In fact, it is known that the function $Z_0(s)$ is meromorphic for

$$\Re s \geq s_1 - \frac{|\log \theta|}{2\|f\|_\infty},$$

$0 < \theta < 1$ being the constant introduced in Section 2. On the other hand, we have $s_2 \geq h_t/2$ and s_2 lies in the above domain if $h_t \|f\|_\infty \leq |\log \theta|$. It is expected that $Z_0(s)$ and $Z_D(s)$ are meromorphic in a larger domain or in the whole complex plan. For $n = 2$ some results in this direction are obtained by Morita [15].

It is interesting to mention that for all $k \in \mathbb{N}$, we have

$$s_k > b_0 = \sup_{\gamma \in \mathcal{P}} \frac{\delta_\gamma}{T_\gamma}. \tag{4.1}$$

In [18] it was established that $b_0 < 0$, so we need to check (4.1) only for $s_k < 0$. In this case the argument of the proof of Proposition 3.2 shows that

$$-s_k T_\gamma + \delta_\gamma \leq \epsilon_k T_\gamma,\ \forall \gamma \in \mathcal{P},$$

with some $\epsilon_k < 0$ and we obtain (4.1). The number b_0 has been introduced in [18] and it is related to the sequence of poles

$$s_{m,\gamma} = \frac{\delta_\gamma}{T_\gamma} + \frac{2m\pi}{T_\gamma}\mathbf{i}, \quad r_{m,\gamma} = \frac{\delta_\gamma}{T_\gamma} + \frac{(2m+1)\pi}{T_\gamma}\mathbf{i}, \quad m \in \mathbb{Z},$$

obtained from the series formed by all iterations of a *fixed* periodic primitive ray γ.

For several strictly convex small obstacles, Ikawa [9] and Stoyanov [25] established the existence of a non-real singularity

$$z_0 = \alpha + \mathbf{i}\frac{\pi}{d_1}, \quad \alpha \in \mathbb{R},$$

of $Z_D(s)$ with d_1 sufficiently close to D_0. Following the analysis in [25, Section 7], we conclude that $s_1 - b_K \leq \alpha < s_1$ with

$$b_K \geq \frac{1}{D_0} \ln\Big(1 + \frac{\kappa_{\min}}{\nu_0} D_0\Big).$$

Here $\kappa_{\min} > 0$ is the minimal normal curvature of ∂K and $\nu_0 > 0$ is a constant depending on d_0, the diameter of K and

$$\chi_0 = \min\{\operatorname{dist}(K_j,\ \text{convex hull}\ (K_i \cup K_l)) : j \neq i, i \neq l, l \neq j\} > 0.$$

For obstacles having sufficiently small diameters, we may arrange the inequality $b_K \geq h_t$. Indeed, it is sufficient to have

$$h_\mu(\sigma_A) \leq \frac{d_0}{D_0} \ln\Big(1 + \frac{\kappa_{\min}}{\nu_0} D_0\Big) \leq b_K \int_{\Sigma_A} f\, d\mu$$

for every σ_A invariant measure $\mu \in \mathcal{M}$. If the diameters of the obstacles are sufficiently small, then $\kappa_{\min}$ is large enough, while $\frac{d_0}{D_0}$ and χ_0 remain bounded from below. Thus in this case we have

$$\sup_{\mu \in \mathcal{M}} h_\mu(\sigma_A) \leq \frac{d_0}{D_0} \ln\Big(1 + \frac{\kappa_{\min}}{\nu_0} D_0\Big)$$

which implies $b_K \geq h_t$. Combining this with (3.1), we obtain immediately

$$s_1 - b_K \leq s_1 - h_t < s_k, \forall k \in \mathbb{N}.$$

Consequently, the line $\Re s = s_k$ lies in the domain where we have complex singularities and this agrees with the conjecture that we must have complex singularities of $Z_D(s)$ close to the line $\Re s = h_\Pi$ or close to the line $\Re s = s_2$.

Acknowledgments The author is grateful to Lachezar Stoyanov for many helpful discussions and comments.

References

[1] C. Bardos, J. C. Guillot and J. Ralston, *La relation de Poisson pour l'équation des ondes dans un ouvert non-borné.* Comm. Partial Differantial Equations **7**(1982), no. 8905–958.

[2] N. Burq, *Controle de l'équation des plaques en présence d'obstacles strictement convexes.* Suppl. Bull. Soc. Math. France, **121**(1993), Mémoire 55.

[3] P. Cvitanović, G. Vattay and A. Wirzba, *Quantum fluids and classical determinants.* In: Classical, Semiclassical and Quantum Dynamics in Atoms, Lecture Notes in Physics 487, Springer, Berlin, pp. 29–62.

[4] D. Dolgopyat, *On decay of correlations of Anosov flows.* Ann. of Math. **147**(1998), no. 2, 357–390.

[5] V. Guillemin and R. Melrose, *The Poisson summation formula for manifolds with boundary.* Adv. in Math. **32**(1979), no. 3, 204–232.

[6] M. Ikawa, *Decay of solutions of the wave equation in the exterior of several strictly convex bodies.* Ann. Inst. Fourier (Grenoble), **38**(1988), no. 2, 113–146.

[7] ———, *On the existence of poles of the scattering matrix for several convex bodies.* Proc. Japan Acad. Ser. A Math. Sci. **64**(1988), no. 4, 91–93.

[8] ———, *On the distribution of poles of the scattering matrix for several convex bodies.* In: Functional-Analytic Methods for Partial Differential Equations, Lecture Notes in Math. 1450, Springer, Berlin, 1990, pp. 210–225.

[9] ———, *Singular perturbation of symbolic flows and poles of the zeta functions.* Osaka J. Math. **27**(1990), no. 2, 281–300; Addendum, Osaka J. Math. **29**(1992), 161–174.

[10] ———, *On scattering by several convex bodies.* J. Korean Math. Soc. **37**(2000), no. 6, 991–1005.

[11] P. Lax and R. Phillips, *Scattering Theory,* 2nd Edition, Pure and Applied Mathematics 26. Academic Press, New York, 1989.

[12] K. Lin and M. Zworski, *Quantum resonances in chaotic scattering.* Chem. Phys. Lett. **355**(2002), no. 1-2, 201–205.

[13] R. Melrose, *Polynomial bound on the distribution of scattering poles.* J. Funct. Anal. **53**(1983), no.3, 287–303.

[14] T. Morita, *The symbolic representation of billiards without boundary condition.* Trans. Amer. Math. Soc. **325**(1991), no.2, 819–828.

[15] ———, *Construction of K-stable foliations for two-dimensional dispersing billiards without eclipse.* J. Math. Soc. Japan **56**(2004), no. 3, 803–831.

[16] F. Naud, *Analytic continuation of a dynamical zeta function under a Diophantine condition.* Nonlinearity, **14**(2001), no. 5, 995–1009.

[17] W. Parry and M. Pollicot, *Zeta functions and the periodic orbit structure of hyperbolic dynamics.* Astérique, no. 187–188, 1990.

[18] V. Petkov, *Analytic singularities of the dynamical zeta function.* Nonlinearity, **12**(1999), no. 6, 1663–1681.

[19] V. Petkov and L. Stoyanov, *Geometry of Reflecting Rays and Inverse Spectral Problems.* Chichester, John Wiley, 1992.

[20] J. Sjöstrand, *A trace formula and review of some estimates for resonances.* In: Microlocal Analysis and Spectral Theory. NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. 490, Kluwer, Dordrecht, 1997, 377–437.

[21] J. Sjöstrand and M. Zworski, *Lower bounds on the number of scattering poles.* Comm. Partial Differential Equations **18**(1993), no. 5-6, 847–857.

[22] L. Stoyanov, *Exponential instability for a class of dispersing billiards.* Ergodic Theory Dynam. Systems **19**(1999), no. 1, 201–226.

[23] ———, *Spectrum of the Ruelle operator and exponential decay of correlation flow for open billiard flows.* Amer. J. Math. **123** (2001), 715–759.

[24] ———, *Ruelle zeta functions and spectra of transfer operators for some Axiom A flows.* Preprint. http://www.maths.uwa.edu.au/~stoyanov

[25] ———, *Scattering resonances for several small convex bodies and the Lax-Phillips conjecture.* To appear in Mem. Amer. Math. Soc. http://www.maths.uwa.edu.au/~stoyanov

[26] ———, *Spectra of Ruelle transfer operators for contact flows on basic sets.* Preprint, 2007. http://www.maths.uwa.edu.au/~stoyanov

[27] G. Vodev, *Sharp bounds on the number of scattering poles for perturbations of the Laplacian.* Comm. Math. Phys. **146**(1992), no. 1, 205–216.

[28] A. Voros, *Spectral functions, special functions and the Selberg zeta function.* Commun. Math. Phys. **110**(1987), no. 3, 437–465.

[29] A. Wirzba, *Quantum mechanics and semi-classics of hyperbolic n-disk scattering systems.* Phys. Rep. **309**(1999), no. 1-2, 1–116.

Département de Mathématiques Appliquées
Université Bordeaux I
351, Cours de la Libération
33405 Talence
France
e-mail: petkov@math.u-bordeaux1.fr