THE KERNEL RELATION FOR A STRICT EXTENSION OF CERTAIN REGULAR SEMIGROUPS

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1. Introduction and summary. Let R be a regular semigroup and denote by $\mathscr{C}(R)$ its congruence lattice. For $\rho \in \mathscr{C}(R)$, the kernel of ρ is the set $\ker \rho = \{a \in R \mid a\rho a^2\}$. The relation K on $\mathscr{C}(R)$ defined by $\lambda K \rho$ if $\ker \lambda = \ker \rho$ is the kernel relation on $\mathscr{C}(R)$. In general, K is a complete $\cap$-congruence but it is not a $\vee$-congruence. In view of the importance of the kernel-trace approach to the study of congruences on a regular semigroup (the trace of ρ is its restriction to idempotents of R), it is of considerable interest to determine necessary and sufficient conditions on R in order for K to be a congruence. This being in general a difficult task, one restricts attention to special classes of regular semigroups. For a background on this subject, consult **[1]**.

The special regular semigroups treated here are of the following form. Let V be a regular semigroup, S be an ideal of V and $Q = V/S$ be the corresponding Rees quotient. In addition, we require that the ideal extension V of S by Q be strict, that is, that the multiplication in V is determined by a partial homomorphism $\varphi: Q^* \to S$. Finally, we assume that Q is an orthogonal sum of 0-simple semigroups and that Q is categorical at zero. With these hypotheses, we are able, in the final theorem of the paper, to determine necessary and sufficient conditions on the ingredients making up V that K be a congruence on $\mathscr{C}(V)$. They involve the same type of condition on S and the 0-simple components of Q as well as on the partial homomorphism φ. On the way to proving this result, we establish several statements of more general interest. For congruences on general ideal extensions of semigroups, see **[2]**.

Section 2 contains some notational conventions and special terminology and Section 3 some general results. The case when Q is 0-simple and categorical at zero is treated in Section 4. The necessary statements leading to the desired generalization are established in Section 5.

2. Notation and terminology. The equality and the universal relations on any set X are denoted by ϵ and ω, or ϵ_X and ω_X, respectively. The restriction of a function or a relation θ to a set X is denoted by $\theta|_X$. If θ is an equivalence relation on X and $x \in X$, then $x\theta$ denotes the θ-class containing X. If also $A \subseteq X$, then

$$A\theta = \{x \in X \mid x\theta a \text{ for some } a \in A\}$$

is the *saturation* of A by θ; if $A\theta = A$, then θ *saturates* A. If X and Y are sets, then $X \backslash Y = \{x \in X \mid x \notin Y\}$.

Let R be any semigroup. By $\mathscr{C}(R)$ we denote the congruence lattice of R. If $A \subseteq R$, $E(A)$ denotes the set of idempotents in A. For $\rho \in \mathscr{C}(R)$,

$$\ker \rho = \{a \in R \mid a\rho e \text{ for some } e \in E(R)\}$$

is the *kernel* of ρ. The *kernel relation* K is defined by

$$\lambda K \rho \quad \text{if} \quad \ker \lambda = \ker \rho \qquad (\lambda, \rho \in \mathscr{C}(R)).$$

If R has an identity, let $R^1 = R$, Otherwise let R^1 stand for R with an identity adjoined. For $a \in R$, $J(a)$ denotes the principal ideal of R generated by a.

Now let R be nontrivial and have a zero. If R_α for $\alpha \in A$ is a system of subsemigroups of R containing the zero of R, whose union is R and satisfying $R_\alpha R_\beta = R_\alpha \cap R_\beta = \{0\}$ whenever $\alpha \neq \beta$, then R is an *orthogonal sum* of semigroups R_α, to be denoted by $\sum_{\alpha \in A} R_\alpha$. Further, R is *categorical at zero* if for any $a, b, c \in R$, $ab \neq 0$ and $bc \neq 0$ implies $abc \neq 0$. Clearly, if R is an orthogonal sum of semigroups R_α, then R is regular (respectively, categorical at zero) if and only if R_α is regular (respectively, categorical at zero) for every $\alpha \in A$. We write $R^* = R \backslash \{0\}$. If $\rho \in \mathscr{C}(R)$ and $0\rho = \{0\}$, then ρ is *0-restricted*. By $\mathscr{C}_0(R)$ we denote the set of all 0-restricted congruences on R. The relation ζ_R defined by

$$a \zeta_R b \quad \text{if} \quad xay = 0 \Leftrightarrow xby = 0 \quad \text{for all} \quad x, y \in R^1$$

is the greatest 0-restricted congruence on R.

Let V be a regular semigroup, S be an ideal of V and $Q = V/S$ be the Rees quotient of V relative to the ideal S. Then V is an (ideal) *extension* of S by Q. A mapping $\varphi : Q^* \to S$ is a partial homomorphism if for any $a, b \in Q^*$, $ab \neq 0$ in Q implies that $(ab)\varphi = (a\varphi)(b\varphi)$. If in addition

$$ab = \begin{cases} (a\varphi)b & \text{if } a \in Q^*, b \in S, \\ a(b\varphi) & \text{if } a \in S, b \in Q^*, \\ (a\varphi)(b\varphi) & \text{if } a, b \in Q^*, ab \in S, \end{cases}$$

then the multiplication in V is *determined by* φ and V is a *strict extension* of S. In such a case, the mapping ψ defined by

$$\psi : a \to \begin{cases} a\varphi & \textit{if } a \in Q^* \\ a & \text{if } a \in S \end{cases}$$

is a retraction of V onto S.

The notation introduced in the preceding paragraph will be fixed throughout the paper, where we take $V = S \cup Q^*$.

We now extract from [**2**, Corollary 1 to Theorem 1 and Proposition 2] the following description of congruences on V. Let $\sigma \in \mathscr{C}(S)$, P be an ideal of Q and $\tau \in \mathscr{C}_0(Q)$ be such that $a, b \in Q^*$, $a\tau b$ implies $a\varphi\sigma b\varphi$. In such a case, (σ, P, τ) is an *admissible triple* for which we define a relation ν on V by

$$a\nu b \Leftrightarrow \begin{cases} a\tau b & \text{if } a, b \in Q \backslash P, \\ a\psi\sigma b\psi & \text{if } a, b \in S \cup P^*. \end{cases}$$

Then ν is a congruence on V and conversely, every congruence on V has this form for unique σ, P and τ.

The notation $\nu = \mathscr{C}(\sigma, P, \tau)$ will always denote the above congruence implicitly implying that (σ, P, τ) is an admissible triple.

3. General results. The first result here is of general interest, the remaining ones will be used later, some of them several times.

PROPOSITION 3.1. *Let R be a regular semigroup such that K is a congruence on $\mathscr{C}(R)$ and let H be a homomorphic image of R. Then K is a congruence on $\mathscr{C}(H)$.*

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Proof. We let θ be a congruence on R and consider $H = R/\theta$. Let $\lambda, \rho, \tau \in \mathscr{C}(H)$ be such that $\lambda K \rho$. For $\delta \in \{\lambda, \rho, \tau\}$, we define a relation $\bar{\delta}$ on R by

$$a\bar{\delta}b \quad \text{if} \quad a\theta\delta b\theta.$$

Then $\bar{\delta}$ is a congruence on R. We show next that $\bar{\lambda}K\bar{\rho}$. Indeed, for $a \in R$, we have

$$\begin{aligned} a \in \ker \bar{\lambda} &\Leftrightarrow a\bar{\lambda}a^2 \Leftrightarrow a\theta\lambda a^2\theta = (a\theta)^2 \\ &\Leftrightarrow a\theta\rho(a\theta)^2 = a^2\theta \Leftrightarrow a\bar{\rho}a^2 \Leftrightarrow a \in \ker \bar{\rho} \end{aligned}$$

and thus $\ker \bar{\lambda} = \ker \bar{\rho}$ whence $\bar{\lambda}K\bar{\rho}$. It follows by hypothesis that $\bar{\lambda} \vee \bar{\tau} K \bar{\rho} \vee \bar{\tau}$.

It is well known that the mapping

$$\gamma \to \hat{\gamma} \qquad (\gamma \in \mathscr{C}(R))$$

where $a\theta\hat{\gamma}b\theta$ $(a, b \in R)$, induces an isomorphism of the interval $[\theta, \omega]$ of $\mathscr{C}(R)$ onto $\mathscr{C}(H)$. Since $\hat{\bar{\delta}} = \delta$ for $\delta \in \{\lambda, \rho, \tau\}$, it follows that $\bar{\lambda} \vee \bar{\tau} = \overline{\lambda \vee \tau}$ and $\bar{\rho} \vee \bar{\tau} = \overline{\rho \vee \tau}$. Hence $\overline{\lambda \vee \tau} K \overline{\rho \vee \tau}$ so that for any $a \in R$,

$$\begin{aligned} a\theta \in \ker(\lambda \vee \tau) &\Leftrightarrow a\theta\lambda \vee \tau(a\theta)^2 = a^2\theta \Leftrightarrow a\overline{\lambda \vee \tau}a^2 \\ &\Leftrightarrow a\overline{\rho \vee \tau}a^2 \Leftrightarrow a\theta\rho \vee \tau a^2\theta = (a\theta)^2 \\ &\Leftrightarrow a\theta \in \ker(\rho \vee \tau) \end{aligned}$$

which proves that $\lambda \vee \tau K \rho \vee \tau$ and K is a congruence on $\mathscr{C}(H)$. □

COROLLARY 3.2. *Assume that K is a congruence on $\mathscr{C}(V)$. Then K is a congruence on both $\mathscr{C}(S)$ and $\mathscr{C}(Q)$.*

Proof. Note that $Q \simeq V/S = V/\rho$ where ρ is the Rees congruence on V relative to the ideal S. Also S is a retract of V under the retraction ψ and is thus a homomorphic image of V. The assertions now follow by Proposition 3.1. □

According to Proposition 3.1 the class $\mathscr{K}$ of all regular semigroups S for which K is a congruence on $\mathscr{C}(S)$ is closed under homomorphic images. That $\mathscr{K}$ is not closed for taking direct products is exhibited on the example of a direct product of a 2-element semilattice by a 2-element group. That $\mathscr{K}$ is not closed for taking of regular subsemigroups can be seen as follows.

For the concepts and results used below, we refer to [5]. Let $S = \mathscr{B}(G, \alpha)$ be a Reilly semigroup where $G = Z_4$, the additive group of integers mod 4, and α is the endomorphism of G mapping $\bar{1}$ onto $\bar{2}$. Then α^2 is the trivial endomorphism and thus $M = \bigcup_{n=1}^{\infty} \ker \alpha^n = G$. Hence condition (iii) of [**5**, Theorem 5.5] is trivially satisfied so that, by condition (vi) of the same reference, K is a congruence on $\mathscr{C}(S)$. Let

$$T = \{(m, g, m) \in S \mid m \leq 1\}.$$

Then T is a semilattice of groups

$$G_i = \{(i, g, i) \mid g \in G\}, \qquad i = 0, 1$$

determined by the homomorphism

$$\varphi:(0,g,0)\to(1,g\alpha,1)=(0,g,0)(1,e,1) \qquad (g\in G).$$

Since φ is not the trivial homomorphism, [**5**, Theorem 4.7] implies that K is not a congruence on $\mathscr{C}(T)$. Here T is a regular subsemigroup of S.

LEMMA 3.3. *For* $\nu=\mathscr{C}(\sigma,P,\tau)$, *we have*

$$\ker\nu=\ker\sigma\cup\{a\in P^*\mid a\varphi\in\ker\sigma\}\cup(\ker\tau)^*.$$

Proof. Since $\sigma=\nu|_S$, we have $\ker\nu\cap S=\ker\sigma$. If $a\in P^*$, then

$$\begin{aligned} a\in\ker\nu &\Leftrightarrow a\nu e \quad \text{for some} \quad e\in E(S\cup P^*)\\ &\Leftrightarrow a\varphi\nu e\psi \quad \text{for some} \quad e\in E(S\cup P^*)\\ &\Leftrightarrow a\varphi\sigma e \quad \text{for some} \quad e\in E(S)\Leftrightarrow a\varphi\in\ker\sigma.\end{aligned}$$

Clearly, for $a\in Q\backslash P$, $a\in\ker\nu\Leftrightarrow a\in\ker\tau$. □

LEMMA 3.4. ([**4**, Theorem 3.6]). *Let* $\nu_i=\mathscr{C}(\sigma_i,P_i,\tau_i)$ *for* $i=1,2$. *Then* $\nu_1\vee\nu_2=\mathscr{C}(\sigma,P,\tau)$ *where* $\sigma=\sigma_1\vee\sigma_2$, $P=(P_1\cup P_2)(\tau_1\vee\tau_2)$ *and* τ *is the* 0*-restricted congruence on* Q/P *satisfying the condition* $\tau|_{Q\backslash P}=(\tau_1\vee\tau_2)|_{Q\backslash P}$.

LEMMA 3.5. *Let* $\nu\in\mathscr{C}(V)$ *and* $a,b\in Q^*$ *be such that* $a\nu b$. *Then* $a\varphi\nu b\varphi$.

Proof. Let $x\in S$. Then $ax\nu bx$ and $xa\nu xb$ and also $ax=(a\varphi)x$ and $xa=x(a\varphi)$ so that $(a\varphi)x\nu(b\varphi)x$ and $x(a\varphi)\nu x(b\varphi)$. Letting $\sigma=\nu|_S$, we note that S/σ is weakly reductive and thus $a\varphi\nu b\varphi$. □

4. The case of Q 0-simple and categorical at zero. In order to treat this case, we need some preliminary lemmas. The second one is stated in somewhat greater generality than necessary.

LEMMA 4.1. *Let* (σ,P,τ) *and* (σ',P',τ') *be admissible triples such that* $\sigma K\sigma'$, $P=P'$ *and* $\tau K\tau'$. *Then* $\mathscr{C}(\sigma,P,\tau)K\mathscr{C}(\sigma',P',\tau')$.

Proof. This follows directly from Lemma 3.3. □

For a partial converse of Lemma 4.1, we have the following result.

LEMMA 4.2. *Assume that in every nonzero* $\mathscr{J}$*-class of* Q *there exists an element* a *such that* $a\rho\in E(S)$ *and* $a^2\in S$. *Let* $\nu=\mathscr{C}(\sigma,P,\tau)$ *and* $\nu'=\mathscr{C}(\sigma',P',\tau')$ *be such that* $\nu K\nu'$. *Then* $\sigma K\sigma'$, $P=P'$ *and* $\tau K\tau'$.

Proof. The assertion $\sigma K\sigma'$ follows by Lemma 3.3. Suppose that $P\neq P'$. By symmetry, we may suppose that $P\backslash P'\neq\varnothing$. Hence $P\backslash P'$ contains a nonzero $\mathscr{J}$-class J. By hypothesis, J contains an element a such that $a\varphi\in E(S)$ and $a^2\in S$. In view of Lemma

3.3, $a \in \ker \nu$ whereas $a \notin \ker \nu'$. But then $\ker \nu \neq \ker \nu'$ contrary to the hypothesis that $\nu K \nu'$. Therefore $P = P'$. Now Lemma 3.3 implies that $\tau K \tau'$. $\square$

LEMMA 4.3. *Let Q be 0-simple and categorical at zero. Assume that for $a \in Q^*$, $a\varphi \in E(S)$ implies $a^2 \in Q^*$. Let $\tau = \zeta_Q \cap \tau'$ where τ' is defined by*

$$a\tau' b \quad \text{if} \quad a, b \in Q^*, a\varphi = b\varphi; \qquad 0\tau' 0.$$

Then $(\epsilon_S, Q, \epsilon)$ and $(\epsilon_S, \{0\}, \tau)$ are admissible triples. Letting $\lambda = \mathscr{C}(\epsilon_S, Q, \epsilon)$ and $\rho = \mathscr{C}(\epsilon_S, \{0\}, \tau)$, we have $\lambda K \rho$ and $\ker \lambda = E(S) \cup \{a \in Q^* \mid a\varphi \in E(S)\}$.

Proof. Clearly τ' is an equivalence relation on Q. In order to see that τ is a congruence on Q, let $a\tau b$ and $c \in Q$ be such that $ac \neq 0$. Since $a\zeta_Q b$, we have $ac\zeta_Q bc$ which implies that $bc \neq 0$ since ζ_Q is 0-restricted. But then

$$(ac)\varphi = (a\varphi)(c\varphi) = (b\varphi)(c\varphi) = (bc)\varphi$$

which shows that $ac\tau bc$. Similarly, if $ac = 0$, then $bc = 0$ since ζ_Q is 0-restricted so that again $ac\tau bc$. Dually, $a\tau b$ implies $ca\tau cb$. Therefore τ is a congruence on Q, and is trivially 0-restricted. If $a, b \in Q^*$ and $x \in S$ are such that $a\tau b$, then $ax = (a\varphi)x = (b\varphi)x = bx$ which shows that $(\epsilon_S, \{0\}, \tau)$ is an admissible triple.

Clearly $(\epsilon_s, Q, \epsilon)$ is an admissible triple. By Lemma 3.3, we have

$$\ker \lambda = E(S) \cup \{a \in Q^* \mid a\varphi \in E(S)\}, \tag{1}$$

$$\ker \rho = E(S) \cup (\ker \tau)^* \tag{2}$$

where

$$\begin{aligned}(\ker \tau)^* &= \{a \in Q^* \mid a\tau a^2\} = \{a \in Q^* \mid a\zeta_Q a^2, a\varphi = a^2\varphi\} \\ &= \{a \in Q^* \mid a\zeta_Q a^2, a\varphi \in E(S)\}.\end{aligned} \tag{3}$$

In order to prove that (1) and (2) are equal, in view of (3), it suffices to show that for $a \in Q^*$, $a\varphi \in E(S)$ implies that $a\zeta_Q a^2$. Hence let $a \in Q^*$ be such that $a\varphi \in E(S)$. By hypothesis $a^2 \in Q^*$. We now consider the semigroup Q. Let $x, y \in Q^1$ and note that $a^2 \neq 0$. If $xay \neq 0$, then xa, a^2 and ay are different from zero which implies that $xa^2y \neq 0$ since Q is categorical at zero. Conversely, if $xa^2y \neq 0$, then xa and ay are different from zero and thus $xay \neq 0$ by the same assumption. We have proved that $a\zeta_Q a^2$. Therefore $\lambda K \rho$, as required. $\square$

We are now ready for the desired result. The theorem below generalizes the main result in [**3**] as well as [**6**, Theorem 7.6].

THEOREM 4.4. *Assume that Q is 0-simple and categorical at zero. Then K is a congruence on $\mathscr{C}(V)$ if and only if*

(i) *K is a congruence both on $\mathscr{C}(S)$ and $\mathscr{C}(Q)$,*

(ii) *either $\varphi : Q^* \to E(S)$ or there exists $a \in Q^*$ such that $a\varphi \in E(S)$ and $a^2 \in S$.*

Proof. Necessity. Part (i) follows by Corollary 3.2. Suppose that the second alternative in part (ii) does not take place. In the notation of Lemma 4.3, we have $\lambda K \rho$. Now let θ be the Rees congruence on V relative to the ideal S, that is $\theta = \mathscr{C}(\omega_S, \{0\}, \epsilon_Q)$. The hypothesis implies that $\lambda \vee \theta K \rho \vee \theta$ which by Lemma 3.4 yields

$\mathscr{C}(\omega_S, Q, \epsilon_S)K\mathscr{C}(\omega_S, \{0\}, \tau)$. It follows by Lemma 3.3 that $Q = \ker \tau$ which by Lemma 4.3 gives that $a\varphi \in E(S)$ for all $a \in Q^*$. Therefore $\varphi : Q^* \to E(S)$.

Sufficiency. We now abbreviate our notation by writing:

$$\mathscr{C}(\sigma, P, \tau) = \begin{cases} [\sigma] & \text{if } P = Q, \\ [\sigma, \tau] & \text{if } P = \{0\}. \end{cases}$$

We first observe that by Lemma 3.3 for $[\sigma]$, $[\sigma, \tau] \in \mathscr{C}(V)$, we have

$$\ker[\sigma] = \ker \sigma \cup \{a \in Q^* \mid a\varphi \in \ker \sigma\}, \qquad \ker[\sigma, \tau] = \ker \sigma \cup (\ker \tau)^*,$$

and thus

$$[\sigma_1]K[\sigma_2] \Leftrightarrow \sigma_1 K \sigma_2,$$
$$[\sigma_1, \tau_1]K[\sigma_2, \tau_2] \Leftrightarrow \sigma_1 K \sigma_2,\ \tau_1 K \tau_2,$$
$$[\sigma_1]K[\sigma_2, \tau_2] \Leftrightarrow (\sigma_1 K \sigma_2;\ a\varphi \in \ker \sigma_1 \Leftrightarrow a \in (\ker \tau_2)^*).$$

We now let $v_i = \mathscr{C}(\sigma_i, P_i, \tau_i)$ for $i = 1, 2$ and using Lemmas 4.1 and 3.3 consider several cases.

Case $[\sigma_1]K[\sigma_2]$. Then $\sigma_1 K \sigma_2$ so by part (i), also $\sigma_1 \vee \sigma_3 K \sigma_2 \vee \sigma_3$ which gives

$$[\sigma_1] \vee \mathscr{C}(\sigma_3, P_3, \tau_3) = [\sigma_1 \vee \sigma_3]K[\sigma_2 \vee \sigma_3] = [\sigma_2] \vee \mathscr{C}(\sigma_3, P_3, \tau_3).$$

Case $[\sigma_1]K[\sigma_2, \tau_2]$. Then $\sigma_1 K \sigma_2$ and $\tau_1 K \tau_2$ so that by part (i), $\sigma_1 \vee \sigma_3 K \sigma_2 \vee \sigma_3$ and $\tau_1 \vee \tau_3 K \tau_2 \vee \tau_3$ which gives

$$[\sigma_1, \tau_1] \vee [\sigma_3] = [\sigma_1 \vee \sigma_3]K[\sigma_2 \vee \sigma_3] = [\sigma_2, \tau_2] \vee [\sigma_3],$$
$$[\sigma_1, \tau_1] \vee [\sigma_3, \tau_3] = [\sigma_1 \vee \sigma_3, \tau_1 \vee \tau_3]K[\sigma_2 \vee \sigma_3, \tau_2 \vee \tau_3]$$
$$= [\sigma_2, \tau_2] \vee [\sigma_3, \tau_3].$$

Case $[\sigma_1]K[\sigma_2, \tau_2]$. Then $\sigma_1 K \sigma_2$ so by part (i), $\sigma_1 \vee \sigma_3 K \sigma_2 \vee \sigma_3$ which gives

$$[\sigma_1] \vee [\sigma_3] = [\sigma_1 \vee \sigma_3]K[\sigma_2 \vee \sigma_3] = [\sigma_2] \vee [\sigma_3].$$

By Lemma 4.2, the second alternative in part (ii) cannot take place in this case. The first alternative in part (ii) implies that $\ker[\sigma] = \ker \sigma \cup Q^*$ for any $Q \in \mathscr{C}(S)$. In particular, the hypothesis for this case implies that $\ker \tau_2 = Q$. We now obtain

$$\ker([\sigma_1] \vee [\sigma_3, \tau_3]) = \ker[\sigma_1 \vee \sigma_3] = \ker(\sigma_1 \vee \sigma_3) \cup Q^*, \tag{4}$$

$$\ker([\sigma_2, \tau_2] \vee [\sigma_3, \tau_3]) = \ker[\sigma_2 \vee \sigma_3, \tau_2 \vee \tau_3]$$
$$= \ker(\sigma_2 \vee \sigma_3) \cup (\ker(\tau_2 \vee \tau_3))^* \tag{5}$$

where $\ker(\sigma_1 \vee \sigma_3) = \ker(\sigma_2 \vee \sigma_3)$ and $\ker(\tau_2 \vee \tau_3) \supseteq \ker \tau_2 = Q$ and the expressions in (4) and (5) are equal. Therefore

$$[\sigma_1] \vee [\sigma_3, \tau_3]K[\sigma_2, \tau_2] \vee [\sigma_3 \tau_3],$$

as required.

This exhausts all the cases and thus shows that K is a congruence on $\mathscr{C}(V)$. $\square$

5. The general case. For the proof of the final result in which Q is an orthogonal sum of 0-simple semigroups categorical at zero, we need a sequence of lemmas.

LEMMA 5.1. *Let Q be 0-simple and let $v \in \mathscr{C}(V)$. Suppose that there exist $b \in Q^*$ and $c \in S$ such that bvc. Then for every $a \in Q^*$, we have $ava\varphi$.*

Proof. Let

$$I = \{x \in Q^* \mid xvy \text{ for some } y \in S\} \cup \{0\}.$$

Then I is an ideal of Q since S is an ideal of V. The hypothesis implies that $I \neq \{0\}$ and thus $I = Q$ since Q is 0-simple. Now let $a \in Q^*$. From the proven statement, it follows that avd for some $d \in S$. Let $x \in S$. Then $axvdx$ and $xavxd$ and since $ax = (a\varphi)x$ and $xa = x(a\varphi)$, we obtain $(a\varphi)xvdx$ and $x(a\varphi)vxd$. Since this holds for all $x \in S$ and $S/(v|_S)$ is weakly reductive, we conclude that $a\varphi vd$. But then $ava\varphi$. $\square$

LEMMA 5.2. *Let $Q = \sum_{\alpha \in A} Q_\alpha$ where Q_α is 0-simple for every $\alpha \in A$. Also let $a, b \in V$, $v \in \mathscr{C}(V)$ and avb. Then $a\psi vb\psi$.*

Proof. We consider several cases.

Case $a, b \in S$. This case is trivial since $a\psi = a$, $b\psi = b$.

Case $a \in S$, $b \in Q^*$. Then $b \in Q_\alpha^*$ for some $\alpha \in A$ which by Lemma 5.1 implies that $bvb\psi$. Hence $a\psi = avbvb\psi$.

Case $a \in Q^*$, $b \in S$. This is dual to the preceding case.

Case $a, b \in Q^*$. This case follows directly from Lemma 3.5. $\square$

In the sequel, $Q = \sum_{\alpha \in A} Q_\alpha$ where for each $\alpha \in A$, Q_α is 0-simple (and regular). For every $\alpha \in A$, let

$$V_\alpha = S \cup Q_\alpha^*$$

so that V_α is an ideal of V. The next result shows that V_α is a retract of V.

LEMMA 5.3. *Fix $\alpha \in A$ and define a mapping χ by*

$$\chi : \begin{cases} a \to a & \text{if } a \in V_\alpha, \\ a \to a\varphi & \text{if } a \in V \backslash V_\alpha. \end{cases}$$

Then χ is a homomorphism of V onto V_α.

Proof. Let $a, b \in V$. If $a \in V_\alpha$ and $b \notin V_\alpha$, then

$$(a\chi)(b\chi) = a(b\varphi) = ab = (ab)\chi.$$

The case $a \notin V_\alpha$ and $b \in V_\alpha$ is dual. If $a, b \notin V_\alpha$, then either $a, b, ab \in Q_\beta^*$ for some $\beta \in A$, in which case

$$(a\chi)(b\chi) = (a\varphi)(b\varphi) = (ab)\varphi = (ab)\chi;$$

or $a \in Q_\beta^*$, $b \in Q_\gamma^*$, $ab \in S$ for some $\beta, \gamma \in A$ in which case

$$(a\chi)(b\chi) = (a\varphi)(b\varphi) = ab = (ab)\chi.$$

The case $a, b \in V_\alpha$ being trivial, we conclude that χ is a retraction of V onto V_α. $\square$

The next result is of indepedent interest.

PROPOSITION 5.4. *Let* $Q = \sum_{\alpha \in A} Q_\alpha$ *where* Q_α *is* 0*-simple for every* $\alpha \in A$ *and let* $v_\alpha \in \mathscr{C}(V_\alpha)$ *be such that* $v_\alpha|_S = v_\beta|_S$ *for any* $\alpha, \beta \in A$. *Define a relation* v *on* V *by: for* $a \in V_\alpha$, $b \in V_\beta$,

$$avb \Leftrightarrow \begin{cases} av_\alpha b & \text{if } \alpha = \beta, \\ a\varphi v_\alpha b\varphi & \text{if } \alpha \neq \beta,\ a \in Q_\alpha^*, av_\alpha a\varphi, b \in Q_\beta^*, bv_\beta b\varphi. \end{cases}$$

Then v *is a congruence on* V. *Conversely, every congruence on* V *can be so represented for unique* V_α.

Proof. Let v be as defined above. Clearly v is reflexive and symmetric. Let $a \in V_\alpha$, $b \in V_\beta$ and $c \in V_\gamma$ be such that avb and bvc. We consider several cases.

Case $\alpha = \beta = \gamma$. Then $av_\alpha bv_\alpha c$ so that avc.

Case $\alpha \neq \beta = \gamma$. Then $a \in Q_\alpha^*$, $b \in Q_\beta^*$, $av_\alpha a\varphi$, $bv_\beta b\varphi$, $a\varphi v_\alpha b\varphi$, $bv_\beta c$. If $c \in Q_\beta^*$, then $a\varphi v_\alpha b\varphi v_\alpha c\varphi$ and $cv_\gamma c\varphi$ so that avc. If $c \in S$, then $c \in V_\alpha$ and $b\varphi v_\beta bv_\beta c$ implies that $b\varphi v_\beta c$ so that $av_\alpha a\varphi v_\alpha b\varphi v_\alpha c$ and thus avc.

Case $\alpha = \beta \neq \gamma$. This is similar to the preceding case.

Case $\alpha \neq \beta \neq \gamma$. Then $a \in Q_\alpha^*$, $b \in Q_\beta^*$ and $c \in Q_\gamma^*$. This splits into two cases.

Subcase $\alpha \neq \gamma$. Then $a\varphi v_\alpha b\varphi v_\beta c\varphi$ and $av_\alpha a\varphi$, $cv_\gamma c\varphi$ imply that avc.

Subcase $\alpha = \gamma$. Then $av_\alpha a\varphi v_\alpha b\varphi v_\beta c\varphi v_\gamma c$ implies that $av_\alpha a\varphi v_\alpha c\varphi v_\alpha c$ so that avc.

Therefore v is transitive and is thus an equivalence relation.

In order to show that v is a congruence, we let $a \in V_\alpha$, $b \in V_\beta$ and $c \in V_\gamma$ be such that avb.

Case $\alpha = \beta$. If $\gamma = \alpha$, then trivially $acvbc$. Otherwise, possibly using Lemma 5.2, we get $a\psi v_\alpha b\psi$ and thus

$$ac = (a\psi)(c\psi)v_\alpha(b\psi)(c\psi) = bc$$

so that again $acvbc$.

Case $\alpha \neq \beta$. Then $a \in Q_\alpha^*$, $b \in Q_\beta^*$, $av_\alpha a\varphi$, $bv_\beta b\varphi$, $a\varphi v_\alpha b\varphi$.

Subcase $\alpha = \gamma$. Then $bc = (b\varphi)(c\varphi) \in S$. This splits further into two subcases.

Subsubcase $ac \in Q_\alpha^*$. Then

$$acv_\alpha(a\varphi)c = (a\varphi)(c\varphi)v_\alpha(b\varphi)(c\varphi) = bc$$

so that $acvbc$.

Subsubcase $ac \in S$. Then

$$ac = (a\varphi)(c\psi)v_\alpha(b\varphi)(c\psi) = bc$$

and again $acvbc$.

Subcase $\beta = \gamma$. This is dual to the preceding case.

Subcase $\alpha \neq \gamma \neq \beta$. Then

$$ac = (a\varphi)(c\psi)v_\alpha(b\varphi)(c\psi) = bc$$

and $acvbc$.

Therefore v is a congruence on V.

Conversely, let ν be a congruence on V. For every $\alpha \in A$, let $\nu_\alpha = \nu|_{V_\alpha}$. Then $\nu_\alpha \in \mathscr{C}(V_\alpha)$ and $\nu_\alpha|_S = \nu|_S = \nu_\beta|_S$. In order to see that ν can be obtained as in the first part of the proposition, it suffices to prove that for $a \in Q^*_\alpha$, $b \in Q^*_\beta$, $\alpha \neq \beta$, $a\nu b$ if and only if $a\nu a\varphi\nu b\varphi\nu b$.

Assume first that $a\nu b$. If a' is an inverse of a, then $a = aa'a\nu ba'a$ where $ba'a \in S$. Hence Lemma 5.1 implies that $a\nu a\varphi$. Also Lemma 3.5 gives $a\varphi\nu b\varphi$. Again Lemma 5.1 provides $b\varphi\nu b$ so that $a\nu a\varphi\nu b\varphi\nu b$. The converse is obvious by transitivity.

Trivially, the congruences $\nu_\alpha = \nu|_{V_\alpha}$ are unique. □

LEMMA 5.5. *Let* $Q = \sum_{\alpha\in A} Q_\alpha$, *where* Q_α *is* 0-*simple for every* $\alpha \in A$, $a \in V_\alpha$, $b \in V\backslash V_\alpha$, $\nu \in \mathscr{C}(V)$, $a\nu b$. *Then* $a\nu b\varphi$.

Proof. First note that $b \in Q^*$. If $a \in S$, then $a\nu b$ by Lemma 5.1 implies that $b\nu b\varphi$ and thus $a\nu b\varphi$. If $a \in Q^*$, then Proposition 5.4 gives $a\nu a\varphi\nu b\varphi$ so again $a\nu b\varphi$. □

LEMMA 5.6. *Let* $Q = \sum_{\alpha\in A} Q_\alpha$ *where* Q_α *is* 0-*simple for every* $\alpha \in A$, $\lambda, \rho \in \mathscr{C}(V)$ *and fix* $\alpha \in A$. *Then* $\lambda|_{V_\alpha} \vee \rho|_{V_\alpha} = (\lambda \vee \rho)|_{V_\alpha}$.

Proof. Write $\lambda' = \lambda|_{V_\alpha}$, $\rho' = \rho|_{V_\alpha}$, $(\lambda\vee\rho)' = (\lambda\vee\rho)|_{V_\alpha}$. Clearly $\lambda', \rho' \subseteq (\lambda\vee\rho)'$ and thus $\lambda'\vee\rho' \subseteq (\lambda\vee\rho)'$. For the opposite inclusion, we let $a(\lambda\vee\rho)'b$. Hence there exists a sequence

$$a\lambda x_1\rho x_2\lambda x_3 \dots x_n\rho b \tag{6}$$

for some $x_1, x_2, \dots, x_n \in V$. We claim that sequence (6) implies the following sequence

$$a\lambda y_1\rho y_2\lambda y_3 \dots y_n\rho b \tag{7}$$

where

$$y_i = \begin{cases} x_i & \text{if } x_i \in V_\alpha, \\ x_i\varphi & \text{otherwise.} \end{cases}$$

The proof of the claim is by induction on i in the following statement

$$a\lambda y_1\rho y_2\lambda y_3 \dots y_i\lambda x_{i+1}\rho \dots x_n\rho b$$

the case $y_i\rho x_{i+1}$ being analogous, where $y_0 = a$ and $x_{n+1} = b$.

The case $i = 0$ is a special case of the general step i. For that step, we have the following cases.

(i) $x_i, x_{i+1} \in V_\alpha$. This case is trivial.

(ii) $x_i \in V_\alpha$, $x_{i+1} \notin V_\alpha$. Then $x_i\lambda x_{i+1}\varphi$ by Lemma 5.5.

(iii) $x_i \notin V_\alpha$, $x_{i+1} \in V_\alpha$. This is dual to the preceding case.

(iv) $x_i, x_{i+1} \notin V_\alpha$. Then $x_i\varphi\lambda x_{i+1}\varphi$ by Lemma 3.5.

By induction, the assertions contained in sequence (7) are proved. Since $y_i \in S$ we have $y_i \in V_\alpha$ for $i = 1, 2, \dots, n$ and thus $a\lambda'\vee\rho' b$. Therefore $(\lambda\vee\rho)' \subseteq \lambda'\vee\rho'$ and equality prevails. □

LEMMA 5.7. *Let* $Q = \sum_{\alpha\in A} Q_\alpha$ *where* Q_α *is* 0-*simple for every* $\alpha \in A$ *and let* $\nu \in \mathscr{C}(V)$. *Then* $\ker \nu = \bigcup_{\alpha\in A} \ker(\nu|_{V_\alpha})$.

Proof. Let $a \in \ker \nu$. Then $a\nu a^2$ and thus, if $a \in V_\alpha$, we have $a\nu|_{V_\alpha} a^2$ so that $a \in \ker(\nu|_{V_\alpha})$. It follows that $\ker \nu \subseteq \bigcup_{\alpha \in A} \ker(\nu|_{V_\alpha})$ and the opposite inclusion is trivial. □

The next result is of independent interest.

PROPOSITION 5.8. *Let $Q = \sum_{\alpha \in A} Q_\alpha$ where Q_α is 0-simple for every $\alpha \in A$. Then K is a congruence on $\mathscr{C}(V)$ if and only if K is a congruence on $\mathscr{C}(V_\alpha)$ for every $\alpha \in A$.*

Proof. Necessity. By Lemma 5.3, for any $\alpha \in A$, V_α is a homomorphic image of V and thus, by Proposition 3.1, K is a congruence on $\mathscr{C}(V_\alpha)$.

Sufficiency. Let $\lambda, \rho, \theta \in \mathscr{C}(V)$ be such that $\lambda K \rho$ and let $\alpha \in A$. Then

$$\ker(\lambda|_{V_\alpha}) = \ker \lambda \cap V_\alpha = \ker \rho \cap V_\alpha = \ker(\rho|_{V_\alpha})$$

and thus $\lambda|_{V_\alpha} K \rho|_{V_\alpha}$. The hypothesis implies that $\lambda|_{V_\alpha} \vee \theta|_{V_\alpha} K \rho|_{V_\alpha} \vee \theta|_{V_\alpha}$ which by Lemma 5.6 gives $(\lambda \vee \theta)|_{V_\alpha} K (\rho \vee \theta)|_{V_\alpha}$. Since this holds for any $\alpha \in A$, Lemma 5.7 implies that $\lambda \vee \theta K \rho \vee \theta$. Therefore K is a congruence on $\mathscr{C}(V)$. □

We are now ready for the main result of the paper.

THEOREM 5.9. *Let V be a regular semigroup and a strict extension of S by Q, with the multiplication determined by $\varphi : Q^* \to S$. Assume that Q is an orthogonal sum of 0-simple semigroups Q_α, $\alpha \in A$, categorical at zero. Then the following statements are equivalent.*

(i) *K is a congruence on $\mathscr{C}(V)$.*

(ii) *Letting $V_\alpha = S \cup Q_\alpha^*$, K is a congruence on $\mathscr{C}(V_\alpha)$ for every $\alpha \in A$.*

(iii) *K is a congruence on $\mathscr{C}(S)$ and for every $\alpha \in A$, K is a congruence on $\mathscr{C}(Q_\alpha)$ and either $\varphi : Q_\alpha^* \to E(S)$ or there exists $a \in Q_\alpha^*$ such that $a\varphi \in E(S)$ and $a^2 \in S$.*

Proof. The equivalence of parts (i) and (ii) is a special case of Proposition 5.8, whereas the equivalence of parts (ii) and (iii) follows directly from Theorem 4.4. □

We conclude by giving an abstract characterization of the semigroups Q appearing in the above theorem.

PROPOSITION 5.10. *Let Q be a nontrivial regular semigroup with zero. Then Q is an orthogonal sum of 0-simple semigroups if and only if for any $e, f \in E(Q^*)$, $e < f$ implies that $e \mathscr{J} f$.*

Proof. Necessity. Let $Q = \sum_{\alpha \in A} Q_\alpha$ where Q_α is 0-simple for every $\alpha \in A$ and let $e, f \in E(Q^*)$ be such that $e < f$. Then $e \in Q_\alpha$ and $f \in Q_\beta$ for some $\alpha, \beta \in A$ and $e = ef$ implies that $\alpha = \beta$. But then $e \mathscr{J} f$ since Q_α is 0-simple.

Sufficiency. Let $a, b \in Q$ be such that $ab \neq 0$, let x be an inverse of a and y be an inverse of ab. Then $(abyax)ab = ab \neq 0$ so that $e = abyax \neq 0$,

$$e^2 = (abyax)(abyax) = (abyab)yax = abyax = e,$$

$$e(ax) = (ax)e = e.$$

Hence letting $f = ax$, we obtain $e, f \in E(Q^*)$ and $e \leq f$. If $e = f$, then $e \mathscr{J} f$. Otherwise $e < f$ and the hypothesis implies that $e \mathscr{J} f$. Now $a \in J(f) = J(e) \subseteq J(ab)$ so that $J(a) \subseteq J(ab)$.

The opposite inclusion always holds and thus $a\mathcal{J}ab$. A similar argument will show that also $b\mathcal{J}ab$ so that $a\mathcal{J}b$.

We have proved that $ab \neq 0$ implies $a\mathcal{J}b\mathcal{J}ab$. By contrapositive, we conclude that Q is an orthogonal sum of nonzero $\mathcal{J}$-classes together with zero, and these are clearly 0-simple. □

REFERENCES

1. F. Pastijn and M. Petrich, Congruences on regular semigroups, *Trans. Amer. Math. Soc.* **295** (1986), 607–633.

2. M. Petrich, Congruences on extensions of semigroups, *Duke Math. J.* **34** (1967), 215–224.

3. M. Petrich, The kernel relation for a retract extension of Brandt semigroups, *Boll. Unione Mat. Ital.* **5-B** (1991), 1–19.

4. M. Petrich, The congruence lattice of an ideal extension of semigroups, *Glasgow Math. J.* **35** (1993), 39–50.

5. M. Petrich, The kernel relation for certain regular semigroups, *Boll. Unione Mat. Ital.* **7-B** (1993), 87–110.

6. M. Petrich, The congruence lattice of an extension of completely 0-simple semigroups, *Acta Math. Hung.* **64** (1994), 409–435.

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