Canad. J. Math. Vol. **63** (2), 2011 pp. 460–480
doi:10.4153/CJM-2011-004-0

Monotonically Controlled Mappings

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Abstract. We study classes of mappings between finite and infinite dimensional Banach spaces that are monotone and mappings which are differences of monotone mappings (DM). We prove a Radó–Reichelderfer estimate for monotone mappings in finite dimensional spaces that remains valid for DM mappings. This provides an alternative proof of the Fréchet differentiability a.e. of DM mappings. We establish a Morrey-type estimate for the distributional derivative of monotone mappings. We prove that a locally DM mapping between finite dimensional spaces is also globally DM. We introduce and study a new class of the so-called UDM mappings between Banach spaces, which generalizes the concept of curves of finite variation.

1 Introduction

The class of real functions defined on a real interval that can be written as a difference of two nondecreasing functions appeared for the first time in works of Camille Jordan. It is a standard result that this class coincides with the class of functions with locally finite variation. The functions of bounded variation were generalized to higher dimensions by Enrico de Giorgi and appeared to be an important tool for the study of geometric variational problems. The theory of monotone operators in Banach spaces was developed in the 1960's as a powerful method for solving nonlinear partial differential equations of elliptic or parabolic type.

The Russian geometer A. D. Alexandrov studied functions of several real variables that are a difference of two convex functions. The concept of such functions was generalized and investigated by L. Veselý and L. Zajíček in the fundamental paper [17] and further developed in [6]. These mappings have many interesting applications in differentiation or optimization theory.

We will study multi-mappings from a Banach space to its dual space that are monotone and that are differences of two monotone mappings (here called DM). We show that in finite-dimensional spaces the global and local definitions of the DM property coincide. The analogous result for monotone mappings holds in an arbitrary dimension. We prove a Morrey-type estimate for the derivative of a monotone mapping in Euclidean spaces, and a Radó-Reichelderfer type estimate for monotone mapping in the finite dimension. These estimates remain valid also for DM mappings, but they are not equivalent to the DM property as we show later. Further, we define the class of UDM mappings, which is a generalization of curves of locally finite variation. These mappings can be defined between two arbitrary Banach spaces even in the more general setting of Banach manifolds (however, this generalization is

Received by the editors October 29, 2008; revised January 10, 2010.
Published electronically February 18, 2011.
AMS subject classification: **26B05**, 46G05.
Keywords: monotone mapping, DM mapping, Radó-Reichelderfer property, UDM mapping, differentiability.

not discussed here). The class of UDM mappings also enjoys the Radó–Reichelderfer property, which is crucial in the proof of the result about the Fréchet differentiability of UDM mappings.

2 Preliminaries

Let X, Y be real Banach spaces. The norm of the Banach space X is denoted by $\|\cdot\|_X$, and for $X = \mathbb{R}^n$ we write simply $|\cdot|$. The symbol $B(x, r)$ stands for the open ball centered at the point x with the radius r. We use $\operatorname{diam} A$ for the diameter of a set $A \subset X$, and if $f\colon X \to Y$ is a mapping, we define $\operatorname{osc}(f, A) := \operatorname{diam} f(A)$. The topological dual space to X is denoted by X^*. The duality pairing between spaces X and X^* (or the scalar product for a Hilbert space X) is denoted by $\langle x^*; x\rangle$. The norm closure of a set $M \subset X$ is denoted by $\overline{M}$. Let $\Omega \subset X$ be an open set and let $f\colon \Omega \to Y$ be a mapping. A bounded linear operator $A\colon X \to Y$ is called a *Fréchet derivative* of the mapping f at a point $x \in \Omega$ provided that

$$\lim_{v \to 0} \frac{f(x+v) - f(x) - Av}{\|v\|_X} = 0$$

in the norm of the space Y. We denote the operator A by $f'(x)$.

Let $f\colon (a; b) \to Y$ be a mapping (curve). For $a < c < d < b$ we define the *variation* of f over $[c; d]$

$$\bigvee_c^d f := \sup \left\{ \sum_{i=1}^{k} \|f(t_i) - f(t_{i-1}\|_Y \right\},$$

where the supremum is taken over all partitions $c = t_0 < t_1 < \cdots < t_k = d$.

The variation of a vector valued measure μ is a nonnegative measure denoted by $|\mu|$. $\mathcal{M}(\Omega; Y)$ means the Banach space of Y-valued Radon measures on an open set $\Omega \subset \mathbb{R}^n$ with finite variation endowed with the norm $\|\mu\|_{\mathcal{M}(\Omega;Y)} := |\mu|(\Omega)$. The Lebesgue measure in $\mathbb{R}^n$ is denoted by $\mathcal{L}^n$ or $|\cdot|$.

Definition 2.1 Let U be an open subset of $\mathbb{R}^n$. We say that $u \in L^1(U)$ is a *function of bounded variation* if the distributional gradient of u is (representable by) a Radon $\mathbb{R}^n$-valued measure in U with finite total variation. We denote this function space by $BV(U)$. The generalization to mappings with values in the space $\mathbb{R}^d$ is immediate.

The following proposition can be viewed as a generalization of the well-known theorem about the Lebesgue points of a locally integrable function.

Proposition 2.2 ([16]) *Let X be a normed linear space and $\mu \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^n; X)$ be a vector measure that is singular with respect to the Lebesgue measure. Then*

$$\lim_{r \to 0+} \frac{|\mu|(B(x, r))}{|B(x, r)|} = 0$$

for $\mathcal{L}^n$-almost all $x \in \mathbb{R}^n$.

3 Monotone Mappings

3.1 Basic Properties

In the sequel the symbol $T\colon M \to 2^V$ will denote that Tm is a subset of V (possibly empty) for every $m \in M$.

Definition 3.1 Let X be a Banach space with the dual space X^* and let $T\colon X \to 2^{X^*}$ be a multi-mapping. Let us denote by $\operatorname{Dom}(T)$ the set of all points $x \in X$ such that $Tx \neq \varnothing$, and let us call it the *effective domain of* T. We say that T is a *monotone multi-mapping* if for every $x_1, x_2 \in \operatorname{Dom}(T)$ and every $x_1^* \in Tx_1, x_2^* \in Tx_2$ the inequality

$$\langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq 0 \tag{3.1}$$

is satisfied. The mapping T is called *maximal monotone* if its graph

$$\operatorname{Gr}(T) := \{(x, x^*); x \in \operatorname{Dom}(T), x^* \in Tx\}$$

is not properly contained in a graph of any monotone mapping.

Definition 3.2 (Minty [13]) Let X be a Hilbert space and let $M \subset X$ be its arbitrary subset. Let $A\colon M \subset X \to 2^X$ be a multi-mapping. The *Cayley transformation*

$$\Gamma\colon X \times X \to X \times X$$

is defined by the formula

$$\Gamma(x_1, x_2) := (x_1 + x_2, -x_1 + x_2).$$

We define the mapping $\Gamma_\sharp A$ via the equality $\operatorname{Gr}(\Gamma_\sharp A) := \Gamma(\operatorname{Gr}(A))$. Further, the mapping $\Gamma_\sharp^{-1} A$ is a mapping whose graph is $\Gamma^{-1}\operatorname{Gr}(A)$.

Proposition 3.3 ([13]) *Let M be an arbitrary subset of a Hilbert space X and let $A\colon M \to 2^X$ be a monotone multi-mapping. Then $\Gamma_\sharp A$ is 1-Lipschitz. On the other hand, for a given 1-Lipschitz mapping $\phi\colon N \subset X \to X$, the multi-mapping $\Gamma_\sharp^{-1}\phi$ is monotone.*

The following propositions are easy facts about monotone mappings and can be found in a more general form in [1].

Proposition 3.4 *Let $u\colon \operatorname{Dom}(u) \to 2^{\mathbb{R}^n}$ be a monotone mapping. Then the set of $x \in \operatorname{Dom}(u)$, where $u(x)$ is not a singleton, is a Lebesgue null set.*

Proposition 3.5 *Let $u\colon \operatorname{Dom}(u) \to 2^{\mathbb{R}^n}$ be a monotone mapping. Then u is locally bounded in* $\operatorname{int}\operatorname{Dom}(u)$.

Definition 3.6 We say that a mapping $a\colon M \to N$, where M, N are abstract sets, is a *selection of a multivalued mapping* $A\colon M \to 2^N$ if $a(m) \in A(m)$ for every $m \in M$. We write briefly $a \in A$ for a being a selection of A.

The following lemma, which is a form of the extension theorem for monotone mappings, will be useful. It is possible that this lemma is known, but since we were not able to find any reference, we sketch the proof.

Lemma 3.7 *Let $\Omega \subset \mathbb{R}^d$ be a bounded set and let $A\colon \Omega \to 2^{\mathbb{R}^d}$ be a bounded monotone multi-mapping. Then there is a monotone multi-mapping $\bar{A}\colon \mathbb{R}^d \to 2^{\mathbb{R}^d}$, with $\operatorname{Dom}(\bar{A}) = \mathbb{R}^d$, which is an extension of A, i.e., $\bar{A}_{|\Omega} = A$. In particular, for $A\colon \Omega \to \mathbb{R}^d$ single-valued there is a single-valued extension.*

Proof We define a multi- mapping $\tilde{A}\colon G := \Omega \cup (\mathbb{R}^d \setminus B(0,R)) \to 2^{\mathbb{R}^d}$, by

$$\tilde{A}(x) := \begin{cases} A(x) & x \in \Omega, \\ x & x \in \mathbb{R}^n \setminus B(0,R), \end{cases}$$

where $B(0,R) \supset \Omega$ and R is so large that the multi-mapping $\tilde{A}$ is monotone on the set G. To show that such R exists we proceed as follows. For $x, y \in \Omega$ or for $x, y \in G \setminus \Omega$ the monotonicity condition is fulfilled trivially. We choose arbitrary points $x \in G \setminus \Omega$ and $y \in \Omega,\ y^* \in A(y)$.

We have

$$\langle \tilde{A}(x) - y^*; x - y\rangle = \langle x - y^*; x - y\rangle \geq |x|^2 - |x||y| - |y^*||x| - |y||y^*|.$$

Since $|y| \leq a,\ |y^*| \leq b$ for some constants a, b, we easily see that this expression can be made nonnegative for $|x| \geq R$ if we choose R sufficiently large. It is well known that there is a maximal monotone extension $\bar{A}$ of $\tilde{A}$ such that $\operatorname{Dom}(\bar{A}) \supset G$. Now it is easy to deduce from [1, Corollary 1.3] that $\operatorname{Dom}(\bar{A}) = \mathbb{R}^d$.

Finally, for A being single-valued, an arbitrary selection of $\bar{A}$, which coincides with A on Ω, is surely a single-valued monotone extension of A. ■

Lemma 3.8 *Let $A\colon \operatorname{Dom}(A) \subset X \to 2^{X^*}$ be a multi-mapping and let $\Omega \subset \operatorname{Dom}(A)$ be a convex set. Then the restriction $A|_\Omega$ is monotone if and only if for each $x, h \in X,\ a \in A|_\Omega$ the function*

$$t \mapsto \varphi(x,h,a;t) := \langle a(x+th) - a(x), h\rangle$$

is nondecrasing on the interval $I_{x,h} := \{t \in \mathbb{R}; x + th \in \Omega\}$.

Proof It is clear that A is a monotone multivalued mapping if and only if each selection $a \in A$ is monotone. Let A be a monotone multi-mapping, $a \in A$ its arbitrary selection, and choose $x, h \in X$, $s < t$ such that $\varphi(x,h,a;t)$ and $\varphi(x,h,a;s)$ are defined. We have

$$\begin{aligned} \varphi(x,h,a;t) - \varphi(x,h,a;s) &= \langle a(x+th) - a(x), h\rangle - \langle a(x+sh) - a(x), h\rangle \\ &= \frac{1}{t-s}\langle a(x+th) - a(x+sh), (t-s)h\rangle \geq 0, \end{aligned}$$

since a is monotone.

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Conversely, let φ be nondecreasing, choose admissible $x, h \in X$ and a selection $a \in A$. We obtain

$$\langle a(x+h) - a(x), h\rangle = \varphi(x, h, a; 1) - \varphi(x, h, a; 0) \geq 0.$$

This gives the monotonicity of the multi-mapping A. ■

Definition 3.9 We say that a multi-mapping $A\colon \Omega \subset X \to 2^{X^*}$ is *locally monotone* in Ω if for every point $x_0 \in \Omega$ there is a (relative) neighborhood $U(x_0)$ of x_0 in Ω such that $A_{|U(x_0)}\colon U(x_0) \to 2^{X^*}$ is a monotone multi-mapping.

Lemma 3.10 *Let $A\colon \operatorname{Dom}(A) \to 2^{X^*}$ be a multi-mapping and $\Omega \subset \operatorname{Dom}(A)$ be a convex set. Then $A_{|\Omega}$ is monotone if and only if it is locally monotone.*

Proof The necessity is obvious.

For the sufficiency recall that, due to Lemma 3.8, it is sufficient to show that for every selection $a \in A$ and every $x, h \in X$ such that the function $\varphi(x, h, a; \cdot)$ is defined, $\varphi(x, h, a; \cdot)$ is nondecreasing on its domain. The assumption of the local monotonicity of A implies the local monotonicity of each selection $a \in A$. This gives that the function $\varphi(x, h, a; \cdot)$ is nondecreasing at every point of its domain. Thus φ is nondecreasing as follows from the standard calculus result. ■

3.2 Differential Theory for Monotone Mappings

Consider a monotone multi-mapping $u\colon \mathbb{R}^n \to 2^{\mathbb{R}^n}$. By Proposition 3.4 the set of points $x \in \operatorname{Dom}(u)$ such that $u(x)$ is a singleton is the set of full Lebesgue measure in $\operatorname{Dom}(u)$.

Theorem 3.11 (Mignot [1, 12]) *Let $u\colon \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be a maximal monotone function and let D be the set of all $x \in \operatorname{Dom}(u)$ such that $u(x)$ is a singleton. Then u is differentiable at almost every $x_0 \in D$,* i.e., *there is a matrix $u'(x_0) \in \mathbb{R}^{n\times n}$ such that*

$$\lim_{\substack{x\to x_0\\ y\in u(x)}} \frac{y - u(x_0) - u'(x_0)(x - x_0)}{|x - x_0|} = 0. \tag{3.2}$$

Remark 3.12 It is easily seen that for a single valued function u the equation (3.2) reduces exactly to the Fréchet differentiability of u at x_0. The fact that for almost every point $x \in \operatorname{Dom}(u)$ there is a matrix $u'(x)$ satisfying (3.2) follows from a combination of Theorem 3.11 and Proposition 3.4. The standard proof of Theorem 3.11 uses the Cayley transformation and the Rademacher theorem. We will provide later an alternative proof for the single valued case that uses the Radó–Reichelderfer property of monotone mappings.

Let Ω be open. It is shown in [1] that every monotone mapping $u\colon \Omega \to \mathbb{R}^n$ is of class L^∞ on every compact subset of Ω. The following result is proved in [1] by using Proposition 3.3.

Theorem 3.13 (see [1]) *Let $u\colon \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be a monotone multi-mapping, and let $\Omega \subset \overline{\Omega} \subset \operatorname{int}\operatorname{Dom}(u)$ be a bounded open set. Then u, understood as an element of $L^\infty(\Omega;\mathbb{R}^n)$, is a mapping of the class $BV(\Omega;\mathbb{R}^n)$. Moreover,*

$$\int_\Omega d|Du| \le C[\operatorname{diam}\Omega + \operatorname{osc}(u,\Omega)]^n, \tag{3.3}$$

where the constant $C = C(n)$ depends only on the dimension n.

The possibility of the following refinement of the estimate (3.3) was remarked in [1]. We provide a proof for the reader's convenience.

Theorem 3.14 *Let $\Omega \subset \mathbb{R}^n$ be an open set, let $u\colon \Omega \to \mathbb{R}^n$ be a monotone mapping, and let $B_0 \subset \overline{B_0} \subset \Omega$ be a ball. Then there are constants $C = C(n, B_0)$ and $\tilde{C} = \tilde{C}(u, n, B_0)$ such that for any ball $B(a,r) =: B \subset B_0$*

$$⨍_B d|Du| \le \frac{C\operatorname{osc}(u,B)}{r} \le \frac{\tilde{C}}{r}. \tag{3.4}$$

Proof Let u be a given monotone function and $B(a,r) =: B \subset B_0$ a ball, let

$$\lambda := \operatorname{osc}(u,B).$$

Choose a point $x_0 \in B$; consider the change of coordinates $2rx' - x_0 = x$, and denote as $B' = B\left(\frac{a+x_0}{2r}, \frac{1}{2}\right)$ the set of all points x' corresponding to all points $x \in B$. We define the monotone function $v(x') := \frac{u(x)}{\lambda}$. Now we have

$$\begin{aligned}\int_B d|Du|(x) &= \frac{\lambda}{2r}\int_B d|Dv|(x) = \frac{\lambda}{2r}\int_{B'}(2r)^n d|Dv|(x') \\ &\le C'\lambda(2r)^{n-1}(\operatorname{osc}(v,B') + \operatorname{diam}B')^n = C''\operatorname{osc}(u,B)(2r)^{n-1},\end{aligned}$$

where we have used the estimate (3.3). The second inequality in (3.4) easily follows, since $\operatorname{osc}(u,B) \le \operatorname{osc}(u,B_0)$. This concludes the proof. ∎

The previous theorem asserts a type of *Morrey estimate* for the derivative of a monotone function. Suppose that Du is representable by a locally integrable function f, then the inequality (3.4) can be read as $f \in M^{1,n-1}(B_0;\mathbb{R}^{n\times n})$. The Morrey spaces of functions have broad applications in the regularity theory of weak solutions of partial differential equations; see [9] for basic facts about Morrey spaces. In our case we have in fact proved the following.

Corollary 3.15 *Let Ω be an open set, let $u\colon \Omega \to \mathbb{R}^n$ be a monotone mapping, and let $B_0 \subset \overline{B_0} \subset \Omega$ be a ball. Then the derivative Du of the mapping u belongs to the space of measures*

$$M_*^{1,n-1}(B_0;\mathbb{R}^{n\times n}) := \left\{\tau \in \mathcal{M}(B_0;\mathbb{R}^{n\times n});\ \sup_{B(x,\rho)\subset B_0}\frac{1}{\rho^{n-1}}\int_{B(x,\rho)} d|\tau| < \infty\right\}.$$

Proof The corollary follows immediately from Theorem 3.14. ∎

The following theorem can be viewed in some sense as a reverse inequality for (3.4). Thus it gives that the expressions appearing in inequality (3.4) are comparable. Another type of L^∞- estimate for monotone mappings is mentioned in [1].

Theorem 3.16 *Let $\Omega \subset \overline{\Omega} \subset \Omega_0 \subset \mathbb{R}^n$, where Ω_0 and Ω are open sets. There exists a constant C, depending only on Ω such that*

$$\frac{\operatorname{osc}(u, B(x_0, r))}{r} \le C \fint_{B(x_0,2r)} d|Du|$$

for every monotone function $u \colon \Omega_0 \to \mathbb{R}^n$ and every ball $B(x_0, r) \subset B(x_0, 2r) \subset \Omega$.

Proof We abbreviate $B_1 := B(x_0, r)$ and $B_2 := B(x_0, 2r)$ and denote $d := \operatorname{osc}(u, B_1)$.

The set $u(B_1)$ can be covered by a finite family $\mathcal{B}$ consisting of $N = N(n)$ balls of the diameter $2\rho := \frac{2}{5}d$. We can find a ball $B := B(z, \rho) \in \mathcal{B}$ such that

$$|B_1 \cap \{u \in B\}| \ge \frac{|B_1|}{N}. \tag{3.5}$$

We denote $E := B_1 \cap \{u \in B\}$. There are two points $y, \tilde{y} \in u(B_1)$ such that $|y - \tilde{y}| \ge 4\rho$. We can suppose that

$$|y - z| \ge 2\rho \tag{3.6}$$

(at least one of the points y, $\tilde{y}$ satisfies this). Let $x \in B_1$ be a point such that $u(x) = y$. We consider the cone

$$U := \Big\{ y'; \langle y' - y; z - y\rangle \le \frac{\sqrt{2}}{2}|y - y'||y - z| \Big\}$$

and define the set

$$E' := B_2 \cap \Big\{ x'; \langle y - z; x' - x\rangle \ge \frac{\sqrt{2}}{2}|x' - x||y - z| \Big\}. \tag{3.7}$$

Take a point $x' \in E'$ and set $y' = u(x')$. We observe

$$\langle y - z; x' - x\rangle \ge \frac{\sqrt{2}}{2}|x' - x||y - z|. \tag{3.8}$$

Since u is monotone, we have

$$\langle y' - y; x' - x\rangle = \langle u(x') - u(x); x' - x\rangle \ge 0. \tag{3.9}$$

Define

$$a := \frac{x - x'}{|x - x'|}, \quad b := \frac{z - y}{|z - y|}, \quad c := \frac{y - y'}{|y - y'|}.$$

Suppose that $\langle c; -b\rangle > \frac{\sqrt{2}}{2}$. Since by (3.8) $\langle b; a\rangle \geq \frac{\sqrt{2}}{2}$, we have

$$\sqrt{2} < \langle b; a-c\rangle \leq |a-c|.$$

Taking the squares and using $|a| = |c| = 1$, we have $\langle a; c\rangle < 0$. This contradicts (3.9). Thus, we have

$$\langle z-y; y'-y\rangle \leq \frac{\sqrt{2}}{2}|z-y||y'-y|. \tag{3.10}$$

The inequality (3.10) means that $y' = u(x') \in U$, consequently $u(E') \subset U$.

We have that B is a ball of radius ρ that is contained in the cone $\mathbb{R}^n \setminus U$. The center z lies on the axis of the cone $\mathbb{R}^n \setminus U$. For each $y'' \in \partial U$ that minimizes the distance from the point z we have

$$|z-y''|^2 = |z-y|^2 - |y''-y|^2$$

and

$$\begin{aligned} |z-y''|^2 &= |z-y|^2 - 2\langle z-y; y''-y\rangle + |y''-y|^2 \\ &= |z-y|^2 + |y''-y|^2 - \sqrt{2}|z-y||y''-y|. \end{aligned}$$

Hence, we conclude $|y''-y|^2 = \frac{1}{2}|z-y|^2$. Thus, by (3.6) $|z-y''|^2 \geq 2\rho^2$. By the triangle inequality we get $\operatorname{dist}(B, U) \geq \rho(\sqrt{2}-1)$, and consequently for any $x \in E$ and any $x' \in E'$ we have

$$|u(x) - u(x')| \geq \rho(\sqrt{2}-1). \tag{3.11}$$

The relations (3.5) and (3.7) imply the existence of a constant $\alpha = \alpha(n)$ depending only on the dimension n such that

$$\alpha|E| \geq |B_2| \tag{3.12}$$

and

$$\alpha|E'| \geq |B_2|. \tag{3.13}$$

Using the estimates (3.12), (3.13), and (3.11) we have

$$\begin{aligned} d = 5\rho &\leq \frac{5}{\sqrt{2}-1}\frac{1}{|E||E'|}\int_E\int_{E'} \operatorname{dist}(U, B)\, dx dx' \\ &\leq \frac{5}{\sqrt{2}-1}\frac{1}{|E||E'|}\int_E\int_{E'} |u(x) - u(x')|\, dx dx' \\ &\leq \frac{k}{|B_2|^2}\int_{B_2}\int_{B_2} |u(x) - u(x')|\, dx\, dx'. \end{aligned} \tag{3.14}$$

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The comparability conditions (3.12) and (3.13) give that the constant k depends only on the dimension.

The right-hand side of (3.14) can be estimated by using the triangle inequality and the Poincaré inequality (see [2]) as

$$\begin{aligned}
&\frac{k}{|B_2|^2}\int_{B_2}\int_{B_2}|u(x)-u(x')|\,dx\,dx' \\
&\quad\leq \frac{k}{|B_2|^2}\int_{B_2}\left(\int_{B_2}|u(x)-u_{B_2}|\,dx+\int_{B_2}|u(x')-u_{B_2}|\,dx\right)dx' \\
&\quad\leq \frac{k}{|B_2|}\int_{B_2}\left(\gamma r \fint_{B_2} d|Du| + |u(x')-u_{B_2}|\right)dx' \\
&\quad\leq 2k\gamma r \fint_{B_2} d|Du|.
\end{aligned}$$

The proof is finished. ■

Definition 3.17 Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $u\colon \Omega \to \mathbb{R}$ be a function. We say that u satisfies a *weak Radó–Reichelderfer condition* if there is a non-negative Radon measure $\mu \in \mathcal{M}^+(\Omega)$ depending only on the function u and the set Ω such that for arbitrary balls $B(a,r) \subset B(a,2r) \subset \Omega$

$$\frac{\operatorname{osc}(u, B(a,r))}{r} \leq \fint_{B(a,2r)} d\mu. \tag{3.15}$$

We will say that u satisfies the weak Radó–Reichelderfer property with the *weight* $\mu \in \mathcal{M}^+(\Omega)$. We use the notation $u \in RR^1_*(\Omega)$. The adaptation for Banach space-valued mappings is straightforward, and we use the notation $u \in RR^1_*(\Omega; Y)$.

Remark 3.18 A class of mappings that enjoy a type of an oscillation bound similar to (3.15) appeared for the first time in the monograph [15], where the authors studied mappings satisfying the bound

$$\left(\frac{\operatorname{osc}(u, B(r))}{r}\right)^p \leq \fint \theta^p,$$

where $0 \leq \theta \in L^p$, $p = n$. The authors also gave applications for change of variables formulas in an integral. The concept was further studied and developed for other ranges of the parameter p in [11] and [3]. The generalization for not necessarily absolutely continuous measures but with $p = 1$ was done in [4].

Theorem 3.16 implies easily the following corollary.

Corollary 3.19 *Let $\Omega \subset \overline{\Omega} \subset \Omega_0 \subset \mathbb{R}^n$, where Ω_0 and Ω are open sets and let $u\colon \Omega_0 \to \mathbb{R}^n$ be a given monotone function. Then $u \in RR^1_*(\Omega; \mathbb{R}^n)$.*

Proof The corollary is an easy reformulation of Theorem 3.16. ∎

Proofs of following two observations are obvious.

Observation 3.20 Let $u\colon \Omega \to \mathbb{R}^n$ be a mapping, let $B_0 \subset \overline{B_0} \subset \Omega$ be a ball, and let $q\colon \mathbb{R}^n \to \mathbb{R}^n$ be a continuously differentiable and Lipschitz continuous mapping. If the mapping u is of the class $RR^1_*(B_0;\mathbb{R}^n)$, then the mapping $q \circ u$ is of the class $RR^1_*(B_0;\mathbb{R}^n)$ as well.

Observation 3.21 Let $u\colon \Omega \to \mathbb{R}^n$ be a mapping, let $B_0 \subset \overline{B_0} \subset \Omega$ be a ball, and let $q\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear mapping. If Du is of the class $M^{1,n-1}_*(B_0;\mathbb{R}^{n\times n})$, then $D(q\circ u)$ is of the class $M^{1,n-1}_*(B_0;\mathbb{R}^{n\times n})$ as well.

We realize in the next proposition that the mappings of the class RR^1_* have very good differentiability properties. The following observation generalizes the result that can be found in [10]. The proof can be done by a similar way as in [10]. We deduce this proposition from the result that is due to Duda [5]. Let $f\colon \Omega \subset X \to Y$, where X, Y are Banach spaces and $\Omega \subset X$ is an open set, be a mapping. Denote

$$S(f) := \left\{ x \in \Omega; \limsup_{y\to x} \frac{\|f(x)-f(y)\|_Y}{\|x-y\|_X} < +\infty \right\}.$$

The set $S(f)$ is a set where the mapping f is pointwise Lipschitz. It is proved in [5] that if the Banach space Y has the Radon–Nikodým property and $X = \mathbb{R}^n$, then f is Fréchet differentiable almost everywhere in $S(f)$.

Assume now that $f \in RR^1_*(\Omega;Y)$ and consider the difference quotient from the definition of the set $S(f)$. If we set $r := |x-y|$, we can estimate this quotient as

$$\limsup_{y\to x} \frac{\|f(x)-f(y)\|_Y}{|x-y|} \le \limsup_{r\to 0+} \frac{\operatorname{osc}(f,B(x,r))}{r} \le \limsup_{r\to 0+} \fint_{B(x,2r)} d\mu.$$

Next, by the Lebesgue–Radon–Nikodým theorem we can write $\mu = \mu^s + \theta\mathcal{L}^n$ for some $\theta \in L^1(\Omega)$. Properties of the Radon–Nikodým derivative and Proposition 2.2 imply that

$$\limsup_{y\to x} \frac{\|f(x)-f(y)\|_Y}{|x-y|} \le \theta(x)$$

for almost every $x \in \Omega$. Thus, almost every $x \in \Omega$ belongs to the set $S(f)$. This combined with a result from [5] gives the following proposition.

Proposition 3.22 *Let $f \in RR^1_*(\Omega;Y)$, where Y is a Banach space having the Radon–Nikodým property. Then f is Fréchet differentiable almost everywhere in Ω.*

Corollary 3.23 *Let $u\colon \Omega \to \mathbb{R}^n$ be a monotone mapping, where $\Omega \subset \mathbb{R}^n$ is an open set. Then u is Fréchet differentiable at almost every point of Ω.*

Proof The corollary follows immediately from Corollary 3.19 and Proposition 3.22. ∎

4 Differences of Monotone Mappings

4.1 Properties of DM Mappings

Definition 4.1 A multi-mapping $A\colon \Omega \subset X \to 2^{X^*}$ is called a *DM multi-mapping* if there exist monotone multi-mappings $A^{\uparrow}, A^{\downarrow}\colon \Omega \to 2^{X^*}$ such that for all $x \in \Omega$ it is

$$Ax \subset A^{\uparrow}x - A^{\downarrow}x.$$

Single-valued DM multi-mappings are called DM mappings. We will also say that the mapping A has the *DM property*.

Remark 4.2 We will work mainly with single-valued DM mappings. It is an easy observation that the class of DM mappings is the smallest linear space generated by the cone of monotone mappings.

Proposition 4.3 *Every Lipschitz mapping from a Hilbert space to itself is a DM mapping.*

Proof Let α be the Lipschitz constant of A. We use the Schwartz inequality to obtain

$$\begin{aligned}\big\langle (\alpha I - A)x - (\alpha I - A)y; x - y\big\rangle &= \alpha\|x-y\|^2 - \|Ax - Ay\|\|x-y\| \\ &\geq \alpha\|x-y\|^2 - \alpha\|x-y\|^2 = 0.\end{aligned}$$

Finally, we set $A = \alpha I - (\alpha I - A)$, and the conclusion follows. ■

Corollaries 3.15 and 3.19 imply two necessary conditions for the DM property of a mapping u.

Proposition 4.4 *Let $u\colon \Omega \to \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$ is an open set, be a DM mapping, and let $B_0 \subset \overline{B_0} \subset \Omega$ be a ball. Then the derivative Du belongs to the space of measures*

$$M_*^{1,n-1}(B_0; \mathbb{R}^{n\times n}),$$

and the mapping u itself belongs to the space $RR_*^1(B_0; \mathbb{R}^n)$.

Proof Let $u = u^{\uparrow} - u^{\downarrow}$, where $u^{\downarrow}$ and $u^{\uparrow}$ are monotone mappings.

The estimates for $u = u^{\uparrow} - u^{\downarrow}$ follow easily from the estimates for the monotone mappings $u^{\uparrow}$ and $u^{\downarrow}$; see Corollaries 3.15 and 3.19. ■

Later we will demonstrate that the conditions from Proposition 4.4 are not sufficient for the DM property.

Corollary 4.5 *Let $\Omega \subset \overline{\Omega} \subset \Omega_0$, where Ω_0 and Ω are open sets, and let $u\colon \Omega_0 \to \mathbb{R}^n$ be a DM mapping. Then u is Fréchet differentiable at almost every point of Ω.*

Proof The corollary follows immediately from Corollary 3.23. ■

Definition 4.6 We say that the mapping $A\colon \Omega \to X^*$ is *locally DM* if for every $x_0 \in \Omega$ there is its neighborhood $U(x_0)$ such that the restriction $A_{|U(x_0)}\colon U(x_0) \to X^*$ is DM.

The next theorem says that in the finite-dimensional case the DM property depends only on the local behavior of a mapping. A similar theorem for d.c. mappings was proved by P. Hartman in [7].

Theorem 4.7 *Let $\Omega \subset \mathbb{R}^d$ be an open convex set, and let $A\colon \Omega \to \mathbb{R}^d$ be a locally DM mapping. Then A is a DM mapping.*

Proof Let $(K_n)_{n\in\mathbb{N}}$ and $(K^n)_{n\in\mathbb{N}}$ be nondecreasing sequences of compact convex subsets of Ω such that

$$\Omega = \bigcup_{n\in\mathbb{N}} K_n = \bigcup_{n\in\mathbb{N}} K^n, \quad K^1 \subset K_1 \subset K^2 \subset K_2 \subset \cdots ,$$

and the distances $\operatorname{dist}(\partial K^j, \partial K_j)$ and $\operatorname{dist}(\partial K_j, \partial K^{j+1})$ are strictly positive.

Consider the set K_1. We find points $x_1^1, \dots, x_{j(1)}^1 \in K_1$ and $r_i^1 > 0$ such that

$$K_1 \subset \bigcup_{i=1}^{j(1)} B\Big(x_i^1, \frac{1}{4} r_i^1\Big)$$

and such that A is DM as the mapping $A\colon B(x_i^1, r_i^1) \to \mathbb{R}^d$. Thus, there are monotone mappings $k_i^1\colon B(x_i^1, r_i^1) \to \mathbb{R}^d$, for $i = 1, \dots, j(1)$ such that $k_i^1 - A\colon B(x_i^1, r_i^1) \to \mathbb{R}^d$ are monotone.

We define the sets $B_i^1 := B(x_i^1, \frac{3}{4} r_i^1) \cap K_1$, where $i = 1, \dots, j(1)$. Consider the restrictions $(k_i^1)_{|B_i^1}$ denoted again as k_i^1. These mappings are bounded (this is a consequence of basic properties of monotone mappings; see [14]) monotone mappings and thus they can be extended to monotone mappings $k_{i,1}\colon \mathbb{R}^d \to \mathbb{R}^d$ by Lemma 3.7. We define

$$k_1 := \sum_{i=1}^{j(1)} k_{i,1}.$$

We claim that this mapping is a monotone mapping such that $k_1 - A$ is monotone on the set K_1.

It is sufficient to realize that $k_1 - A$ satisfies the monotonicity inequality (3.1) for arbitrary two points that are closer than $\varepsilon := \frac{1}{4} \min\{r_i^1; i = 1, \dots, j(1)\}$. But this is easy: take x, y, $|x - y| \le \varepsilon$, then there is a ball $B(x_i^1, \frac{1}{4} r_i^1)$ that contains x, and therefore y is contained in $B(x_i^1, \frac{1}{2} r_i^1)$. The monotone mapping $k_{i,1}$ enjoys the property that $k_{i,1} - A$ is a monotone mapping on the set $B(x_i, \frac{1}{2} r_i^1)$. This gives that k_1 has the desired property as well.

Now we claim the existence of a mapping k^1 with the properties:

(i) $k^1\colon \mathbb{R}^d \to \mathbb{R}^d$ is a monotone mapping,
(ii) $k^1 - A$ is a monotone mapping on the set K_2,
(iii) $k^1 = k_1$ on the set K^1.

Assume for a moment that we have constructed such a mapping k^1. Then, by induction, we construct a sequence $(k^n)_{n\in\mathbb{N}}$ such that

(in) $k^n\colon \mathbb{R}^d \to \mathbb{R}^d$ is a monotone mapping,
(ii$n$) $k^n - A$ is a monotone mapping on the set K_{n+1},

(iii$n$) $k^n = k^{n+1}$ on the set K^{n+1}.

Hence the limit $\lim_{n\to\infty} k^n =: k$ exists uniformly on compact subsets of Ω and is monotone. Since $k = k^j$ on K^i for $j \geq i$, we have that $k - A$ is a monotone mapping on the whole set Ω.

It remains to construct the mapping k^1. For given sets $K^1 \subset K_1 \subset K^2 \subset K_2$ such that the mapping $k_1 \colon \mathbb{R}^d \to \mathbb{R}^d$ is monotone and $k_1 - A$ is monotone on the set K_1, we are looking for a mapping k^1 with the properties (i)–(iii). Consider the set $\overline{K_2 \setminus K_1}$ and the mapping $k_1 - A$. This mapping is locally DM on this set, and thus this set can be covered by a finite number of open balls $G_1^1, \dots, G_{m(1)}^1$ such that there exist monotone mappings $\varrho_1^1, \dots, \varrho_{m(1)}^1$ with the property that $\varrho_i^1 + k_1 - A$ is monotone on the set G_i^1. We can suppose that $\operatorname{dist}(G_i^1, K^1) > \varepsilon_0 > 0$, $i = 1, \dots, m(1)$, since $\operatorname{dist}(\partial K^1, \partial K_1) > 0$ by the assumption. We can again assume that ϱ_i^1 is bounded on the set G_i^1.

We define the sets $G_{1,i} := G_i^1 \cap (K_2 \setminus K_1)$. Let $i \in \{1, \dots, m(1)\}$ be fixed. Consider the mapping

$$\varrho_{1,i}(x) = \begin{cases} \varrho_i^1(x) - c_i & x \in G_{1,i}\,, \\ 0 & x \in K^1, \end{cases}$$

where the vector $c_i \in \mathbb{R}^d$ is chosen such that the mapping $\varrho_{1,i} \colon K^1 \cup G_{1,i} \to \mathbb{R}^d$ is monotone.

Let us find such vector c_i. This vector is chosen suitably if and only if it satisfies

$$\langle \varrho_i^1(x) - c_i; x - z \rangle \geq 0,\ (x, z) \in G_{1,i} \times K^1.$$

Thus, it suffices to take c_i, which solves the inequality

$$\langle c_i; x - z \rangle \leq -|\varrho_i^1(x)||x - z|,\ (x, z) \in G_{1,i} \times K^1. \tag{4.1}$$

Since the mapping ϱ_i^1 is bounded on the sets $G_{1,i}$ and the sets $K^1, G_{1,i}$ are bounded, we have that the right-hand side of (4.1) can be estimated from below by some $\alpha < 0$. The Hahn–Banach theorem (applied on the compact convex sets K^1 and $\overline{G_i^1}$) enables us to find $d_i \in \mathbb{R}^d$ and $\epsilon > 0$ such that

$$\langle d_i; x - z \rangle \leq -\epsilon,\ (x, z) \in G_{1,i} \times K^1.$$

The desired vector c_i is $-\frac{\alpha}{\epsilon} d_i$.

Let us define the mappings $\rho_{1,i} \colon \mathbb{R}^d \to \mathbb{R}^d$ as monotone extensions of $\varrho_{1,i} \colon G_i^1 \to \mathbb{R}^d$ (see Lemma 3.7). Set $\rho_1 := \sum_{i=1}^{m(1)} \rho_{1,i}$. Finally $k^1 := \rho_1 + k_1$ is the desired mapping satisfying (i)–(iii). ■

Corollary 4.8 *Let $A \colon \mathbb{R}^d \to \mathbb{R}^d$ be a locally Lipschitz mapping. Then A is a DM mapping.*

Proof This is a direct combination of Theorem 4.7 and Proposition 4.3. ■

Let us realize one consequence of the construction from the proof of Theorem 4.7. A mapping $A \colon \Omega \subset \mathbb{R}^d \to \mathbb{R}^d$ is called *potential mapping* if there exists a Gateaux differentiable function on $\mathbb{R}^d$ whose Gateaux differential is A.

Proposition 4.9 *Let $A\colon \Omega \to \mathbb{R}^d$ be a locally DM mapping and assume that for every point $x \in \Omega$ there is a neighborhood $U(x) \subset \Omega$ and a monotone control mapping k_x, which is potential, such that $(k_x - A)_{|U(x)}$ is monotone. Then there is a potential monotone mapping $k\colon \Omega \to \mathbb{R}^d$ such that $k - A$ is monotone.*

Proof Since the proof is similar to the proof of Theorem 4.7, let us only describe main changes.

The potential monotone mappings $k_{i,1}$ are gained from the well-known fact that every convex L-Lipschitz function defined on a bounded convex set can be extended to a convex (L-Lipschitz) function defined on the whole space. Further, the potential monotone mappings ϱ_i^1 that can be written as a selection of the subdifferential of some convex function $\varphi_i^1\colon G_i^1 \to \mathbb{R}$ are modified to functions $\rho_{1,i}\colon \mathbb{R}^d \to \mathbb{R}^d$ in the following way. We extend the function φ_i^1 to a convex function $\varphi^{1,i}\colon \mathbb{R}^d \to \mathbb{R}$. By adding an appropriate convex function, which is equal to zero on some large ball containing the set K^3, we can suppose that the function $\varphi^{1,i}$ has superlinear growth. Now we can find by the geometric Hahn–Banach theorem an affine function $a_i\colon \mathbb{R}^d \to \mathbb{R}$ such that

$$(\varphi^{1,i} + a_i)_{|K^1} < \min_{G_i^1}\{\varphi^{1,i} + a_i\} - 10.$$

We define

$$\varphi_{1,i} := \max\big\{\, \varphi^{1,i} + a_i; \min_{G_i^1}\{\varphi^{1,i} + a_i\} - 5\big\}\,.$$

The mappings $\rho_{1,i}$ are suitable selections of the subdifferential of $\varphi_{1,i}$. The other steps from the proof of Theorem 4.7 are the same. ■

The motivation for the following simple observation comes from analogous criteria for d.c. mappings; see [17].

Theorem 4.10 *Let $D \subset X$ be an open convex set. Then the mapping $T\colon D \to X^*$ is DM if and only if there is a monotone mapping $A\colon D \to X^*$ such that for every line segment $L = [L_0, L_0 + L_1]$,*

$$\bigvee_0^1 T_L^\star \le \bigvee_0^1 A_L^\star, \tag{4.2}$$

where $A_L^\star(t) := \langle A(L_0 + tL_1); L_1\rangle$, and $T_L^\star$ is defined analogously.

Proof Lemma 3.8 implies that the function $A_L^\star$ is nondecreasing. We set $B := A - T$ and show that B is monotone. Then it suffices to write $T = A - (A - T)$ for the proof of the DM property of T.

The monotonicity of B will be proved by showing that the function $B_L^\star$ is nondecreasing. So let $0 \le t_1 < t_2 \le 1$ be chosen. We have by the monotonicity of A, the assumption (4.2) and the definition of the variation that

$$B_L^\star(t_2) - B_L^\star(t_1) = A_L^\star(t_2) - A_L^\star(t_1) - (T_L^\star(t_2) - T_L^\star(t_1)) \ge \bigvee_{t_1}^{t_2} A_L^\star - \bigvee_{t_1}^{t_2} T_L^\star \ge 0.$$

In fact, we used the relation (4.2) for the line segment $[L_0+t_1L_1; L_0+t_2L_1]$. This gives the monotonicity of B.

For the reverse implication we set $A := T^{\uparrow} + T^{\downarrow}$, where $T = T^{\uparrow} - T^{\downarrow}$ is supposed to be DM. For arbitrary line segment L we have

$$\bigvee_0^1 T_L^\star = \bigvee_0^1 (T_L^{\uparrow\star} - T_L^{\downarrow\star}) \leq \bigvee_0^1 T_L^{\uparrow\star} + \bigvee_0^1 T_L^{\downarrow\star} = \bigvee_0^1 (T_L^{\uparrow\star} + T_L^{\downarrow\star}) = \bigvee_0^1 A_L^\star,$$

where we used that the functions $T_L^{\uparrow\star}$, $T_L^{\downarrow\star}$ are nondecreasing. ■

4.2 UDM mappings

Definition 4.11 Let X, Y be Banach spaces, and let $C \subset X$ be a set. We say that a mapping $F\colon C \to Y$ is a *UDM mapping* if there is a monotone operator $f\colon C \to X^*$ such that for every $Q \in B_{\mathcal{L}(Y,X^*)}$ the mapping $Q \circ F + f\colon C \to X^*$ is monotone. The monotone operator f is called the *control mapping for F*.

Remark 4.12 It is an easy observation that if $|a| \leq b$ and F is controlled by f, then aF is controlled by bf. Thus if we consider in Definition 4.11 the operators Q with the norm bounded by some $c > 0$, we obtain an equivalent definition, and it is also obvious that the class of UDM mappings forms a linear space. Further, it is easy to see that if $F\colon C \subset X \to Y$ is a UDM mapping controlled by f and $L\colon Y \to Z$ is a continuous affine mapping, then $L \circ F\colon C \to Z$ is a UDM mapping controlled by $\operatorname{lip}(L) f$. Finally, notice that for the case $Y = X^*$ every UDM mapping is a DM mapping. Indeed, we can write

$$F = \tfrac{1}{2}(f + F) - \tfrac{1}{2}(f - F).$$

Definition 4.13 Let $\Omega \subset X$ be an open convex subset of a Hilbert space X. Let $u\colon \Omega \to X$ be a mapping. We say that u is a δ-*monotone mapping* if there is a number $\delta > 0$ such that for all $x, y \in \Omega$

$$\langle u(x) - u(y); x - y\rangle \geq \delta \|u(x) - u(y)\|_X \|x - y\|_X. \tag{4.3}$$

Remark 4.14 The class of the δ-monotone mappings is studied in papers by L. Kovalev in detail (see for instance [8]).

Proposition 4.15 *Every δ-monotone mapping is a UDM mapping. Consequently each linear combination of δ-monotone mappings is a UDM mapping.*

Proof Let $Q \in B_{\mathcal{L}(X)}$ be arbitrary. We use (4.3) to estimate

$$\langle Q \circ u(x) - Q \circ u(y); x - y\rangle \leq \|u(x) - u(y)\|_X \|x - y\|_X \leq \Big\langle \frac{u}{\delta}(x) - \frac{u}{\delta}(y); x - y \Big\rangle.$$

Hence, u/δ is a control mapping for u, and consequently, u is a UDM mapping. ■

The following lemma will be useful.

https://doi.org/10.4153/CJM-2011-004-0 Published online by Cambridge University Press

Lemma 4.16 *Let X, Y be Banach spaces, and let $x \in X$ and $y \in Y$ be fixed. Then*

$$\|x\|_X \|y\|_Y = \sup\{\langle Qy; x\rangle; Q \in B_{\mathcal{L}(Y,X^*)}\}.$$

Proof We can suppose that $\|x\|_X = \|y\|_Y = 1$. The Hahn–Banach theorem implies the existence of functionals $x^* \in X^*$, $y^* \in Y^*$ with the norms equal to one such that $\langle x; x^*\rangle = \langle y; y^*\rangle = 1$.

We define the operator $Q_{x,y}z := \langle y^*; z\rangle x^*$. It is evident that $\|Q\| \le 1$. We have

$$\langle Q_{x,y}y; x\rangle = \langle y^*; y\rangle\langle x^*; x\rangle = 1.$$

This concludes the proof. ∎

Corollary 4.17 *Let $C \subset X$ be a set, and let $F\colon C \to Y$ and $f\colon C \to X^*$ be mappings. Then the following assertions are equivalent:*

(i) *F is UDM with a control mapping f;*

(ii) *the estimate*

$$\|F(x_1) - F(x_2)\|_Y \|x_1 - x_2\|_X \le \langle f(x_1) - f(x_2); x_1 - x_2\rangle \tag{4.4}$$

is satisfied for all $x_1, x_2 \in C$.

We will call the inequality (4.4) *the* control inequality.

Proof The corollary is an immediate consequence of Lemma 4.16. ∎

Corollary 4.18 *Let $F_n\colon C \subset X \to Y$ and $f_n\colon C \to X^*$, $n \in \mathbb{N}$ be sequences of mappings. Assume that each F_n is a UDM mapping with a control mapping f_n and that for every $x \in C$ it is $F_n(x) \to F(x)$, $f_n(x) \to f(x)$ weakly-$*$ as $n \to \infty$. Then F is a UDM mapping with the control mapping f.*

Proof We pass directly to the limit in the inequality

$$\|F_n(x) - F_n(y)\|_Y \|x - y\|_X \le \langle f_n(x) - f_n(y); x - y\rangle. \qquad ∎$$

Corollary 4.19 *Let C be an open convex subset of $\mathbb{R}^n$, and let Y be a Banach space. Assume that $F\colon C \subset \mathbb{R}^n \to Y$ is a UDM mapping. Then $F \in RR^1_*(C; Y)$.*

Proof The proof is an easy combination of Lemma 4.17 and the definition of the class RR^1_* (Definition 3.17). Let f be a control mapping for F. Let $\mu \in \mathcal{M}^+(C)$ be a weight from the RR^1_* property of the mapping f. Let $B(x_0, r) \subset B(x_0, 2r) \subset C$ be balls. We have for $x, y \in B(x_0, r)$ that

$$\frac{\|F(x) - F(y)\|_Y}{r} \le \frac{|f(x) - f(y)|}{r} \le \frac{\operatorname{osc}(f, B(x_0, r))}{r} \le \fint_{B(x_0,2r)} d\mu,$$

and we pass to the supremum for $x, y \in B(x_0, r)$. ∎

Proposition 4.20 *Let Y be a Banach space and let $F\colon (a; b) \to Y$ be a mapping. Then F is a UDM mapping if and only if F has locally finite variation.*

Proof Let F be a UDM mapping, $f\colon (a;b) \to \mathbb{R}$ be its nondecreasing control mapping and let $a < c < d < b$ be arbitrary. By using the Hahn–Banach theorem and by the control property we obtain

$$\begin{aligned}\bigvee_c^d F &= \sup\Big\{\sum_{i=1}^k \|F(t_i) - F(t_{i-1})\|_Y; c = t_0 < t_1 < \cdots t_k = d\Big\}\\ &= \sup\Big\{\sum_{i=1}^k \sup\{\langle F(t_i) - F(t_{i-1}); y^*\rangle; \|y^*\|_{Y^*} \le 1\}; c = t_0 < t_1 < \cdots t_k = d\Big\}\\ &\le \sup\Big\{\sum_{i=1}^k |f(t_i) - f(t_{i-1})|; c = t_0 < t_1 < \cdots t_k = d\Big\} = \bigvee_c^d f < +\infty.\end{aligned}$$

Conversely, let $F\colon (a;b) \to Y$ be a mapping of locally finite variation. Choose arbitrary $c \in (a;b)$ and define

$$f(t) = \begin{cases} \bigvee_c^t F & t \ge c,\\ -\bigvee_t^c F & t < c.\end{cases}$$

If $y^* \in B_{Y^*}$, $a < s < t < b$ are given, we have

$$\langle F(t) - F(s); y^*\rangle + f(t) - f(s) \ge f(t) - f(s) - \|F(t) - F(s)\|_Y \ge 0.$$

Thus, f is a control mapping for F. ■

The following two theorems present simple results about compositions of UDM mappings. Let V, X be Banach spaces, and let $L \in \mathcal{L}(V,X)$ be a bounded linear operator. The adjoint operator of L is denoted by L_*. The operator $L \in \mathcal{L}(V,X)$ is called *bounded from below* if there is $\epsilon > 0$ such that for every $v \in V$ there holds

$$\|Lv\|_X \ge \epsilon \|v\|_V.$$

Theorem 4.21 *Let V, X, Y be Banach spaces and let $D \subset V$ and $C \subset X$ be open convex sets. Let $L \in \mathcal{L}(V,X)$ be a bounded linear operator which is bounded from below and which fulfills $LD \subset C$. Let $F\colon C \to Y$ be a UDM mapping. Then the composition $F \circ L\colon D \to Y$ is a UDM mapping.*

Proof Let $\epsilon > 0$ be a non-negative number from the boundedness from below of the operator L. Let $f\colon C \to X^*$ be a monotone control mapping for F, and let $u, v \in D$ be arbitrary. We have

$$\begin{aligned}\|F\circ L(u) - F\circ L(v)\|_Y \|u - v\|_V &\le \frac{1}{\epsilon}\|F\circ L(u) - F\circ L(v)\|_Y \|Lu - Lv\|_X\\ &\le \frac{1}{\epsilon}\langle f\circ L(u) - f\circ L(v); Lu - Lv\rangle\\ &= \frac{1}{\epsilon}\langle L_*\circ f\circ L(u) - L_*\circ f\circ L(v); u - v\rangle.\end{aligned}$$

Thus we see by Corollary 4.17 that $\frac{1}{\epsilon}L_* \circ f \circ L\colon D \to V^*$ is a monotone control mapping for $F \circ L$. ■

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Theorem 4.22 *Let X, Y, Z be Banach spaces, let C, D be an open convex set of X, Y respectively. Assume that $F\colon C \to D$ is a UDM mapping with a control mapping f, and let $G\colon D \to Z$ be Lipschitz continuous with a constant ℓ_G. Then the composition mapping $G \circ F\colon C \to Z$ is a UDM mapping with the control mapping $\ell_G f$.*

Proof Let $Q \in B_{\mathcal{L}(Z,X^*)}$ and $x, y \in C$ be arbitrary. We use Corollary 4.17 and the Lipschitz continuity of G to estimate

$$\begin{aligned}\langle Q \circ G \circ F(x) - Q \circ G \circ F(y); x - y\rangle &\le \ell_G \|F(x) - F(y)\|_Y \|x - y\|_X \\ &\le \langle \ell_G f(x) - \ell_G f(y); x - y\rangle.\end{aligned}$$

Thus, $\ell_G f$ is a control mapping for $G \circ F$, and hence $G \circ F$ is a UDM mapping. ∎

The inspiration for the following theorem is the well-known fact that every curve of locally finite variation, which takes values in a Banach space with the Radon–Nikodým property, is Fréchet differentiable almost everywhere.

Theorem 4.23 *Let $C \subset \mathbb{R}^n$ be an open convex set, and let Y be a Banach space having the Radon-Nikodým property. If $F\colon C \to Y$ is a UDM mapping, then F is Fréchet differentiable almost everywhere in C.*

Proof This theorem is a consequence of Corollary 4.19 and Proposition 3.22. ∎

Remark 4.24 The straightforward generalization enables us to consider not necessarily UDM mappings but only mappings that are locally UDM, *i.e.*, for every $x \in C$ there is a ball $U(x)$ such that $F_{|U(x)}$ is a UDM mapping.

4.3 Some Examples of DM and UDM Mappings

Example 4.25 We show that in contrast to the one-dimensional case there is a mapping $u \in BV(\Omega; \mathbb{R}^n)$ that is not DM. Let $L = [L_0; L_0 + L_1] \subset \Omega$ be a closed line segment, and let $v\colon \Omega \to \mathbb{R}^n$ be a DM mapping. We denote $v_L^\star(t) = \langle v(L_0 + tL_1); L_1\rangle$. We have $\bigvee_0^1 v_L^\star < \infty$ by Lemma 3.8. Suppose that we have a function $u_1 \in BV(\Omega)$ for which $\bigvee(u_1, L) = \infty$ holds, where $L := [0, \mathbf{e}_1]$. Then we put $u := (u_1, 0, \dots, 0)$. Thus, we have $u_L^\star(t) = u_1(t\mathbf{e}_1)$ which gives $\bigvee_0^1 u_L^\star = \infty$. Thus u cannot be DM. It is well known that a mapping defined on an open subset Ω of $\mathbb{R}^n$ is a mapping of bounded variation if and only if it has a representative that has the bounded variation over almost all line sections of Ω by the lines parallel with the coordinate axis. Examples of functions of bounded variations that do not have the finite variation on all line segments are known. We construct such an example by using the results about monotone mappings.

For the transparency we will work only in the two-dimensional space but it will be clear that a similar construction can be done in a space of an arbitrary finite dimension. Let $[a; b] \subset \mathbb{R}$ be a compact interval and let $f\colon [a; b] \to \mathbb{R}$ be an arbitrary bounded function which does not have the finite variation $\bigvee_a^b f$. Consider the mapping $u\colon D := [a; b] \times \{0\} \subset \mathbb{R}^2 \to \mathbb{R}^2$ defined by the formula

$$u(x_1, x_2) = (x_1, f(x_1)).$$

https://doi.org/10.4153/CJM-2011-004-0 Published online by Cambridge University Press

It is easy to see that this mapping $u\colon D \to \mathbb{R}^2$ is monotone and bounded. Lemma 3.7 enables us to find a mapping $v\colon \mathbb{R}^2 \to \mathbb{R}^2$ that is a monotone extension of the mapping u. Using Theorem 3.13 we have that v is a mapping of locally bounded variation. Let $Q\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear mapping given by the matrix $Q = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. This linear mapping, which is in fact the anti-clock-wise rotation by the angle $\pi/2$, is obviously Lipschitz continuous; thus the composition $-w(x) := Q \circ v(x)$ is a mapping of bounded variation (see [2]). Since $\langle w(t\mathbf{e}_1); \mathbf{e}_1\rangle = -f(t)$ the mapping w cannot be DM again by Lemma 3.8.

Let us realize that this example also gives a counterexample of a DM mapping that is not a UDM mapping, and it also demonstrates that satisfying the Radó–Reichelderfer condition by the mapping v and the Morrey condition by the measure Dv is not sufficient for posing the DM property by the mapping v. Indeed, first observe that v is a UDM mapping. Theorem 4.22 asserts that $w = Q \circ v$ is a UDM mapping as well. But this is a contradiction, since w is not DM as we have ensured. Further, the mapping v fulfills the Radó–Reichelderfer condition by Corollary 3.19, and the measure Dv fulfills the Morrey estimate by Corollary 3.15. Observations 3.20 and 3.21 imply that the same conclusion holds for the mapping $w = Q \circ v$ and the measure Dw. The absence of the DM property for the mapping w was already discussed.

Finally, if we consider the mapping $z(x) := v \circ Q(x)$, we obtain for $-t \in [a; b]$,

$$\langle z(t\mathbf{e}_2); \mathbf{e}_2\rangle = \langle u(-t, 0); \mathbf{e}_2\rangle = f(-t).$$

This demonstrates the non-stability of DM mappings with respect to inner compositions.

Example 4.26 This example demonstrates some effects for monotone and UDM mappings that have no counterpart in the case of convex and d.c. functions.

It is easily seen that if $\Psi\colon X \to \mathbb{R}$ is a convex function, then Ψ is a d.c. function with the control function Ψ. We show that there is a monotone mapping $F\colon \mathbb{R}^2 \to \mathbb{R}^2$, no multiple of which can be a control mapping for F in the sense of Definition 4.11.

Let us define $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ by the formula

$$F(x) := \begin{cases} \frac{x}{|x|} & x \neq 0, \\ 0 & x = 0, \end{cases}$$

where $|\cdot|$ stands for the Euclidean norm. Let us notice that the mapping

$$x \mapsto \begin{cases} \frac{x}{|x|^\lambda} & x \neq 0, \\ 0 & x = 0, \end{cases}$$

where $\lambda < 1$, is shown to be δ-monotone (consequently by Proposition 4.15, UDM) in [8] (in a bit of a tricky way).

At first we realize that F is a monotone mapping. This follows from the fact that F is a selection of $\partial|\cdot|$.

We show that there is no $K > 0$ such that

$$\Big|\frac{x}{|x|} - \frac{y}{|y|}\Big| |x-y| \le K\Big\langle \frac{x}{|x|} - \frac{y}{|y|}; x-y\Big\rangle \tag{4.5}$$

is fulfilled for all $x, y \in \mathbb{R}^2$. We can do an analytic computation, but we can proceed in a more geometric way. The inequality (4.5) in fact means that the angle between the vectors $x-y$ and $F(x)-F(y)$ is less than or equal to $\pi/2-\epsilon$, where $\epsilon = \epsilon(K) > 0$. For $x_1, y_1 \in S^1$ set $x_t := tx_1$ and $y_s := sy_1$. For fixed $t > 1$ the angle between the vectors $x_t - y_s$ and $x_1 - y_1$ can be arbitrary close to $\pi/2$ by taking $s > 0$ and $|x_1 - y_1|$ sufficiently small . Thus, the inequality (4.5) cannot be fulfilled for any $K > 0$.

Remark 4.27 Let us note that the previous example is not too surprising. Let X be a Hilbert space, and let $\Omega \subset X$ be an open convex set. If a monotone mapping $F\colon \Omega \to X$ is a UDM mapping with some multiple of F as a control mapping, then F is necessarily δ-monotone. It is proved in [8] (the proof is not obvious) that every δ-monotone mapping is locally Hölder continuous with an exponent λ depending only on δ. This is why the discontinuous mapping cannot be controlled by its multiple in Example 4.26.

Example 4.28 We show an example of a UDM mapping which fails to be continuous in an interior point of its domain. Let us define $F(x) := \mathbf{1}_{\{0\}}$, *i.e.*, $F(0) = 1$ and $F(x) = 0$ for $\mathbb{R}^2 \ni x \neq 0$. Let $f := x/|x|$ for $x \neq 0$ and $f(0) := 0$. We have already realized in Example 4.26 that $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a monotone mapping.

We prove that F is a UDM mapping with the control mapping f. Since for $x, y \neq 0$ the control inequality

$$|F(x) - F(y)||x-y| \le \langle f(x) - f(y); x-y\rangle$$

is trivial, we need to check this inequality for $x \neq 0 = y$. But this is easy, since we have

$$|F(x) - F(0)||x| = |x| \le \Big\langle \frac{x}{|x|}; x\Big\rangle .$$

This gives the desired conclusion.

This example can be easily modified. Let $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}^2$ be a countable dense subset of $\mathbb{R}^2$, and let $F_n(x) := \frac{1}{n^2}\mathbf{1}_{\{a_n\}}$ and

$$f_n(x) := \begin{cases} \frac{x-a_n}{n^2|x-a_n|} & x \neq a_n,\\ 0 & x = a_n. \end{cases}$$

We can show as in the first part of the example that F_n is a UDM mapping with the control mapping f_n. The series $f := \sum_{n=1}^{\infty} f_n$, converges by the Weierstrass criterion, and the mapping $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a monotone mapping. The mapping $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F := \sum_{n=1}^{\infty} F_n$ is also correctly defined. We conclude by Corollary 4.18 that F is a UDM mapping with control mapping f. Thus we have found a UDM mapping that is discontinuous on the dense set.

Acknowledgments I am grateful to Jan Malý for suggesting the theme and valuable conversations. I would like also thank to Nečas Center for Mathematical Modeling financed by the Czech Ministry of Youth and Education (LC06052) for the support during work on this paper.

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