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A MAPPING PROBLEM AND J_p-INDEX. I

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1. Introduction. Indices of normal spaces with countable basis for equivariant mappings have been investigated by Bourgin [**4**; **6**] and by Wu [**11**; **12**] in the case where the transformation groups are of prime order p. One of us has extended the concept to the case where the transformation group is a cyclic group of order p^t and discussed its applications to the Kakutani Theorem (see [**10**]). In this paper we will define the J_p-index of a normal space with countable basis in the case where the transformation group is a cyclic group of order n, where n is divisible by p. We will decide, by means of the spectral sequence technique of Borel [**1**; **2**], the J_p-index of $\mathrm{SO}(n)$ where n is an odd integer divisible by p. The method used in this paper can be applied to find the J_p-index of a classical group G whose cohomology ring over J_p has a system of universally transgressive generators of odd degrees.

2. Preliminaries.

2.1. Throughout this paper, n is a positive integer divisible by a prime number p, that is, $n = p^t n''$, where $(p, n'') = 1$, and let $S = \{1, s, \ldots, s^{n-1}\}$ be a cyclic transformation group of order n *acting properly discontinuously* on a simplicial complex K. That is, for any simplex σ in K, $s^i(\sigma) \neq \sigma$ for $i = 1, 2, \ldots, n-1$.

Let Π: $K \to K' = K/S$ be a natural projection of K onto its orbit space K'. We define $\bar{\Pi}$: $C^r(K, G) \to C^r(K', G)$ by

$$(\bar{\Pi} f^r)([\sigma]_S) = \sum_{i=1}^{n} f^r(s^i\sigma)$$

for each f^r in $C^r(K, G)$, where G is an abelian group. It is clear that $\bar{\Pi}$ is onto since S acts properly discontinuously on K.

2.2. *Definition.*

$$\begin{aligned} \tau &= 1 + s + \ldots + s^{n-1}, \\ \gamma &= 1 - s, \\ s(2i) &= \tau, \\ s(2i+1) &= \gamma. \end{aligned}$$

We use τ for τ^*, τ_*, $\tau^\#$, and $\tau_\#$ and the same holds for γ, and $s(i)$. It is easy to show that Ker γ = Im τ and Ker τ = Im γ.

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3. The J_p-Smith classes of a simplicial complex.

3.1. Let J_p be the ring of integers modulo p. Let f'^0 be the unit 0-cocycle in $C^0(K', J_p)$. Since $\bar{\Pi}$ is onto, we can find f^0 in $C^0(K, J_p)$ such that $\bar{\Pi} f^0 = f'^0$.

3.2. LEMMA. *We can find a system of cochains f^i in $C^i(K, J_p)$ such that $\delta f^i = s(i+1) f^{i+1}$ for $i \geqq 0$.*

Proof. Notice that δf^0 is in Ker τ. Hence there exists f^1 such that $\delta f^0 = \gamma f^1$. Suppose that it is true for $i = k-1$, that is, $\delta f^{k-2} = s(k-1) f^{k-1}$. Since $s(k-1)\delta f^{k-1} = \delta s(k-1) f^{k-1} = \delta\delta f^{k-1} = 0$, there exists f^k such that $\delta f^{k-1} = s(k) f^k$.

3.3. LEMMA. *$f'^i = \bar{\Pi} f^i$ is a cocycle.*

Proof. Let i be an even integer; then

$$\delta f'^i([c_{i+1}]_S) = \sum_{j=1}^{n} \delta f^i(s^j c_{i+1}) = \sum_{j=1}^{n} \gamma f^{i+1}(s^j c_{i+1}) = 0.$$

Let i be an odd integer; then

$$\delta f'^i([c_{i+1}]_S) \equiv \sum_{j=1}^{n} \tau f^{i+1}(s^j c_{i+1}) \equiv \sum_{j=1}^{n} \sum_{m=1}^{n} f^{i+1}(s^{j+m} c_{i+1}) \equiv 0 \pmod{p}.$$

3.4. LEMMA. *Let $\mathscr{A}_p^i(K, s)$ be the class of f'^i defined as above in $H^i(K', J_p)$. Then $\mathscr{A}_p^i(K, s)$ is independent of the choice of f^i.*

Proof. Let $\bar{\Pi} f_1^0 = f_2^0 = \bar{\Pi} f'^0$. Then

$$0 = (\bar{\Pi} f_1^0 - \bar{\Pi} f_2^0)([c_0]_S) = \tau(f_1^0 - f_2^0)(c_0).$$

Hence, there exists c^0 in $C^0(K, J)$ such that $f_1^0 - f_2^0 = \gamma c^0$. Let $\delta f_1^0 = \gamma f_1^1$ and $\delta f_2^0 = \gamma f_2^1$, then $\gamma(f_1^1 - f_2^1) = \delta(f_1^0 - f_2^0) = \gamma \delta c^0$. Hence, there exists c^1 in $C^1(K, J_p)$ such that $f_1^1 - f_2^1 - \delta c^0 = \tau c^1$. By an inductive argument, we can show that $f_1^i - f_2^i = \delta c^{i-1} + s(i+1) c^i$. Notice that $\bar{\Pi}\tau \equiv 0 \pmod{p}$ and that $\bar{\Pi}\gamma = 0$. Thus we have $f_1'^i - f_2'^i = \bar{\Pi}(f_1^i - f_2^i) = \delta \bar{\Pi} c^{i-1}$.

3.5. *Definition.* *$\mathscr{A}_p^i(K, s)$ is called the ith J_p-Smith class of the system (K, S).*

3.6. Given two systems (K, S) and (L, S), a simplicial map g of K into L is called *an equivariant map* of the systems if $gs = sg$. An equivariant map g induces a cell map g' of K' into L' with the following commutative diagrams,

$$\begin{array}{ccc} K & \xrightarrow{g} & L \\ \Pi \big\downarrow & & \big\downarrow \Pi \\ K' & \xrightarrow{g'} & L' \end{array} \qquad \text{and} \qquad \begin{array}{ccc} C^\#(L) & \xrightarrow{g^\#} & C^\#(K) \\ \bar{\Pi} \big\downarrow & & \big\downarrow \bar{\Pi} \\ C^\#(L') & \xrightarrow{g'^\#} & C^\#(K') \end{array}$$

It is clear that $g'^*(\mathscr{A}_p^i(L, s)) = \mathscr{A}_p^i(K, s)$.

4. The J_p-index and the total index of a simplicial complex.

4.1. *ρ refers either to τ or to γ and then $\bar{\rho}$ refers either to γ or to τ.* We will write ${}^{\rho}C(K, G)$ for $\operatorname{Im} \rho$ and ${}^{\rho^{-1}}C(K, G)$ for $\operatorname{Ker} \rho$. Since $\operatorname{Ker} \rho = \operatorname{Im} \bar{\rho}$, we have ${}^{\bar{\rho}}C(K, G) \cong {}^{\rho^{-1}}C(K, G)$. Since s is simplicial, we have,

$$\text{(A)} \qquad {}^{\rho}H^*(K, G) \cong {}^{\bar{\rho}^{-1}}H^*(K, G).$$

By [**8**, p. **70**] the following sequence is exact,

$$\text{(B)} \qquad \to {}^{\rho^{-1}}H^s(K, G) \xrightarrow{i^*} H^s(K, G) \xrightarrow{\rho^*} {}^{\rho}H^s(K, G) \xrightarrow{{}_{\rho}\delta} {}^{\rho^{-1}}H^{s+1}(K, G) \to,$$

where ${}_{\rho}\delta[\rho f] = [\delta f]$. Since there is no fixed point in K, we have,

$$\text{(C)} \qquad H^s(K/S) \cong {}^{\gamma^{-1}}H^s(K) \cong {}^{\tau}H^s(K).$$

λ: $H^*(K', G) \to {}^{\tau}H^*(K, G)$ is defined by $(\lambda f')(c) = f'([c]_S)$ for each f' in $H^*(K', G)$, and λ is an isomorphism onto. Notice that $\lambda\overline{\Pi} = \tau$.

$$\mu\colon {}^{\gamma}H^*(K, G) \to {}^{\tau}H^*(K, G_p)$$

is defined by $\mu\gamma f = \mathscr{P}\tau f$, where $\mathscr{P}$: $G \to G_p = G/pG$ is a natural surjection.

4.2. *Definition. The Smith homomorphism* $\mathbf{s}(m)$ *is defined by*

$$\mathbf{s}(2m) = \lambda^{-1}{}_{\gamma}\delta \ldots {}_{\tau}\delta\lambda\colon H^i(K', G) \to H^{i+2m}(K', G),$$

and by $\mathbf{s}(2m+1) = \lambda^{-1}\mu_{\tau}\delta \ldots {}_{\tau}\delta\lambda$: $H^i(K', G) \to H^{i+2m+1}(K', G_p)$.

4.3. In (4.1) and (4.2) let $G = J_p$, then $G_p = J_p$ and we have,

$$\text{(D)} \qquad \mathbf{s}(m)\mathscr{A}_p{}^0(K, s) = \mathscr{A}_p{}^m(K, s).$$

Hence, our Smith homomorphism is an extension of that of [**5**, p. 329]. Simple calculation will show that [**5**, 134.2 (a)–(c)] holds in our case. That is,

$$\text{(E)} \qquad \begin{aligned} \mathscr{A}_p{}^{2m}(K, s) &= (\mathscr{A}_p{}^2(K, s))^m, \\ \mathscr{A}_p{}^{2m+1}(K, s) &= \mathscr{A}_p{}^1(K, s)(\mathscr{A}_p{}^2(K, s))^m, \end{aligned}$$

where powers are in the sense of cup products, if K is a finite complex.

4.4. Let X be a normal space with countable basis and let $S = \{1, s, \ldots, s^{n-1}\}$ be *a properly discontinuous group* of X onto itself, that is, $s^i(x) \neq x$ for each x in X and for $0 < i < n$. Let $\Gamma = \{U_a\colon a \in A\}$ be an open cover of X such that for each $a, b \in A$, (i) $U_a \neq U_b$ if $a \neq b$, (ii) $U_a \neq \emptyset$, (iii) if U_a is in Γ, then sU_a is in Γ, and (iv) either $U_a \cap U_b = \emptyset$ or $U_a \cap s^iU_b = \emptyset$ for $i = 1, 2, \ldots, n-1$. A covering of X satisfying the above condition is called *a P-covering* of X. With the aid of [**9**, § 3] and the paracompactness of X, the existence of such covering can easily be shown. S will induce a properly discontinuous transformation group on the nerve complex of Γ. Moreover, since the system of P-coverings of X is cofinal in the system of all open coverings of X, we get the ith J_p-Smith class $\mathscr{A}_p{}^i(X, s)$ in the Čech cohomology group $H^i(X/S, J_p)$.

4.5. We define the J_p-*index* of (X, S) as the least integer i such that $\mathscr{A}_p^i(X, s) = 0$, if it exists. We write $\nu_p(X, S) = i$. If there is no such i, we define $\nu_p(X, S) = \infty$. Let $n = p_1^{\alpha_1}p_2^{\alpha_2} \ldots p_m^{\alpha_m}$ be the prime decomposition of a positive integer n such that $0 < p_1 < \ldots < p_m$. We can define the J_{p_i}-index of (X, S) for each i. The m-tuple $\nu(X, S) = (\nu_{p_1}(X, S), \ldots, \nu_{p_m}(X, S))$ is called *the total index* of the system (X, S).

4.6. We may define the $J(p)$-*index* of X by using the coefficient group J, the ring of integers, for even dimensions in (3.1) and (3.2). That is, define $\rho(i): J \to J(i)$ to be the natural map where $J(2i) = J$ and $J(2i + 1) = J_p$. Thus Lemma 3.3 reads: "$f'^i = \rho(i)\Pi f^i$", Lemma 3.4 reads: "*Let* $\mathscr{A}^i(p)(K, s)$ *be the class of* f'^i *defined as above in* $H^i(K', J(i))$. *Then* $\mathscr{A}^i(p)(K, s) \ldots$," and so on in the case of $J(p)$-indices. Therefore, the $\Phi(k)$-index of $\mathbf{R}^N$ defined in [**10**, § III] is equal to the $J(p)$-index of $\mathbf{R}^{Nk} - \mathbf{F}_k$, where $k = p^t$, S is of period k, and $\mathbf{F}_k$ is the set of fixed points. On the other hand, as noted in [**7**], the indices considered in [**10**, § IV] are J_p-indices. The justification for this is that the proof in [**10**, § III, Lemma 3.4] implies that

$$\mathscr{A}_p^j(\mathbf{R}^{Nk} - \mathbf{F}_k, s) = 0$$

for $j \geqq N(k - 1)$ as well as $\mathscr{A}^j(p)(\mathbf{R}^{Nk} - \mathbf{F}_k, s) = 0$ for $j \geqq N(k - 1)$.

5. The J_p-index of SO(n).

5.1. In this section we calculate the J_p-index of SO(n), where $n = 2m + 1$ is an odd integer. Hence, *throughout this section, the coefficient group is* J_p. Let $n = p^t n''$ such that $(p, n'') = 1$. We also denote n/p by n'. Let

$$S = \{1, s, \ldots, s^{n-1}\}$$

be the transformation group acting on SO(n) as follows:

$$s(w_1, w_2, \ldots, w_n) = (w_2, \ldots, w_n, w_1),$$

where $(w_1, \ldots, w_n) = \bar{w}$ is an arbitrary point of SO(n); that is, $\bar{w}$ is an orthonormal n-tuple. Let $Q = \{[\mathrm{SO}(n) \times E_S]_S, B_S, \Pi\}$, where E_S is the N-universal space of S for a sufficiently large N and $B_S = E_S/S$. We may assume that E_S is compact [**3**, Chapter IV]. The bracket notation indicates cosets with respect to S. Π: $[\mathrm{SO}(n) \times E_S]_S \to B_S$ is the projection. Then Q is a principal bundle. We also have the projection

$$\Pi_1\colon [\mathrm{SO}(n) \to E_S]_S \to \mathrm{SO}(n)/S.$$

By the Vietoris-Begle theorem we have:

(F) $$H^*(\mathrm{SO}(n)/S) = H^*([\mathrm{SO}(n) \times E_S]_S).$$

Moreover, there is a spectral sequence of Π such that

$$E_\infty \cong H^*(\mathrm{SO}(n)/S),$$

(G) and that,

$$E_2^{i,j} \cong H^i(B_S, H^j(\mathrm{SO}(n))) \cong H^i(B_S) \otimes H^j(\mathrm{SO}(n)),$$

where the coefficient group is J_p. For the sake of convenience we write $H^*(B_L, J_p)$ as $B_L{}^*$. Let T be the maximal torus in $\mathrm{SO}(n)$ [**1**] and let G be the subgroup of elements of order n in T. The following results are known [**1**; **2**]:

$$H^*(\mathrm{SO}(n), J_p) \cong \wedge(u_3, u_7, \ldots, u_{4m-1}), \quad \dim u_i = i,$$
$$B^*_{\mathrm{SO}(n)} \cong J_p(v_4, v_8, \ldots, v_{4m}), \quad \dim v_i = i,$$
$$B_T{}^* \cong J_p(t_1, t_2, \ldots, t_m), \quad \dim t_i = 2,$$
$$B_G{}^* = \wedge(a_1, a_2, \ldots, a_m) \otimes J_p(b_1, b_2, \ldots, b_m), \quad \dim a_i = 1, \dim b_i = 2,$$
$$B_S{}^* = \wedge(a) \otimes J_p(b), \quad \dim a = 1, \dim b = 2,$$

where $J_p(\ )$ and $\wedge(\ )$ refer to the polynomial and to the exterior algebra, respectively. Also, $\{u_{4i-1}: i = 1, 2, \ldots, m\}$ is the set of universally transgressive generators of $H^*(\mathrm{SO}(n), J_p)$ and v_{4i} is the image of u_{4i-1} by the transgression. Let M be a compact connected group and let L be a subgroup of M.

The projection $\rho(L, M)$ of B_L onto B_M induces $\rho^*(L, M): B_M{}^* \to B_L{}^*$. By [**1**, p. 200] we have:

$$\rho^*(T, \mathrm{SO}(n))(B^*_{\mathrm{SO}(n)}) = J_p\left(\prod_{i=1}^m (1 + t_i^2)\right).$$

The passage from $B_T{}^*$ to $B_G{}^*$ is a monomorphism which replaces t_i by b_i. The passage from $B_G{}^*$ to $B_S{}^*$ is obtained by replacing b_i by ib [**6**; **7**; **10**]. Let A be the constant number in [**6**, Lemma 1]. Let $k = p^t$ and let $k' = k/p$. Also let $p = 2h + 1$.

5.2. LEMMA. $\prod_{j=1}^m (1 + (jb)^2) \equiv [1 + \mathrm{A}b^{2hk'}]^{n''} \pmod p$.

Proof. Notice that there are $n'h$ numbers of integers which are not divisible by p between 1 and m. Also notice that $(h - i)^2 \equiv (h + i + 1)^2 \pmod p$. Hence,

$$\prod_{j=1}^m (1 + (jb)^2) \equiv \prod_{j=1}^h (1 + (jb)^2))^{n'} \equiv \left[\prod_{j=1}^h (1 + j^2 b^{2k'})\right]^{n''} \equiv [1 + \mathrm{A}b^{2hk'}]^{n''} \pmod p.$$

5.3. According to [**2**, Proposition 10.3] we have:

$$E_\infty = H^*(\mathrm{SO}(n)/S) = \wedge(a) \otimes J_p(b)/\mathscr{I}(b^{2hk'}) \otimes P',$$

where $\mathscr{I}(b^{2hk'})$ is the ideal generated by $b^{2hk'}$ and

$$P' = \bigwedge(u_3, u_7, \ldots, \hat{u}_{4hk'-1}, u_{4hk'+3}, \ldots, u_{4m-1}),$$

since $\rho^*(S, \mathrm{SO}(n))v_{4i} = 0$ for $i < hk' = n''Ab^{2hk'}$ for $i = hk'$.

5.4. Since $\tau \equiv \gamma(1 + 2s + \ldots + (n-1)s^{n-2}) \pmod p$, the inclusion map of ${}^{\tau}C(X, J_p)$ to ${}^{\gamma}C(X, J_p)$ induces the map η: ${}^{\tau}H^m(X, J_p) \to {}^{\gamma}H^m(X, J_p)$. In fact, we have the following commutative diagram (cf. [**5**, p. 328]):

$$\begin{array}{ccccc}
{}^{\gamma}H^m(X, J_p) & \xrightarrow{\gamma\delta} & {}^{\tau}H^{m+1}(X, J_p) & \xrightarrow{\tau\delta} & {}^{\gamma}H^{m+2}(X, J_p) \\
\uparrow \eta & & \uparrow \mu & & \uparrow \eta \\
{}^{\tau}H^m(X, J_p) & \xrightarrow{\tau\delta} & {}^{\gamma}H^{m+1}(X, J_p) & \xrightarrow{\gamma\delta} & {}^{\tau}H^{m+2}(X, J_p)
\end{array}$$

Assume that

$$\mathscr{A}_p{}^1(\mathrm{SO}(n), s) = \lambda^{-1}\mu_\tau\delta\lambda\mathscr{A}_p{}^0(\mathrm{SO}(n), s) = 0.$$

Then $\mu_\tau\delta\lambda\mathscr{A}_p{}^0(\mathrm{SO}(n), s) = 0$ since λ^{-1} is an isomorphism. By the commutativity of the above diagram, we have $0 = {}_\gamma\delta\eta\lambda\mathscr{A}_p{}^0(\mathrm{SO}(n), s)$. Let $\rho = \gamma$ and $s = 0$ in (4.1) (B); we then have

$$\to {}^{\gamma}Z^0(\mathrm{SO}(n)) \overset{\gamma\delta}{\to} {}^{\gamma^{-1}}H^1(\mathrm{SO}(n)) \to H^1(\mathrm{SO}(n)) = 0.$$

Since ${}^{\gamma}Z^0(\mathrm{SO}(n)) \cong {}^{\gamma^{-1}}H^1(\mathrm{SO}(n)) \cong J_p$ and the exactness of the above sequence, ${}_\gamma\delta$ is an isomorphism. Hence $\eta\lambda\mathscr{A}_p{}^0(\mathrm{SO}(n), s) = 0$, which is a contradiction.

Assume that $\mathscr{A}_p{}^2(\mathrm{SO}(n), s) = \lambda^{-1}{}_\gamma\delta\,{}_\tau\delta\,\lambda\mathscr{A}_p{}^0(\mathrm{SO}(n), s) = 0$. Because of the following exact sequence (4.1) (B),

$$\to H^1(\mathrm{SO}(n)) \to {}^{\gamma}H^1(\mathrm{SO}(n)) \overset{\gamma\delta}{\to} {}^{\gamma^{-1}}H^2(\mathrm{SO}(n)) \to,$$

${}_\gamma\delta$ is an isomorphism onto. Hence, ${}_\tau\delta\lambda\mathscr{A}_p{}^0(\mathrm{SO}(n), s) = 0$. Hence

$$\mathscr{A}_p{}^1(\mathrm{SO}(n), s) = 0,$$

which is a contradiction. Hence we may consider $\mathscr{A}_p{}^1(\mathrm{SO}(n), s) = a$ and $\mathscr{A}_p{}^2(\mathrm{SO}(n), s) = b$. Hence, according to (4.3) and (4.5) we have:

$$\nu_p(\mathrm{SO}(n), S) = 4hk' = \nu_p(\mathrm{SO}(k), W),$$

where $(\mathrm{SO}(k), W)$ is a system of period $k = p^t$ (cf. [**7**, Theorem 7]).

Therefore, we have the following theorem.

5.5. Theorem. *If $n = p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_q^{\alpha_q}$ is an odd integer and*

$$p_i = 2h_i + 1 < p_j = 2h_j + 1$$

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for $i < j$, then,

$$\nu(\mathrm{SO}(n), S) = (4h_1k_1', 4h_2k_2', \ldots, 4h_qk_q'),$$

where $k_i' = p_i^{\alpha_i}/p_i$.

6. A comment on $\nu_p((\mathbf{R}^N)_*^k)$.

6.1. Let $k = p^t$ and let $S = \{1, s, \ldots, s^{k-1}\}$ act on $(\mathbf{R}^N)_*^k$ as in [**10**, p. 411]. Then, as indicated in (4.6), $I(k)(\mathbf{R}^N) = \nu_p((\mathbf{R}^N)_*^k)$, where $(\mathbf{R}^N)_*^k = \mathbf{R}^{Nk} - \mathbf{F}_k$. In [**10**, Theorem 3.2] we have $\nu_p((\mathbf{R}^N)_*^k) \leqq N(k-1)$. As a matter of fact, we have a better upper bound.

6.2. THEOREM. $\nu_p((\mathbf{R}^N)_*^k) \leqq N(p-1)p^{t-1}$.

Proof. Let $S_p = \{1, s^{k'}, \ldots, s^{(p-1)k'}\}$, where $k' = p^{t-1}$. Let I^N be the N-cube. Then we may assume that $(\mathbf{R}^N)^k = \operatorname{int} I^{Nk}$ so that $(\mathbf{R}^N)_*^k \subset (I^N)_*^k$. The inclusion is equivariant. Hence we have $\nu_p((\mathbf{R}^N)_*^k) \leqq \nu_p((I^N)_*^k)$. (Indeed we have $\nu_p((\mathbf{R}^N)_*^k) = \nu_p((I^N)_*^k)$ since i: $(I^N)_*^k \subset (\mathbf{R}^N)_*^k$ is equivariant.) $(I^N)_*^k = (I^{Nk'})_*^p = I^{Nk} -$ (the set of the fixed points under S_p). Let K be the cell complex of $I^{Nk'}$, i.e., $|K| = I^{Nk'}$. Let $K^p = K \times \ldots \times K$ (p factors) be the p-fold product complex of K. Let K_*^p be the subcomplex of K^p which consists of all cells $\sigma_1 \times \ldots \times \sigma_p$ ($\sigma_i \in K$) with no vertex of K common to all these σ_i. Then by [**11**, Theorem 1] or the method in [**10**, § II, Theorem 2.1] we may show that $|K_*^p|$ is a deformation retract of $(I^{Nk'})_*^p$. Since K_*^p is of a dimension $Nk'(p-1)$, $H^i(|K_*^p|) = 0 = H^i((I^{Nk'})_*^p)$ for $i \geqq Nk'(p-1)$. On the other hand, we have

$$\frac{(I^N)_*^k}{S} = \frac{((I^N)_*^k/S_p)}{(S/S_p)} = \frac{((I^{Nk'})_*^p/S_p)}{(S/S_p)}.$$

Applying [**3**, p. 44, Theorem 5.2] twice to the above equation, we have $H^i((I^N)_*^k/S) = 0$ for $i \geqq Nk'(p-1)$. *A fortiori*, $\mathscr{A}_p^i((I^N)_*^k, s) = 0$ for $i \geqq Nk'(p-1)$. Hence $\nu_p((I^N)_*^k) \leqq Nk'(p-1)$.

Remark 1. The results in [**10**, § IV] can be strengthened accordingly. For example, [**10**, § IV, Corollary 2.9] can be replaced by

$$\text{“}\dim D \geqq \tfrac{1}{2}p^{t-1}(p^{t+1} - 3p + 2)\text{”}.$$

Remark 2. It may be shown that a deformation retraction in question can be taken to be equivariant. However, this is not required for the proof of Theorem 6.2 by virtue of [**3**, p. 44].

For applications to mapping problems of the above type, Theorem 6.2 is sufficient. However, it may be of interest to find the exact value of the index of $(R^N)_*^k$.

6.3. LEMMA. *If S acts on S^q, a q-sphere, without fixed points, then $\mathscr{A}_p^i(S^q, s) \neq 0$ for $i \leqq q$.*

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Proof. This is immediate from (4.1) (B) and the commutative diagram in (5.4).

6.4. THEOREM. $\nu_p((\mathbf{R}^N)_*^k) = N(p-1)p^{t-1}$.

Proof. Because of Theorem 6.2, it suffices to show that $\mathscr{A}_p{}^i((\mathbf{R}^N)_*^k, s) \neq 0$ for $i \leqq N(p-1)p^{t-1} - 1$. Since $\mathbf{R}^N$ can be considered as a vector space, $(\mathbf{R}^N)_*^k$ is the space of ordered k vectors, $(v_1, \ldots, v_k)$ of $\mathbf{R}^N$ with the set of all the points of the type $(v_1, \ldots, v_{k'}, \ldots, v_1, \ldots, v_{k'})$ deleted, where, of course, $k' = p^{t-1} = k/p$. In $(\mathbf{R}^N)_*^k$ we define a subspace X by the following relations:

$$\text{(I)} \qquad \sum_{j=0}^{p-1} v_{i+jk'} = 0 \qquad \text{for } i = 1, 2, \ldots, k',$$

and

$$\text{(II)} \qquad \sum_{j=1}^{k} |v_j|^2 = 1.$$

Since X is an $(Nk'(p-1)-1)$-sphere invariant under S, $\mathscr{A}_p{}^i(X, s) \neq 0$ for $i \leqq Nk'(p-1) - 1$. Since $X \subset (\mathbf{R}^N)_*^k$, $\mathscr{A}_p{}^i(X, s) \neq 0$ induces

$$\mathscr{A}_p{}^i((\mathbf{R}^N)_*^k, s) \neq 0$$

for $i \leqq Nk'(p-1) - 1$.

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