FAILURE OF BETH'S THEOREM IN RELEVANCE LOGICS

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Abstract. Beth's theorem equating explicit and implicit definability fails in all logics between Meyer's basic logic **B** and the logic **R** of Anderson and Belnap. This result has a simple proof that depends on the fact that these logics do not contain classical negation; it does not extend to logics such as **KR** that contain classical negation. Jacob Garber, however, showed that Beth's theorem fails for **KR** by adapting Ralph Freese's result showing that epimorphisms may not be surjective in the category of modular lattices. We extend Garber's result to show that the Beth theorem fails in all logics between **B** and **KR**.

§1. Introduction. In a paper dedicated to the memory of Helena Rasiowa, the present author showed [24] that Beth's theorem equating implicit and explicit definability fails in all of the logics between $\mathbf{B} + 22$ and **R**, where **B** is the basic relevant logic defined by Meyer [2, sec. 48.7] and **22** is the transitivity axiom

$$[(A \to B) \wedge (B \to C)] \to (A \to C).$$

Blok and Hoogland [5] later improved this result by replacing $\mathbf{B} + 22$ with the basic logic **B**.

The results described in the previous paragraph have simple proofs that depend on the fact that the logics in question do not contain classical negation, and so we can exploit the fact that relative complements in distributive lattices are implicitly but not explicitly definable. These proofs do not work in logics containing classical negation, such as the logic **KR** that results by adding the axiom *ex falso quodlibet*, $(A \wedge \neg A) \to B$, to **R**.

The paper [24] by the present author suggested a way in which to extend the results to logics containing classical negation, such as **KR**. It is known that Beth's theorem in a logic L is closely linked to the question of whether epimorphisms are surjective in the category of algebras corresponding to L. (Blok and Hoogland [5] give an excellent exposition of the connection.) In the case that L is **KR**, the corresponding category is that of *Boolean monoids*. It follows that if we can show that epimorphisms are not necessarily surjective in the category of Boolean monoids, then we have shown that Beth's theorem fails in **KR**.

A technique for solving this last problem was provided by a construction of Roger Maddux [16]. Maddux showed how to represent any modular lattice as a lattice of

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commuting equivalence elements of some relation algebra, answering a query [13] of Bjarni Jónsson. Maddux's construction was rediscovered by the present author in [25].

Ralph Freese showed [8] that epimorphisms are not necessarily surjective in the category of modular lattices; this leads to the conjecture in [24, 25] that his result could be adapted to show the corresponding result for Boolean monoids. This plan was carried out successfully by Jacob Garber in a recent excellent contribution [9]. In the present paper, we give our version of Garber's proof as part of a demonstration showing that the failure of Beth's property can be proved for any logic intermediate between **R** and **KR**. Combining this with the earlier result of Blok and Hoogland [5], this demonstrates the failure of the Beth property for any logic between **B** and **KR**.

§2. De Morgan and Boolean monoids.

DEFINITION 2.1. *A* De Morgan monoid *is an algebra* $\langle S, +, \wedge, \vee, \neg, 0, \top, \bot \rangle$, *where we define* $a \to b = \neg(a + \neg b)$, *and* $a \leq b \Leftrightarrow a \vee b = b$, *and the algebra satisfies the postulates*:

1. $\langle S, \wedge, \vee, \top, \bot \rangle$ *is a distributive lattice with largest element* $\top$ *and least element* $\bot$.
2. $\langle S, +, 0 \rangle$ *is a commutative monoid with* 0, *so that* $0 + a = a$, *for all* $a \in S$, *and in addition*, $a \leq a + a$, *for all* $a \in S$.
3. *For all* $a, b \in S$, $\neg\neg a = a$ *and* $\neg(a \vee b) = \neg a \wedge \neg b$.
4. *For* $a, b, c \in S$, $a + (b \vee c) = (a + b) \vee (a + c)$ *and* $a + (a \to b) \leq b$.

A Boolean monoid *is a De Morgan monoid satisfying* $a \vee \neg a = \top$, *for all* $a \in S$, *so that* $\langle S, \wedge, \vee, \neg, \top, \bot \rangle$ *forms a Boolean algebra*.

A *morphism* in the category of Boolean monoids is a Boolean monoid homomorphism. The *category of Boolean monoids* is the category whose objects are Boolean monoids, and the morphisms are Boolean monoid homomorphisms.

We say that a Boolean monoid $\langle S, +, \wedge, \vee, \neg, 0, \top, \bot \rangle$ is *complete and atomic* if the underlying Boolean algebra $\langle S, \wedge, \vee, \neg, \top, \bot \rangle$ is complete and atomic. In a complete and atomic Boolean monoid, we can define an infinitary analog of the linear sum $a + b$: for $F \subseteq S$,

$$\sum F = \bigvee \{a + b : a, b \in F\}.$$

We use the $\sum$ notation below for linear joins of subsets of complete lattices.

The category $\mathcal{M}$ has as objects the complete atomic Boolean monoids, and as morphisms, the complete morphisms in the category of Boolean monoids. If K, L are complete lattices, then K is a *complete sublattice of* L if $\bigwedge X \in K$ and $\sum X \in K$ for all $X \subseteq K$, where $\bigwedge X$ and $\sum X$ are formed in L.

Boolean monoids are described under the name "**KR**-algebras" in [23]. In the literature of relevance logic, the operation $a + b$ is described as "fusion," and written as $a \circ b$, whereas in the literature of relation algebras, $a + b$ corresponds to relative product $a;b$. Boolean monoids can also be described as dense symmetric relation algebras.

The constant 0 corresponds to the constant t in the literature of relevance logic [1, p. 342], where t is interpreted as the conjunction of all logical truths. In the context of relation algebras, 0 corresponds to the constant $1'$, interpretable as the identity relation. In the literature of linear logic [10, p. 182] the corresponding constant is **1**.

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We use the notation $a + b$ to distinguish between the Boolean join $\vee$, and linear join $+$. This notation is also used in the context of modular lattice theory [17], [8]. In addition, we employ juxtaposition ab as an alternative notation for $a \wedge b$, and $\overline{a}$ for $\neg a$.

DEFINITION 2.2. *Let A be a formula of* **R**, *and $\mathcal{D}$ a De Morgan monoid. An* interpretation *of A in $\mathcal{D}$ is a function φ mapping subformulas of A into $\mathcal{D}$ that satisfies the following conditions*: $\varphi(A \to B) = \varphi(A) \to \varphi(B)$, $\varphi(A \wedge B) = \varphi(A) \wedge \varphi(B)$, $\varphi(A \vee B) = \varphi(A) \vee \varphi(B)$, $\varphi(A + B) = \varphi(A) + \varphi(B)$, $\varphi(\neg A) = \neg\varphi(A)$, $\varphi(t) = 0$, $\varphi(\top) = \top$ *and* $\varphi(\bot) = \bot$.

Let $\mathcal{M}$ be a family of De Morgan monoids. A formula A is said to be *valid in* $\mathcal{M}$ if $\varphi(A) \geq 0$ for all interpretations $\varphi(A)$ in a De Morgan monoid belonging to $\mathcal{M}$. For the notion of validity just defined, we have the following fundamental completeness theorem, due to Dunn [7],[1, sec. 28.2].

THEOREM 2.1. *A formula is provable in* **R** *if and only if it is valid in the class of all De Morgan monoids.*

For Boolean monoids, a similar result holds the following theorem.

THEOREM 2.2. *A formula is provable in* **KR** *if and only if it is valid in the class of all Boolean monoids.*

Proof. This can be proved by an extension of Dunn's completeness theorem in Theorem 2.1, since the axiom $(A \wedge \neg A) \to B$ implies that the Lindenbaum algebra of **K***R* is a Boolean algebra. □

A lattice is *modular* if it satisfies the implication

$$x \geq z \Rightarrow x \wedge (y + z) = (x \wedge y) + z.$$

For background on modular lattice theory, the reader can consult the texts of Birkhoff [4] or Grätzer [11].

DEFINITION 2.3. *Let $\mathfrak{A}$ be a De Morgan monoid. The family $\mathcal{L}(\mathfrak{A})$ is defined to be the elements of $\mathfrak{A}$ that are ≥ 0 and idempotent, that is to say, $a \in \mathcal{L}(\mathfrak{A})$ if and only if $a + a = a$ and $0 \leq a$.*

THEOREM 2.3. *If $\mathfrak{A}$ is a De Morgan monoid, then $\mathcal{L}(\mathfrak{A})$, ordered by containment, forms a lattice, with least element* 0, *and the lattice operations of meet and join defined by $a \wedge b$ and $a + b$. If $\mathfrak{A}$ is Boolean, then $\mathcal{L}(\mathfrak{A})$ is modular.*

Proof. The lattice properties of $\mathcal{L}(\mathfrak{A})$ are easily proved from the assumptions $0 \leq a$ and $a + a = a$, for all $a \in \mathcal{L}(\mathfrak{A})$. For the modular law, see the proof of Lemma 5.1 below. □

The following lemma, due to Robert K. Meyer, provides a useful characterization of the elements of $\mathcal{L}(\mathfrak{A})$.

LEMMA 2.1. *Let $\mathfrak{A}$ be a De Morgan monoid. Then the following conditions are equivalent*:

1. $a \in \mathcal{L}(\mathfrak{A})$.
2. $a = (a \to a)$.
3. $\exists b[a = (b \to b)]$.

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Proof. ($\mathbf{1} \Rightarrow \mathbf{2} \Rightarrow \mathbf{3}$): Since $0 \leq a$, we have $0 \leq (a \rightarrow a) \rightarrow a$, $0 + (a \rightarrow a) \leq a$, hence $(a \rightarrow a) \leq a$. Since $a + a \leq a$, $a \leq (a \rightarrow a)$, so $a = (a \rightarrow a)$, proving the second and hence the third condition.

($\mathbf{3} \Rightarrow \mathbf{1}$): First, we have $0 \leq (b \rightarrow b) = a$. Second, $(b \rightarrow b) \leq (b \rightarrow b) \rightarrow (b \rightarrow b)$, so $(b \rightarrow b) + (b \rightarrow b) \leq (b \rightarrow b)$, that is to say, $a + a \leq a$, so $a + a = a$. □

2.1. Duality for Boolean monoids.

DEFINITION 2.4. *A* geometrical frame $\mathcal{F} = \langle S, C, 0 \rangle$ *is a* 3*-place relation* C *on a set containing a distinguished element* 0, *satisfying the postulates*:

1. $C0xy \Leftrightarrow x = y$.
2. $Cxxx$.
3. $Cxyz \Rightarrow (Cyxz \;\&\; Cxzy)$ (*total symmetry*).
4. $(Cxyz \;\&\; Czuv) \Rightarrow \exists w(Cxuw \;\&\; Cwyv)$ (*Pasch's postulate*).

In a geometrical setting, Pasch's postulate states that if a line intersects two sides of a triangle, then it also intersects the third side, a postulate introduced by Moritz Pasch in his rigorous axiomatization [18] of classical geometry. Geometrical frames were called "**KR** model structures" in [25]. We have renamed them here to emphasize their connection with geometry; these frames can be built from a projective space by adding a zero element with some added postulates, a construction due to Roger Lyndon [15], and employed in various papers [21–23] by the present author. The relation $Cxyz$ can be interpreted as "x, y, z are collinear."

Given a geometrical frame $\mathcal{F} = \langle S, C, 0 \rangle$, we can define an algebra $\mathfrak{A}(\mathcal{F})$ as follows.

DEFINITION 2.5. *The algebra* $\mathfrak{A}(\mathcal{F}) = \langle \mathcal{P}(S), \cap, \cup, \neg, \top, \bot, \mathbf{0}, + \rangle$ *is defined on the Boolean algebra* $\langle \mathcal{P}(S), \cap, \cup, \neg, \top, \bot \rangle$ *of all subsets of* S, *where* $\mathbf{0} = \{0\}$, *and the operator* $A + B$ *is defined by*

$$A + B = \{c \mid \exists a \in A, b \in B(Cabc)\}.$$

LEMMA 2.2. *If* $\mathcal{F}$ *is a geometrical frame, then* $\mathfrak{A}(\mathcal{F})$ *is a complete atomic Boolean monoid.*

Proof. As mentioned above, the operator $A + B$ is the algebraic counterpart of the fusion connective in relevance logics. Hence, the proof that $\mathfrak{A}(\mathcal{F})$ is a Boolean monoid can be adapted from the soundness proofs [19][2, sec. 48] for relevance logics relative to the ternary relational semantics of Routley and Meyer, with some added details to take account of the characteristic axiom of **KR**. □

DEFINITION 2.6. *Let* $\mathcal{F}_1 = \langle S_1, C_1, 0 \rangle$ *and* $\mathcal{F}_2 = \langle S_2, C_2, 0 \rangle$ *be geometrical frames. A mapping* ψ *from* S_1 *to* S_2 *is said to be a* geometric morphism *if the following conditions are satisfied*:

1. $\psi(x) = 0 \Leftrightarrow x = 0$.
2. $C_1xyz \Rightarrow C_2\psi(x)\psi(y)\psi(z)$.
3. $C_2xy\psi(z) \Rightarrow \exists uv(C_1uvz \;\&\; \psi(u) = x \;\&\; \psi(v) = y)$.

LEMMA 2.3. *Let* $\mathcal{F}_1 = \langle S_1, C_1, 0 \rangle$ *and* $\mathcal{F}_2 = \langle S_2, C_2, 0 \rangle$ *be geometrical frames, and* ψ *a geometric morphism from* $\mathcal{F}_1$ *to* $\mathcal{F}_2$. *For* $A \subseteq S_2$, *define* $\varphi(A) = \psi^{-1}(A)$. *Then*

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1. *φ is a complete morphism from $\mathfrak{A}(\mathcal{F}_2)$ to $\mathfrak{A}(\mathcal{F}_1)$ in the category of complete atomic Boolean monoids.*
2. *If ψ is surjective, then the mapping φ is a complete embedding of $\mathfrak{A}(\mathcal{F}_2)$ in $\mathfrak{A}(\mathcal{F}_1)$.*

Proof. This is again a straightforward verification. We prove the equation $\varphi(A + B) = \varphi(A) + \varphi(B)$, where $A, B \subseteq S_2$.

If $z \in \varphi(A) + \varphi(B)$, then there are $x, y \in S_1$ so that $x \in \varphi(A)$, $y \in \varphi(B)$ and C_1xyz. By condition 2, we have $C_2\psi(x)\psi(y)\psi(z)$. Since $\psi(x) \in A$ and $\psi(y) \in B$, it follows that $\psi(z) \in A + B$, so that $z \in \varphi(A + B)$.

Conversely, assume that $z \in \varphi(A + B)$, so that $\psi(z) \in A + B$, showing that there are $x, y \in S_2$ so that $x \in A$, $y \in B$ and $C_2xy\psi(z)$. By condition 3, there are $u, v \in S_1$ where C_1uvz, $\psi(u) = x$, and $\psi(v) = y$. Hence, $u \in \varphi(A)$, $v \in \varphi(B)$, showing that $z \in \varphi(A) + \varphi(B)$. □

§3. Modular lattices and geometric morphisms.

3.1. Geometric frames from modular lattices. We require a few basic lattice-theoretic definitions here. A *chain* in a lattice $L = \langle S, \wedge, +, 0\rangle$ is a totally ordered subset of L; the length of a finite chain C is $|C| - 1$. A chain C in a lattice L is *maximal* if for any chain D in L, if $C \subseteq D$ then $C = D$. If L is a lattice, $a, b \in L$ and $a \leq b$, then the *interval* $[a, b]$ is defined to be the sublattice $\{c : a \leq c \leq b\}$.

Let L be a lattice with least element 0. We define the *height* function: for $a \in L$, let $h(a)$ denote the length of a longest maximal chain in $[0, a]$ if there is a finite longest maximal chain; otherwise put $h(a) = \infty$. If L has a largest element 1, and $h(1) < \infty$, then L has *finite height*.

Let L be a modular lattice with 0 of finite height. Then for $a \in L$, $h(a)$ is the length of *any* maximal chain in $[0, a]$. In addition, the height function in L satisfies the condition

$$h(a) + h(b) = h(a \wedge b) + h(a + b),$$

for all $a, b \in L$. For a lattice of finite height, this condition is equivalent to modularity; see Grätzer [11, chap. IV, sec. 2].

The next construction, due to Roger Maddux [16], was rediscovered by the present author in [25].

Definition 3.1. *Let L be a lattice with least element* 0. *Define a ternary relation C_L on the elements of L by*:

$$C_L abc \Leftrightarrow a + b = b + c = a + c,$$

and let $\mathcal{F}(L)$ be $\langle L, C_L, 0\rangle$.

Theorem 3.1. *$\mathcal{F}(L)$ is a geometric frame if and only if L is modular.*

Proof. This is a straightforward verification, using the modularity of L to validate the Pasch postulate. Theorem 2.7 of the paper [25] contains a detailed proof of the equivalence. □

Definition 3.2. *If L is a lattice, then an* ideal *of L is a non-empty subset I of L such that*

1. *If $a, b \in I$ then $a + b \in I$.*
2. *If $b \in I$ and $a \leq b$, then $a \in I$.*

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The family of ideals of a lattice L, ordered by containment, forms a complete lattice $I(L)$. The original lattice L is embedded in $I(L)$ by mapping an element $a \in L$ into the *principal ideal* containing a, $(a]_L = \{b \in L \mid b \leq a\}$. The mapping $a \longmapsto (a]_L$ is a lattice isomorphism between L and a sublattice of $I(L)$. Theorem 2.9 of [25] proves the next result.

THEOREM 3.2. *Let L be a modular lattice with least element 0, and $\mathcal{F}(L) = \langle L, C, 0\rangle$ the geometric frame constructed from L. Then $\mathcal{L}(\mathfrak{A}(\mathcal{F}(L)))$ is identical with the lattice of ideals of L.*

3.2. Geometric morphisms between modular lattices. Let K, L be complete modular lattices with K a complete sublattice of L; K and L have a common least element 0. This is because

$$0_K = \sum_K \emptyset = \sum_L \emptyset = 0_L.$$

Similarly, they have a common greatest element 1. We shall show that $\mathcal{F}(K)$ is the image of a geometric morphism defined on $\mathcal{F}(L)$.

DEFINITION 3.3. *Let K, L be complete modular lattices with K a complete sublattice of L. We define a* projection function *π from L to K: for $x \in L$,*

$$\pi(x) = \bigwedge S_x = \bigwedge\{b \in K : x \leq b\}.$$

Definition 3.3 is known in the context of Boolean algebras with operators. Jónsson and Tarski [14] employ the corresponding operation in their characterization (Definition 1.20) of closed elements in complete atomic Boolean algebras. If we think of S_x as the family of elements approximating x in K, then $\pi(x)$ is the closest approximation to x in K.

The use of the term "projection function" can be justified as follows. For F a field, let F^I and F^J, where $J \subseteq I$, be vector spaces over F, and K and L the lattices of subspaces of F^J and F^I. For $f \in F^I$, let $\psi(f) = f \restriction J$ be the restriction map from F^I to F^J, so that K is embedded in L by the mapping $\varphi(Z) = \psi^{-1}(Z)$, for $Z \in K$. Then if $X \in L$, $\pi(X)$ is the projection of X in the sublattice $\varphi(X)$.

LEMMA 3.1. *The projection function π from L to K satisfies the following conditions. For $x, y \in L$:*

1. $x \leq \pi(x)$.
2. $\pi(x) = x$ *if and only if* $x \in K$.
3. *If* $x \leq y$, *then* $\pi(x) \leq \pi(y)$.
4. *If* $x \leq y$ *and* $y \in K$, *then* $\pi(x) \leq y$.
5. $\pi(x + y) = \pi(x) + \pi(y)$.

Proof.
1: For $x \in L$, x is a lower bound for S_x, so $x \leq \bigwedge S_x = \pi(x)$.
2: If $\pi(x) = x$, then $x \in K$; conversely, if $x \in K$, then $S_x = [x)_K$, so $\pi(x) = \bigwedge S_x = x$.
3: If $x \leq y$, then $S_y \subseteq S_x$, hence $\pi(x) = \bigwedge S_x \leq \bigwedge S_y = \pi(y)$.
4: This follows from the second and third conditions.
5: Since $x \leq \pi(x)$ and $y \leq \pi(y)$, we have $x + y \leq \pi(x) + \pi(y)$, so by the third condition, $\pi(x + y) \leq \pi(\pi(x) + \pi(y)) = \pi(x) + \pi(y)$, since $\pi(x) + \pi(y) \in K$. By the

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third condition, we have $\pi(x) \leq \pi(x+y)$ and $\pi(y) \leq \pi(x+y)$, so $\pi(x)+\pi(y) \leq \pi(x+y)$, showing that $\pi(x+y) = \pi(x)+\pi(y)$. □

LEMMA 3.2. *The projection map π from L to K is a geometric morphism from $\mathcal{F}(L)$ to $\mathcal{F}(K)$.*

Proof. For the first condition of Definition 2.6, we recall that K and L have a common least element 0, so $\pi(0) = \bigwedge S_0 = \bigwedge K = 0$.

For the second condition of Definition 2.6, we have by Lemma 3.1,

$$\begin{aligned} C_L(x,y,z) &\Leftrightarrow x+y = x+z = y+z \\ &\Rightarrow \pi(x+y) = \pi(x+z) = \pi(y+z) \\ &\Rightarrow \pi(x)+\pi(y) = \pi(x)+\pi(z) = \pi(y)+\pi(z) \\ &\Leftrightarrow C_K(\pi(x),\pi(y),\pi(z)). \end{aligned}$$

For the third condition, assume that $C_K(x,y,\pi(z))$, for $x,y \in K$ and $z \in L$, so that $x+y = x+\pi(z) = y+\pi(z)$. Define $u := (y+z) \wedge x$ and $v := (x+z) \wedge y$. We need to show that $\pi(u) = x$, $\pi(v) = y$ and $C_L(u,v,z)$.

By Lemma 3.1(1), we have

$$\begin{aligned} u+y &= [(y+z) \wedge x] + y \\ &= (y+z) \wedge (x+y) && \text{Modularity} \\ &= (y+z) \wedge (y+\pi(z)) \\ &= (y+z). \end{aligned}$$

Since $u \leq x$, $u \leq x \wedge \pi(u)$, hence $z \leq y+z = u+y \leq (x \wedge \pi(u))+y$, so that by Lemma 3.1(4), $\pi(z) \leq (x \wedge \pi(u))+y$ and $\pi(z)+y \leq (x \wedge \pi(u))+y$. Consequently, $x+y = y+\pi(z) \leq (x \wedge \pi(u))+y$, so $x+y = ((x \wedge \pi(u))+y)$. Hence,

$$\begin{aligned} x &= (x+y) \wedge x \\ &= [(x \wedge \pi(u))+y] \wedge x \\ &= (x \wedge \pi(u)) + (y \wedge x) && \text{Modularity} \\ &= x \wedge \pi(u), \end{aligned}$$

since $y \wedge x \leq u \leq x \wedge \pi(u)$, so that $x \leq \pi(u)$. We have $\pi(u) \leq x$ by Lemma 3.1, part 4, hence $\pi(u) = x$. A symmetrical proof shows that $\pi(v) = y$.

To show that $C_L(u,v,z)$, we compute

$$\begin{aligned} u+z &= [(y+z) \wedge x] + z \\ &= (y+z) \wedge (x+z) && \text{Modularity} \\ &= [(x+z) \wedge y] + z && \text{Modularity} \\ &= v+z, \end{aligned}$$

so that $u+z = (x+z) \wedge (y+z) = v+z$. Since $u \leq x$ and $z \leq \pi(z)$, we have $u+z \leq x+\pi(z) = x+y$. Hence,

$$\begin{aligned} u+z &= (x+y) \wedge (u+z) \\ &= (x+y) \wedge (x+z) \wedge (y+z) \\ &= [x+(y \wedge (x+z))] \wedge (y+z) && \text{Modularity} \end{aligned}$$

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$$= [x \wedge (y+z)] + [y \wedge (x+z)] \qquad \text{Modularity}$$
$$= u + v,$$

completing the proof of the third condition. □

§4. Garber's theorem. In this section, we give a proof of Garber's theorem [9] that epimorphisms are not necessarily surjective in the category of Boolean monoids. The proof is based on the results from §3, also due to Garber; the starting point of the proof is the following result of Freese [8, p. 297].

THEOREM 4.1. *Complete epimorphisms in the category of complete modular lattices and complete homomorphisms are not necessarily surjective.*

In proving this result, Freese constructs two modular lattices K and L, with K a sublattice of L, where the inclusion map from K into L is an epimorphism, although it is not surjective. The lattices are of finite length, and so are complete, and in addition, K is a complete sublattice of L.

By Theorem 3.1, $\mathcal{F}(K)$ and $\mathcal{F}(L)$ are geometric frames, so that by Lemma 2.2, $\mathfrak{A}(K) = \mathfrak{A}(\mathcal{F}(K))$ and $\mathfrak{A}(L) = \mathfrak{A}(\mathcal{F}(L))$ are complete atomic Boolean monoids, and by Lemma 2.3, the map $\varphi(X) = \pi^{-1}(X)$, for $X \subseteq K$ is a complete embedding of $\mathfrak{A}(K)$ in $\mathfrak{A}(L)$.

THEOREM 4.2 (Garber). *Assume that*:

1. *L is a complete modular lattice, and $K \subseteq L$ a complete sublattice.*
2. *$(K]$ and $(L]$ are the images of K and L under the principal ideal map from L to $\mathcal{L}(\mathfrak{A}(L))$.*
3. *$\mathfrak{K}$ is the subalgebra of $\mathfrak{A}(L)$ generated by $(K]$, while $\mathfrak{L}$ is the subalgebra of $\mathfrak{A}(L)$ generated by $(L]$.*

Then:

1. *If K is an epic sublattice of L, then $\mathfrak{K}$ is a epic subalgebra of $\mathfrak{L}$.*
2. *The embedding map from $\mathfrak{K}$ to $\mathfrak{L}$, restricted to $(K]$, is a lattice embedding of $(K]$ in $(L]$.*

Proof. (1): Assume that K is an epic sublattice of L. K is isomorphic to $(K]$ and L to $(L]$ under the principal ideal map, so that $(K]$ is an epic sublattice of $(L]$. We shall show that any morphism defined on $\mathfrak{L}$ is uniquely determined by its values on $\mathfrak{K}$, showing that the inclusion map $\mathfrak{K} \mapsto \mathfrak{L}$ is an epimorphism.

If $f : \mathfrak{L} \to \mathfrak{M}$ is a morphism from $\mathfrak{L}$ to a Boolean monoid $\mathfrak{M}$, then it is determined by its values on $(L]$, since $(L]$ generates $\mathfrak{L}$, so $f \restriction (L]$ is a lattice epimorphism from $(L]$ to $\mathcal{L}(\mathfrak{M})$.

By assumption, the inclusion map is an epimorphism from $(K]$ to $(L]$. Since $f \restriction (L]$ is determined by its values on $(K]$, $f : \mathfrak{L} \to \mathfrak{M}$ is determined by its values on $\mathfrak{K}$, showing that $\mathfrak{K}$ is an epic subalgebra of $\mathfrak{L}$.

(2): We now show that $\mathfrak{K}$ is a proper subalgebra of $\mathfrak{L}$, by proving that the embedding $\varphi = \pi^{-1} : \mathfrak{K} \hookrightarrow \mathfrak{L}$ induces a corresponding embedding on the lattices $(K]$ and $(L]$. Suppose that for $a \in L$, $(a] = \varphi(A)$, for some $A \subseteq K$. Then $\pi(a) \in A$, and $a \leq \pi(a)$ by Lemma 3.1(1). Since $\pi(a) \in K$, we have $\pi(\pi(a)) = \pi(a)$ by Lemma 3.1(2), so $\pi(a) \in \varphi(A) = (a]_L$, hence $\pi(a) \leq a$. Consequently, $a = \pi(a) \in K$, by Lemma 3.1(2).

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If K is a proper sublattice of L, then for some $a \in L \setminus K$, $(a]_L \in \mathfrak{L}$. However, by the preceding proof, $(a]_L \notin \mathfrak{K}$ showing that $\mathfrak{K}$ is a proper subalgebra of $\mathfrak{L}$. □

THEOREM 4.3. *Epimorphisms are not necessarily surjective in the category of Boolean monoids.*

Proof. This follows from Theorems 4.1 and 4.2. □

§5. Generalizing Garber's theorem.

In this section, we generalize Garber's result to the category of De Morgan monoids. This requires a somewhat more logic-oriented approach to the problem.

5.1. Modular elements in De Morgan monoids. An element a in a De Morgan monoid $\mathfrak{M}$ is *modular* if $\mathfrak{M}$ satisfies the universal implication:

$$\forall b, c \in \mathfrak{M}\, [a \geq c \Rightarrow a(b + c) = ab + c].$$

An essential tool in the generalization is the following lemma, closely related to a result of Chin and Tarski [6, theorem 2.18], [20, p. 268]; see also [13, p. 463].

LEMMA 5.1. *Let* a *be an element of a De Morgan monoid, satisfying the equations* $a\overline{a} = \bot$ *and* $a + \overline{a} = \overline{a}$. *Then* a *is modular.*

Proof. If $a \geq c$, then:

$$\begin{aligned} a(b + c) &= a[(ab \vee \overline{a}b) + c] \\ &= a[(ab + c) \vee (\overline{a}b + c)] \\ &\leq a[(ab + c) \vee (\overline{a} + a)] \\ &= a[(ab + c) \vee \overline{a}] \\ &= a(ab + c) \vee a\overline{a} \\ &\leq ab + c \end{aligned}$$

Hence, since $ab + c \leq a(b + c)$, $a(b + c) = ab + c$. □

5.2. Beth's property for equational theories. Let T be an equational theory in a language L. If $\Gamma \cup \{\sigma = \tau\}$ is a set of equations in L, then a *derivation of* $\sigma = \tau$ is a sequence of lattice identities, each of which is either in $T \cup \Gamma$ or is derived from earlier steps by the rules of equational logic. We write $\Gamma \vdash_T \sigma = \tau$ if there is a derivation of $\sigma = \tau$ in the theory T.

DEFINITION 5.1. *Let T be an equational theory in the language L, Γ a set of equations of L containing a variable* x, *and* $\Gamma[y/x]$ *the result of substituting* y *for* x *in the equations in* Γ.

1. Γ implicitly defines *the variable* x *if* $\Gamma \cup \Gamma[y/x] \vdash_T x = y$, *where* y *does not occur in* Γ.
2. Γ explicitly defines *the variable* x *if* $\Gamma \vdash_T x = \rho$ *for some term* ρ *in* L *containing only variables in* Γ *other than* x.
3. *The theory T has the* Beth property *if whenever T implicitly defines a variable* x, *then it explicitly defines* x.

As an example, the theory of Boolean algebras has the Beth property – this is a formulation of Beth's theorem [3]. On the other hand, the theory of distributive

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lattices does not, since relative complements in distributive lattices are implicitly but not explicitly definable. We make use of this latter example below in discussing Freese's results [8] in the theory of modular lattices. The main result of this section is that Beth's property fails for any variety of algebras between the variety of De Morgan monoids and the variety of Boolean monoids.

Fix the language L_1 to be the language of De Morgan monoids, that is to say, the equational language with the basic functions $\{+, \wedge, \vee, \neg\}$ and constants $\{0, \top, \bot\}$, and L_2 to be the sublanguage of L_1 containing only $\wedge$, $+$ and 0. A term in L_2 is a *lattice term*; a *lattice identity* has the form $\sigma = \tau$, where σ and τ are lattice terms. A *theory* is a family of equational identities closed under logical deduction and uniform substitution. We use the abbreviation ML for the theory of modular lattices with 0, and DMM for the theory of De Morgan monoids.

We can use Lemma 5.1 to show that derivations in the theory of modular lattices ML can be simulated exactly in the theory DMM of De Morgan monoids.

Definition 5.2. *Define Δ to be a family of equations in the language L_1 containing the following*:

1. $x + x = x$ *and* $0 \wedge x = 0$ *for* x *a variable.*
2. $\rho \wedge \overline{\rho} = \bot$ *and* $\rho + \overline{\rho} = \overline{\rho}$, *for* ρ *a lattice term.*

Lemma 5.2. *Let* $\Gamma \cup \{\sigma = \tau\}$ *be a family of lattice equations. Then*

$$\Gamma \vdash_{ML} \sigma = \tau \;\Leftrightarrow\; \Gamma \cup \Delta \vdash_{DMM} \sigma = \tau.$$

Proof. ($\Rightarrow$): Assume that $\Gamma \vdash_{ML} \sigma = \tau$, so that there is a derivation of $\sigma = \tau$ from Γ in the theory of modular lattices. Starting from the assumptions $x + x = x$ and $0 \wedge x = 0$ for x a variable, we can prove in the theory DMM that the lattice terms form a lattice with least element 0. If the derivation involves the application of a modular identity from the theory ML, then we employ Lemma 5.1 to derive the appropriate identity from $\rho \wedge \overline{\rho} = \bot$ and $\rho + \overline{\rho} = \overline{\rho}$ in Δ, for some lattice term ρ.

($\Leftarrow$): Assume that $\Gamma \nvdash_{ML} \sigma = \tau$. We shall show that there is a Boolean monoid $\mathfrak{M}$ so that $\mathfrak{M} \models \Gamma \cup \Delta$ and $\mathfrak{M} \not\models \sigma = \tau$. By assumption, there is a modular lattice L with 0 so that $L \models \Gamma$ and $L \not\models \sigma = \tau$. Let $\mathfrak{A}(L) = \mathfrak{A}(\mathcal{F}(L))$ be the Boolean monoid constructed from L using Theorem 3.1 and Lemma 2.2. By Theorem 3.2, the lattice L is embedded in the lattice $\mathcal{L}(\mathfrak{A}(L))$. The identity $a \wedge \overline{a} = \bot$ holds in any Boolean monoid, and for $a \in \mathcal{L}(\mathfrak{A}(L))$, we have $a + \overline{a} = \overline{a}$ by Lemma 2.1. Consequently, $\mathfrak{A}(L) \models \Gamma \cup \Delta$, and $\mathfrak{A}(L) \not\models \sigma = \tau$. □

5.3. Freese's lattices. In §4, we made use of Freese's result [8, theorem 3.3] that epimorphisms in the category of modular lattice and lattice homomorphisms are not necessarily surjective. Using the general framework of §5.2, we examine Freese's proof in more detail; it is based on Lemma 5.3.

In a lattice L, we denote by a/b a pair of elements $a, b \in L$, with $a \geq b$; a/b is called a *quotient of* L [11, p. 129]; the interval $[b, a]$ is a *quotient lattice* of L. If a/b and c/d are quotients of L, then we write $a/b \nearrow c/d$ if $b = a \wedge d$ and $c = a + d$; we say a/b is *perspective to* c/d (in the up direction). For the definition of an n-frame, the reader can consult von Neumann [17, p. 118] or Freese [8, pp. 278–279].

Lemma 5.3. *Let M be a modular lattice, satisfying the conditions*:

1. *M contains* 4*-frames* $\{a_i, c_{1j}\}$ *and* $\{a'_i, c'_{1j}\}$ *of characteristic* p *and* q *respectively, for distinct primes* p *and* q.

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2. $a_1 + a_2/a_1a_2 \nearrow a_1' + a_2'/a_1'a_2'$.
3. $a_1' = a_1 + a_1'a_2'$, $a_2' = a_2 + a_1'a_2'$, *and* $c_{12}' = c_{12} + a_1'a_2'$.

Then a_1/a_1a_2 *is a distributive sublattice of* M.

Proof. This lemma is Freese's Corollary 3.2 [8, p. 297]. □

The next theorem is essentially equivalent to Theorem 4.1, but we provide a more detailed account of the proof in terms of equational derivations.

THEOREM 5.1. *The Beth property fails in the variety of modular lattices.*

Proof. In accordance with Definition 5.1, we show that there is a set of lattice equations Γ containing a variable x so that Γ implicitly defines x, but does not explicitly define x. We construct Γ using the lattices K and L already described in §4.

Freese begins the proof of his main result [8, sec. 2] by constructing a modular lattice of finite length. Let F and G be countably infinite fields with char $F = p$ and char $G = q$ for distinct primes p and q. Let L_p be the lattice of subspaces of the F-vector space F^4, and similarly for L_q. L_p contains a 4-frame of characteristic p and L_q a 4-frame of characteristic q. (The notion of characteristic for n-frames uses Von Neumann's definition of addition in an n-frame [17, p. 142], for $n \geq 4$.) Freese constructs a lattice M by using the Hall–Dilworth construction [12] to glue L_p to L_q, where the gluing takes place on two-dimensional quotient lattices determined by the frames. The lattice M is Freese's central tool in proving the main result of [8].

Define K as $\{(x, y) \in M \times M : x \leq y\}$ and L as the sublattice of $M \times M$ generated by K and $(a_1, \mathbf{0})$, where $\mathbf{0}$ is a_1a_2 (not necessarily the zero element of M). The lattice M is embedded into K, and hence into L by the diagonal embedding $x \longmapsto (x, x)$. It follows that L contains elements $(a_1, a_1), (c_{1j}, c_{1j}), \ldots$ that satisfy the conditions of Lemma 5.3, so that the quotient lattice $(a_1, a_1)/(\mathbf{0}, 0)$ in L is distributive.

Now define the set of equations Γ to consist of the following:

1. The equations stating that K satisfies the conditions of Lemma 5.3.
2. $(\mathbf{0}, a_1) \wedge x = (\mathbf{0}, \mathbf{0})$ and $(\mathbf{0}, a_1) + x = (a_1, a_1)$.

Since the quotient lattice $(a_1, a_1)/(\mathbf{0}, 0)$ is distributive, the value of $x = (a_1, \mathbf{0})$ in L is uniquely determined by Γ, that is to say, Γ implicitly defines x.

However, Γ does not define x explicitly. The element $(a_1, \mathbf{0})$ is not in K, so that there is no lattice term t containing only variables in Γ other than x so that $\Gamma \vdash_{ML} x = t$. □

THEOREM 5.2. *The Beth property fails in all the varieties between the varieties of De Morgan monoids and Boolean monoids.*

Proof. Let V be a variety intermediate between the variety DMM of De Morgan monoids and the variety BM of Boolean monoids, and $T(V)$ the theory of V (the set of equations valid in V).

Let Γ be the set of lattice equations defined in Theorem 5.1. Then

$$\Gamma \cup \Gamma[y/x] \vdash_{ML} x = y.$$

The lattice-theoretic computations forming the derivation can be found in Freese's proof [8, pp. 295–297] of his Theorem 3.1. Hence

$$\Gamma \cup \Gamma[y/x] \cup \Delta \vdash_{DMM} x = y$$

by Lemma 5.2, so there is a finite subset Δ^* of Δ so that $\Gamma \cup \Delta^*$ implicitly defines x.

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To show that $\Gamma \cup \Delta^*$ does not explicitly define x, we consider the Boolean monoids $\mathfrak{A}(K) = \mathfrak{A}(\mathcal{F}(K))$ and $\mathfrak{A}(L) = \mathfrak{A}(\mathcal{F}(L))$ constructed from the modular lattices K and L by Theorem 3.1 and Lemma 2.2. The principal ideal map $x \longmapsto (x]$ is an embedding of K and L in $\mathcal{L}(\mathfrak{A}(K))$ and $\mathcal{L}(\mathfrak{A}(L))$. The principal ideal $((a_1, \mathbf{0})]$, where $(a_1, \mathbf{0}) \in L$, is the unique solution to the equations

$$((\mathbf{0}, a_1)] \wedge x = ((\mathbf{0}, \mathbf{0})] \text{ and } ((\mathbf{0}, a_1)] + x = ((a_1, a_1)].$$

However, by Theorem 4.2, $((a_1.\mathbf{0})] \notin \mathfrak{A}(K)$ so Γ does not explicitly define x. □

§6. Logical consequences. The earlier sections have expounded the results in lattice-theoretic and algebraic terms. However, it is easy to translate them into a logical form; here, we adopt the terminology of Anderson and Belnap [1].

Definition 6.1. *If τ is a De Morgan term, then the* translation $T(\tau)$ *into the language of relevance logic is defined inductively as follows*:

1. *If x_i is a variable, then $T(x_i) = p_i$.*
2. $T(\sigma \wedge \tau) = T(\sigma) \,\&\, T(\tau)$.
3. $T(\sigma \vee \tau) = T(\sigma) \vee T(\tau)$.
4. $T(\sigma + \tau) = \overline{T(\sigma) \circ T(\tau)}$.
5. $T(\neg\sigma) = \overline{T(\sigma)}$.
6. $T(0) = t$, $T(\top) = T$ *and* $T(\bot) = F$.

If $\mathcal{F}$ is a family of De Morgan monoids, then a formula A is *valid in* $\mathcal{F}$ if $\mathfrak{M} \models 0 \leq \varphi(A)$ for all interpretations in any algebra $\mathfrak{M}$ belonging to $\mathcal{F}$.

Lemma 6.1. *For σ, τ De Morgan terms, and V a variety of De Morgan monoids, $T(\sigma) \leftrightarrow T(\tau)$ is valid in V if and only if $\sigma = \tau$ holds in all algebras in V.*

Proof. For $\mathfrak{M}$ a De Morgan monoid, σ, τ De Morgan terms, and φ an interpretation of σ and τ in $\mathfrak{M}$, $\mathfrak{M} \models 0 \leq \varphi[T(\sigma) \leftrightarrow T(\tau)]$ if and only if $\varphi(\sigma) = \varphi(\tau)$. The Lemma follows immediately from this equivalence. □

Using Lemma 6.1, we can translate Definition 5.1 into logical terms (see [24] for a detailed version).

Theorem 6.1. *The Beth property fails for all logics between* **B** *and* **KR**.

Proof. For logics **L** intermediate between **B** and **R**, the theorem follows from the results of Blok and Hoogland [5]. For logics intermediate between **R** and **KR**, the logical version of Theorem 5.2 demonstrates the failure of the Beth property. □

The proof of Theorem 6.1 divides into two parts, of which the second is much more complicated. Both parts ultimately rely on the fact that the Beth property fails in the theory of distributive lattices, but the first exploits this property directly, whereas the second part relies on the much more sophisticated result of Freese showing the existence of large distributive intervals in certain modular lattices.

I would like to express my thanks to Jacob Garber for his fine solution to the problem that I posed in [24] and [25], and to Roger Maddux for interesting and informative correspondence on the proof of Garber's theorem and its relation to the literature of Boolean algebras with operators.

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