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LATTICE PARTITIONS WITH A STRAIGHT LINE

BY

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ABSTRACT. In [1], the solution of a problem of distinct digital filter enumeration was expressed in terms of enumerating partitions of a rectangular set of lattice points with a straight line, under certain restrictions. Here, firstly, an explicit expression is derived for the number of such partitions in that and a more general case. Secondly, the asymptotic ratio of partitions to square of lattice dimensions is derived for a square lattice.

1. **Partitions for a given Λ_{MN}.** Let Λ_{MN} be a M by N lattice of points

$$\{p(x_s, y_t);\ s = 1, \dots, N;\ t = 1, \dots, M\}$$

in the xy plane with unit spacing in x and y.

Let the line $L(x,y) = ax + by + c = 0$ not pass through any lattice point, so that $L(p) \neq 0$ for all $p \in \Lambda_{MN}$.

Gradients between points in Λ_{MN} are given by

$$Gp = \begin{cases} g(i,j) = \dfrac{i}{j};\ i = -(M-1), \dots, (M-1);\ j = -(N-1), \dots, (N-1), & j \neq 0 \\ +\infty \text{ if } i > 0, j = 0 \qquad i,j \text{ coprime} \\ -\infty \text{ if } i < 0, j = 0 \end{cases}$$

Any $L(x,y)$ such that $-a/b \notin Gp$ can parition Λ_{MN} in one of $MN + 1$ ways.

DEFINITION. The partition of Λ_{MN} by $L(x, y)$ is

$$\mathbb{P}(L, \Lambda_{MN}) = \{(x_s, y_t) \colon (x_s, y_t) \in \Lambda_{MN},\ L(x_s, y_t) < 0\}$$

Hence, for all $(x_s, y_t) \in (\Lambda_{MN} - \mathbb{P}(L, \Lambda_{MN}))$, $L(x_s, y_t) > 0$.

When $-a/b$ increases from $(i/j - \epsilon)$ to $(i/j + \epsilon)$, then for sufficiently small $\epsilon > 0$, the possible partitions of Λ_{MN} (depending on c) change.

For any subset of q collinear points at gradient i/j, the order in which they can be added to $\mathbb{P}(L, \Lambda_{MN})$ is reversed. That is, if a line of gradient $(i/j - \epsilon)$, moved from left to right over the points, produces partitions

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$$\Phi; P1; P1, P2; \ldots; P1, P2, \ldots, Pq$$

then a line of gradient $(i/j + \epsilon)$, moved from left to right over the points, produces partitions

$$\Phi; Pq; P(q-1), Pq; \ldots; P2, \ldots, Pq; P1, P2, \ldots, Pq$$

That is, $(q-1)$ new partitions have been created. That is, the number of new partitions is equal to the number of immediate neighbor pairs with separation $\sqrt{i^2 + j^2}$.

Now if $(i/j) = (ai'/aj')$ for integer i, j, i', j', a and $a > 1$ then i/j spuriously provides enumeration of new partitions as $(-a/b)$ increases through (i/j), since these partitions are included in those enumerated for (i'/j').

Thus only (i/j): i, j coprime need be considered. That is, $g(i,j) = (i/j) \in Gp$.

The nearest neighbour to a point at gradient i/j (i, j coprime) is distance $|j|$ in the x-direction and distance $|i|$ in the y-direction, giving an immediate neighbour pair. Thus for a given row of points there are $(N - |j|)$ such pairs. Also, for a given column of points there are $(M - |i|)$ such pairs. Hence, for $(M - |i|)$ rows of $(N - |j|)$ pairs, given i, j coprime, the number of immediate neighbour pairs in Λ_{MN} at gradient (i/j) is $(M - |i|)(N - |j|)$.

Firstly consider L only of negative slope and $a > 0$, $b > 0$. Then $(-a/b)$ can change through $(-i/j)$, $i = 1, \ldots, (M-1)$, $j = 1, \ldots, (N-1)$ and so the number of partitions $\mathbb{P}(L, \Lambda_{MN})$ is

$$Z_{MN} = MN + 1 + \sum_{i=1}^{M-1} \sum_{j=1}^{N-1} (M-i)\cdot(N-j)\cdot b(i,j)$$

where

$$b(i,j) = \begin{cases} 1 & \text{if } i, j \text{ coprime} \\ 0 & \text{otherwise} \end{cases}$$

If L can have slope in $[-g, g]$, $g > M$, and $b > 0$, then the number of partitions is

$$\begin{aligned} Y_{MN} &= MN + 1 + \sum_{i=1}^{M-1} \sum_{j=1}^{N-1} (M-i)\cdot(N-j)\cdot b(i,j) \\ &\quad + M(N-1)\,\{\text{number of new pairs as } (-a/b) \text{ increases thro' } 0\} \\ &\quad + \sum_{i=1}^{M-1} \sum_{j=1}^{N-1} (M-i)\cdot(N-j)\cdot b(i,j) \\ &= 2MN - M + 1 + 2\cdot \sum_{i=1}^{M-1} \sum_{j=1}^{N-1} (M-i)\cdot(N-j)\cdot b(i,j) \end{aligned}$$

If L can have slope from $(-g)$ to (g), $g > M$, and a or b can be positive or negative, then

$$L(x,y) = ax + by + c = 0$$

and

$$L'(x,y) = -ax - by - c = 0$$

give different partitions. The the number of possible partitions is

$$\begin{aligned}
X_{MN} = & \sum_{i=1}^{M-1}\sum_{j=1}^{N-1}(M-i)\cdot(N-j)\cdot b(i,j) && a/b < 0,\ a < 0 \\
& + M(N-1) && \text{grad. from } 0- \text{ to } 0+ \\
& + \sum_{i=1}^{M-1}\sum_{j=1}^{N-1}(M-i)\cdot(N-j)\cdot(b(i,j) && a/b > 0,\ a > 0 \\
& + N(M-1) && \text{grad. from } >g \text{ to } <(-g) \\
& + \sum_{i=1}^{M-1}\sum_{j=1}^{N-1}(M-i)\cdot(N-j)\cdot b(i,j) && a/b < 0,\ a > 0 \\
& + M(N-1) && \text{grad. from } 0- \text{ to } 0+ \\
& + \sum_{i=1}^{M-1}\sum_{j=1}^{N-1}(M-i)\cdot(N-j)\cdot b(i,j) && a/b > 0,\ a < 0 \\
& + N(M-1) && \text{grad. from } >g \text{ to } <(-g) \\
= & \ 4MN - 2M - 2N + 4\sum_{i=1}^{M-1}\sum_{j=1}^{N-1}(M-i)\cdot(N-j)\cdot b(i,j)
\end{aligned}$$

Thus the number of partitions, $\mathbb{P}(L, \Lambda_{MN})$, when partitions with all or no points are included, is

$$T_{MN} = 2\left(2MN - (M+N) + 2\cdot\sum_{i=1}^{M-1}\sum_{j=1}^{N-1}(M-i)\cdot(N-j)\cdot b(i,j)\right) + 2$$

If the 'sense' of the line is ignored, then the number of possible partitions (with 'all' and 'none' a single possibility) is

$$R_{MN} = 2\cdot M\cdot N - (M+N) + 1 + 2\cdot\sum_{i=1}^{M-1}\sum_{j=1}^{N-1}(M-i)\cdot(N-j)\cdot b(i,j)$$

A particular case of interest, [1], is the lattice Λ^0_{MN}, which is centred on the origin. That is, point coordinates are

$$\left\{\left(s - \frac{(N+1)}{2}, t - \frac{(M+1)}{2}\right): s = 1,\ldots,N,\ t = 1,\ldots,M\right\}$$

In this case, the number of distinct members of a given class (weighted median filters [1]) is given by the number of ways, F_{MN}, of partitioning Λ^0_{MN} with a straight line of nonpositive slope, passing below and to the left of the origin. If M and N are both odd, one point of Λ^0_{MN} is at the origin and

$$F_{MN} = \frac{(Z_{MN})}{2} = \frac{\left(M\cdot N + 1 + \sum_{i=1}^{M-1}\sum_{j=1}^{N-1}(M-i)\cdot(N-j)\cdot b(i,j)\right)}{2}$$

If M or N is even, then a sufficiently small shift of an L passing through the origin (not now belonging to Λ^0_{MN}) will not change $\mathbb{P}(L, \Lambda^0_{\mathrm{MN}})$. Thus taking $F_{MN} = (Z_{MN})/2$ in such cases would only include half the number of such partitions, all of which should be included in F_{MN}.

Let B_{MN} be the number of lines $L(x,y) = ax + by + c = 0$ that pass through the origin and for which each pair

$$L'(x,y) = ax + by + c - \epsilon = 0$$

and

$$L''(x,y) = ax + by + c + \epsilon = 0$$

gives

$$\mathbb{P}(L', \Lambda^0_{\mathrm{MN}}) = \mathbb{P}(L'', \Lambda^0_{\mathrm{MN}})$$

for some sufficiently small $\epsilon > 0$. Then, as the coordinates of points in Λ_{MN} are of the form

$$\left(\frac{(2\cdot(u-1) + c(N))}{2}, \frac{(2\cdot(v-1) + c(M))}{2}\right),$$

where

$$u = -\left\lfloor\frac{N}{2}\right\rfloor + s, \quad s = 1,\ldots,N$$

$$v = -\left\lfloor\frac{M}{2}\right\rfloor + t, \quad t = 1,\ldots,M$$

$$c(X) = \begin{cases} 1 \text{ if } X \text{ even integer} \\ 0 \text{ if } X \text{ odd integer} \end{cases}$$

then

$$B_{MN} = c(M\cdot N) + \sum_{i=1}^{[\frac{N}{2}]}\sum_{j=1}^{[\frac{M}{2}]} b(2\cdot i + c(N), 2\cdot j + c(M))$$

where $\lfloor X\rfloor$ is the greatest integer less than or equal to X. (N.B. Consistent as if M and N odd, $B_{MN} = 0$.)

Thus

$$F_{MN} = \frac{(Z_{MN} + B_{MN})}{2}$$

2. **Asymptotic Value for Λ_{MM}.** Consider

$$b(i,j) = \begin{cases} 1 \text{ if } i,j \text{ coprime} \\ 0 \text{ otherwise} \end{cases}$$

$$f(i,j,M) = \frac{(M-i)\cdot(M-j)}{(M-1)^2}$$

If $(M-1) = K\cdot D$, $K > 0$, $D > 0$, K and D integer then for any k,l integer: $k = 1,\ldots,K$, $l = 1,\ldots,K$ let

$$U_{kl} = \max(f(i,j,M)),\ L_{kl} = \min(f(i,j,M))$$
$$(k-1)\cdot D + 1 \leq i \leq k\cdot D,\ (l-1)\cdot D + 1 \leq j \leq l\cdot D$$

then

$$U_{kl} = \frac{(M-((k-1)\cdot D+1))\cdot(M-((l-1)\cdot D+1))}{(M-1)^2}$$

$$L_{kl} = \frac{(M-k\cdot D)\cdot(M-l\cdot D)}{(M-1)^2}$$

and

$$U_{kl} - L_{kl} = \frac{(D-1)\cdot(2\cdot M + D - 1 - D\cdot(k+l))}{(M-1)^2}$$
$$\leq \frac{(D-1)\cdot(2\cdot M + D - 1 - D\cdot(1+1))}{(M-1)^2}$$
$$= U_{11} - L_{11} = \frac{2}{K}\cdot\left(1 - \frac{1}{2K} - \frac{1}{D} + \frac{1}{KD} - \frac{1}{2KD^2}\right) < \frac{2}{K}$$

Therefore, for any $\epsilon > 0$, K_0 can be chosen s.t. $(2/K_0) < \epsilon$ and so

$$U_{kl} - L_{kl} < \epsilon \quad (\text{if } K \geq K_0)$$

Thus

$$L_{kl} \sum_{i=(k-1)D+1}^{kD} \sum_{j=(l-1)D+1}^{lD} \frac{b(i,j)}{D^2} \leq \sum_{i=(k-1)D+1}^{kD} \sum_{j=(l-1)D+1}^{lD} \frac{f(i,j,M)\cdot b(i,j)}{D^2}$$
$$< (L_{kl} + \epsilon) \sum_{i=(k-1)D+1}^{kD} \sum_{j=(l-1)D+1}^{lD} \frac{b(i,j)}{D^2}$$

Now

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$$E_M = \sum_{i=1}^{M-1} \sum_{j=1}^{M-1} \frac{b(i,j)}{(M-1)^2} = \left(2 \cdot \sum_{i=1}^{M-1} \sum_{j=1}^{i} \frac{b(i,j)}{(M-1)^2}\right) - 1$$

as

$$b(i,i) = \begin{cases} 1 \text{ if } i = 1 \\ 0 \text{ otherwise} \end{cases}$$

Therefore

$$E_M = \frac{2}{(M-1)^2} \sum_{i=1}^{M-1} \phi(i) - 1, \quad \phi(i) \text{ is Euler function}$$

and

$$E_M = \frac{2}{(M-1)^2} \Phi(M-1) - 1$$

and from thm. 330 in [2]

$$E_M = \frac{6}{\pi^2} + 0\left(\frac{\log(M-1)}{(M-1)}\right)$$

That is,

$$\lim_{M\to\infty} E_M = 2 \cdot \lim_{M\to\infty} \frac{\Phi(M-1)}{(M-1)^2} - 1 = \frac{6}{\pi^2}$$

Thus for any $\zeta > 0$ and for sufficiently large D_1, then for any $D > D_1$,

$$\left| \frac{\sum_{i=1}^{kD} \sum_{j=1}^{kD} b(i,j) - \sum_{i=1}^{(k-1)D} \sum_{j=1}^{(k-1)D} b(i,j)}{(k^2 \cdot D^2 - (k-1)^2 \cdot D^2)} - \frac{6}{\pi^2} \right| < \zeta$$

That is,

$$\left| \frac{2 \cdot \sum_{i=(k-1)D+1}^{kD} \sum_{j=1}^{i} b(i,j)}{(2 \cdot k - 1) \cdot D^2} - \frac{6}{\pi^2} \right| < \zeta$$

But for any i, the unit values of $b(i,j)$ are approximately uniformly distributed in j. Thus

$$\left| \frac{\sum_{i=(k-1)D+1}^{kD} \sum_{j=(l-1)D+1}^{lD} b(i,j)}{D^2} - \frac{\sum_{i=(k-1)D+1}^{kD} \sum_{j=1}^{i} b(i,j)}{(k-\frac{1}{2}) \cdot D^2} \right|$$

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can be made as small as required for sufficiently large D.
Thus for any $\delta > 0$, there exists D_0 s.t. for all $D \geqslant D_0$

$$\left| \sum_{i=(k-1)D+1}^{kD} \sum_{j=(l-1)D+1}^{lD} \frac{b(i,j)}{D^2} - \frac{6}{\pi^2} \right| < \delta$$

So

$$L_{kl}\left(\frac{6}{\pi^2} - \delta\right) < \sum_{i=(k-1)D+1}^{kD} \sum_{j=(l-1)D+1}^{lD} \frac{f(i,j,M)\cdot b(i,j)}{D^2} < (L_{kl} + \epsilon)\cdot\left(\frac{6}{\pi^2} + \delta\right)$$

Now,

$$\frac{1}{K^2}\sum_{k=1}^{K}\sum_{l=1}^{K} L_{kl} = \frac{1}{K^2}\sum_{k=1}^{K}\sum_{l=1}^{K} \frac{(M - k\cdot D)\cdot(M - l\cdot D)}{(M-1)^2}$$

$$= \frac{K^2\cdot(2\cdot M - (K+1)\cdot D)^2}{4\cdot K^2\cdot(M-1)^2} = \frac{1}{4}\left(1 - \frac{1}{K} + \frac{2}{KD}\right)^2$$

So

$$\frac{1}{K^2}\sum_{k=1}^{K}\sum_{l=1}^{K} L_{kl}\left(\frac{6}{\pi^2} - \delta\right) = \frac{1}{4}\left(1 - \frac{1}{K} + \frac{2}{KD}\right)^2\cdot\left(\frac{6}{\pi^2} - \delta\right) = \mathcal{L}$$

and

$$\frac{1}{K^2}\sum_{k=1}^{K}\sum_{l=1}^{K} (L_{kl} + \epsilon)\cdot\left(\frac{6}{\pi^2} + \delta\right) = \left(\frac{1}{4}\left(1 - \frac{1}{K} + \frac{2}{KD}\right)^2 + \epsilon\right)\cdot\left(\frac{6}{\pi^2} + \delta\right) = \mathcal{R}$$

and as ϵ, δ can be as small as required for large enough K, D (e.g. choose $\epsilon = (2/K)$) then if $K = D$, then as $M \to \infty$, $K \to \infty$ and $D \to \infty$.
Thus $\mathcal{L}$ and $\mathcal{R}$ can be made as close to $(6/4\pi^2) = (3/2\pi^2)$ as required.
So

$$\frac{1}{K^2}\sum_{k=1}^{K}\sum_{l=1}^{K}\sum_{i=(k-1)D+1}^{kD}\sum_{j=(l-1)D+1}^{lD} \frac{(M-i)\cdot(M-j)\cdot b(i,j)}{D^2\cdot(M-1)^2}$$

$$= \sum_{i=1}^{M-1}\sum_{j=1}^{M-1} \frac{(M-i)\cdot(M-j)\cdot b(i,j)}{(M-1)^4} \to \frac{1}{4}\cdot\frac{6}{\pi^2} = \frac{3}{2\pi^2}$$

as $M \to \infty$ with $K = D$ and $KD = (M-1)$.
So

$$\lim_{M\to\infty} \frac{\sum_{i=1}^{M-1}\sum_{j=1}^{M-1} f(i,j,M)\cdot b(i,j)}{\sum_{i=1}^{M-1}\sum_{j=1}^{M-1} f(i,j,M)} = \frac{6}{\pi^2}$$

and as M increases

$$T_{MM} \sim \frac{6 \cdot M^4}{\pi^2} \qquad Y_{MM} \sim \frac{3 \cdot M^4}{\pi^2}$$

$$Z_{MM} \sim \frac{3 \cdot M^4}{2 \cdot \pi^2} \qquad F_{MM} \sim \frac{3 \cdot M^4}{4 \cdot \pi^2}$$

References

1. Brownrigg, D. R. K., *The Weighted Median Filter*, CACM, Vol. 27, No. 8, (1984), pp. 807–818.

2. Hardy, G. H., and Wright, E. M., *An Introduction to the Theory of Numbers*, O.U.P. (1979).

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