ASYMPTOTIC EXPANSIONS II

LEO MOSER AND MAX WYMAN

1. Introduction. In a previous paper **(1)** the authors considered the problem of finding an asymptotic formula for numbers or functions $B_{n,m}$ whose generating function is of the form

$$\exp(P_m(x)) = \sum_{n=0}^{\infty} B_{n,m} \frac{x^n}{n!}, \tag{1.1}$$

where $P_m(x)$ is a polynomial of degree m in x given by

$$P_m(x) = \sum_{k=1}^{m} a_k x^k, \qquad a_m \neq 0. \tag{1.2}$$

The above-mentioned paper contained the restriction that $a_k \geqslant 0$. In our present paper we remove this restriction and allow the coefficients a_k to be positive, negative or zero. However we do retain $a_m \neq 0$. In **(1)** it was shown that there is no loss in generality in assuming that the greatest common divisor of the values of k, for which $a_k \neq 0$, is one. This assumption we shall also retain throughout the present paper.

Since the degree m of the polynomial $P_m(x)$ is fixed we shall, wherever possible, suppress m from our notation. Iterated exponential functions occur throughout the paper. For this reason we shall use alternative notations e^x or $\exp(x)$ to denote the exponential function. In this notation we write (1.1) and (1.2) as

$$\exp(P(x)) = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \tag{1.3}$$

$$P(x) = \sum_{k=1}^{m} a_k\, x^k, \qquad a_m \neq 0. \tag{1.4}$$

2. Trigonometric polynomials. In our previous paper **(1)** the trigonometric polynomial $S(R, \theta)$ associated with $P(x)$ by means of

$$S(R, \theta) = \Re[P(Re^{i\theta})] = \sum_{k=1}^{m} a_k\, R^k \cos k\theta \tag{2.1}$$

played a very important role. We shall call this function the dominant function of $P(x)$ and throughout the paper R shall be considered as large and positive.

When the coefficients $a_k \geqslant 0$ the dominant function has a greatest maximum at $\theta = 0$. Further in a sense, which we shall explain later in the paper, this greatest maximum is unique in the range $0 \leqslant \theta \leqslant \pi$. However when we do not restrict the sign of the a_k the greatest maximum will, in general, occur at

Received June 7, 1956.

some value of θ which depends on R. There may also be other maxima which we shall define to be equivalent to the greatest maximum. From the point of view of finding an asymptotic formula for the B_n these equivalent maxima are important. In fact the asymptotic formula is obtained by expansions about the angles which give the greatest maximum and its equivalents.

For the above reasons we shall, in this section, discuss and classify the maxima of the dominant function $S(R, \theta)$. Throughout the section we restrict θ to be in the range $0 \leqslant \theta \leqslant \pi$.

Denoting differentiation with respect to θ by a prime then the values of θ which make $S(R, \theta)$ a maximum are contained in the solutions of the equation

$$S'(R, \theta) = -\sum_{k=1}^{m} k\, a_k\, R^k \sin k\,\theta = 0. \tag{2.2}$$

For a large positive value of R it is easily shown that all of the solutions of (2.2) for θ must have the form

$$\theta_r(R) = \frac{r\pi}{m} + \sum_{s=1}^{\infty} b_{s,r}\, R^{-s}, \tag{2.3}$$

where r is an integer in the range $0 \leqslant r \leqslant m$ and $b_{s,r}$ are constants.

The only values of r for which θ_r will give a maximum of $S(R, \theta)$ are those values that will satisfy the equation

$$a_m \cos(r\pi) = |a_m|, \tag{2.4}$$

which of course implies that consecutive maxima are separated by an angular displacement of approximately $2\pi/m$.

Definition 2.1 The angles θ_r of (2.3) which yield maxima of $S(R, \theta)$ shall be called the *asymptotic angles* of $S(R, \theta)$. The reason for the name will become apparent later in the paper.

Definition 2.2 Let θ_i, θ_j be two asymptotic angles of $S(R, \theta)$. These shall be called *equivalent* asymptotic angles if and only if

$$\lim_{R\to\infty} (S(R, \theta_i) - S(R, \theta_j)) = \text{finite constant}. \tag{2.5}$$

Definition 2.3 The maxima associated with equivalent asymptotic angles are called *equivalent* maxima.

THEOREM 2.1. *There is no other asymptotic angle equivalent to $\theta = 0$ when $\theta = 0$ is an asymptotic angle of $S(R, \theta)$. A similar theorem holds for $\theta = \pi$.*

Proof. Let us assume there is an asymptotic angle θ_r contained in (2.3) which is equivalent to $\theta = 0$. Two possibilities exist either θ_r is independent of R and hence $b_{s,r} = 0$ or else θ_r depends on R and there exists at least one $b_{s,r} \neq 0$.

For the first possibility $\theta_r = r\pi/m$ and

$$S(R, 0) - S(R, \theta_r) = \sum_{k=1}^{m} a_k\, R^k\, (1 - \cos k\,\theta_r). \tag{2.6}$$

https://doi.org/10.4153/CJM-1957-023-3 Published online by Cambridge University Press

Hence

$$\lim_{R\to\infty} [S(R, 0) - S(R, \theta_r)]$$

is finite if and only if

(2.7) $$\cos k\,\theta_r = 1$$

for all values of k, $1 \leqslant k \leqslant m$, for which $a_k \neq 0$. Since the greatest common divisor of the above-mentioned values of k is one it is easily shown that the only solution of equations (2.7) in the range $0 \leqslant \theta_r \leqslant \pi$ is $\theta_r = 0$.

Let us now assume that the constants $b_{s,r}$ are not all zero. This implies that there must exist a value of k in (2.2), for which $a_k \neq 0$, and such that $\sin k(r\pi/m) \neq 0$. Otherwise $\theta_r = r\pi/m$ would be a solution of (2.2) and we would be back to the case where θ_r is independent of R. Let us suppose that $k = q$ is the largest value of k such that

(2.8) $$\sin(qr\pi/m) \neq 0, \quad a_q \neq 0, \qquad 1 \leqslant q \leqslant m - 1.$$

Under these conditions we can easily show from (2.2) that

(2.9) $$\theta_r = \frac{r\pi}{m} + O(1/R^{m-q}).$$

We shall now consider values of $k > q$ for which $a_k \neq 0$. For all such values of k we have by assumption that

(2.10) $$\sin(kr\pi/m) = 0.$$

Hence by using (2.8), (2.9) and expanding $\cos(k\theta_r)$ we have

(2.11) $$\cos(k\theta_r) = \cos(kr\pi/m) + O(1/R^{2m-2q}).$$

This implies that

(2.12) $$a_k R^k \cos k\theta_r = a_k R^k \cos(kr\pi/m) + O(R^{k+2q-2m}).$$

Since $k \leqslant m$ and $q \leqslant m - 1$, $k + 2q - 2m \leqslant q - 1$. Hence the order term cannot affect any terms of order R^s, $s \geqslant q$. A similar result can be obtained for the single term $k = q$. This means that

(2.13) $$\sum_{k=1}^{m} a_k R^k \cos k\theta_r = \sum_{k=q}^{m} a_k R^k \cos(kr\pi/m) + O(R^{q-1}).$$

From (2.13) we can conclude that

$$\lim_{R\to\infty} [S(R, 0) - S(R, \theta_r)]$$

is finite only if

(2.14) $$\cos(kr\pi/m) = 1$$

for all values of k, $q \leqslant k \leqslant m$, such that $a_k \neq 0$. However $a_q \neq 0$ by assumption. Hence (2.14) implies $\cos(qr\pi/m) = 1$. This in turn means $\sin(qr\pi/m) = 0$ which contradicts (2.8). Hence if $\theta = 0$ is an asymptotic angle of $S(R, \theta)$ it has no equivalent asymptotic angles. A similar proof can be given for $\theta = \pi$.

THEOREM 2.2. *Let θ_i and θ_j be two non-equivalent asymptotic angles of $S(R, \theta)$. Then*

$$S(R, \theta_i) - S(R, \theta_j) = O(R^q), \tag{2.15}$$

where q is some integer in the range $1 \leqslant q \leqslant m - 1$.

Proof. Using (2.3) one can easily see that $S(R, \theta_i)$, $S(R, \theta_j)$ have expansions in the powers of R of the form

$$S(R, \theta_i) = \sum_{-\infty}^{m} C_s R^s, \tag{2.16}$$

$$S(R, \theta_j) = \sum_{-\infty}^{m} \bar{C}_s R^s, \tag{2.17}$$

where C_s and $\bar{C}_s$ are constants such that $C_m = \bar{C}_m$. Since the maxima are not equivalent there must be a value of s for which $C_s \neq \bar{C}_s$, $1 \leqslant s \leqslant m - 1$. If we denote by q the largest such value of s then

$$S(R, \theta_i) - S(R, \theta_j) = O(R^q). \tag{2.18}$$

Definition 2.4. Let $\phi_1 < \phi_2 < \ldots < \phi_t$ be the subset of asymptotic angles which includes the asymptotic angle for which $S(R, \theta)$ is equal to its greatest maximum and all asymptotic angles which are equivalent to this latter asymptotic angle. The asymptotic angles ϕ_i $(i = 1, 2, \ldots, t)$ are called the *maximal asymptotic angles* of $S(R, \theta)$.

THEOREM 2.3. *The set of maximal asymptotic angles consists only of one angle if either $\theta = 0$ or $\theta = \pi$ is a maximal asymptotic angle.*

This follows immediately from Theorem 2.1.

THEOREM 2.4. *Let us assume that the set of maximal angles consists of a single angle ϕ_1, such that $0 < \phi_1 < \pi$. If ϵ is defined by*

$$\epsilon = R^{(1-4m)/8}, \tag{2.19}$$

then

$$S(R, \theta) \leqslant S(R, \phi_1 - \epsilon) \tag{2.20}$$

for all values of θ in the range $0 \leqslant \theta \leqslant \phi_1 - \epsilon$ or $\phi_1 + \epsilon \leqslant \theta \leqslant \pi$.

The proofs for the two ranges of θ are similar so that we shall consider only the range $0 \leqslant \theta \leqslant \phi_1 - \epsilon$. In this range we note that there must always exist at least one minimum of $S(R, \theta)$. This is easily seen because $\theta = 0$ is always a solution of (2.2). Hence $S(R, 0)$ is either a maximum or minimum of $S(R, \theta)$. If $S(R, 0)$ is a minimum then there is nothing more to prove. If $S(R, 0)$ is a maximum then there is a minimum in the range $0 \leqslant \theta \leqslant \phi_1$ because $S(R, \phi_1)$ is also by assumption a maximum. However all the maxima and minima are determined by (2.3), hence the angles at which they occur are always a finite distance apart. Since $\epsilon \to 0$ as $R \to \infty$ we must have that there always exists at least one minimum of $S(R, \theta)$ in the range $0 \leqslant \theta \leqslant \phi_1 - \epsilon$.

If there is only one minimum of $S(R, \theta)$ in the range $0 \leqslant \theta \leqslant \phi_1 - \epsilon$ and if it occurs at $\theta = 0$ then the proof of the theorem is trivial. In this case $S(R, \theta)$ is an increasing function in the range $0 \leqslant \theta \leqslant \phi_1 - \epsilon$ and hence for this range $S(R, \theta) \leqslant S(R, \phi_1 - \epsilon)$.

Let us now assume that there are other minima of $S(R, \theta)$ in this range and denote by $\psi > 0$ the angle closest to ϕ_1 for which $S(R, \theta)$ is a minimum. Since $S(R, \theta)$ is increasing in the range $\psi \leqslant \theta \leqslant \phi_1 - \epsilon$ we must have $S(R, \theta) \leqslant S(R, \phi_1 - \epsilon)$ in this range. Hence to complete the proof we need only consider the range $0 \leqslant \theta \leqslant \psi$. Again it is easily shown that in this later range a maximum of $S(R, \theta)$ must exist. We denote by α the angle at which the greatest maximum of $S(R, \theta)$ occurs for the range $0 \leqslant \theta \leqslant \psi$. We note of course that α is an asymptotic angle. Clearly for the range $0 \leqslant \theta \leqslant \psi$, $S(R, \theta) \leqslant S(R, \alpha)$. Hence

$$S(R, \phi_1 - \epsilon) - S(R, \theta) \geqslant S(R, \phi_1 - \epsilon) - S(R, \alpha). \tag{2.21}$$

Expanding $S(R, \phi_1 - \epsilon)$ in a Taylor's expansion, and remembering that $S'(R, \phi_1) = 0$ because $S(R, \phi_1)$ is a maximum, we must have

$$S(R, \phi_1 - \epsilon) = S(R, \phi_1) + \tfrac{1}{2}S''(R, \phi_1)\, \epsilon^2 + \ldots . \tag{2.22}$$

Since $S(R, \phi_1)$ is a maximum $S''(R, \phi_1)$ is negative and is of order $O(R^m)$ in R. Hence using (2.19)

$$S(R, \phi_1 - \epsilon) = S(R, \phi_1) - O(R^{1/4}). \tag{2.23}$$

From (2.21) and (2.23)

$$S(R, \phi_1 - \epsilon) - S(R, \theta) \geqslant S(R, \phi_1) - S(R, \alpha) - O(R^{1/4}). \tag{2.24}$$

However, ϕ_1, α are non-equivalent asymptotic angles, hence, by Theorem 2.2,

$$S(R, \phi_1) - S(R, \alpha) = O(R^q). \tag{2.25}$$

where q is an integer in the range $1 \leqslant q \leqslant m - 1$. Thus (2.24) and (2.25) imply

$$S(R, \phi_1 - \epsilon) - S(R, \theta) \geqslant O(R^q). \tag{2.26}$$

By assumption $S(R, \phi_1)$ is the greatest maximum of $S(R, \theta)$ in the range $0 \leqslant \theta \leqslant \pi$. Hence the order term must be positive. When this is so (2.26) gives

$$S(R, \theta) \leqslant S(R, \phi_1 - \epsilon). \tag{2.27}$$

This completes the proof for the range $0 \leqslant \theta \leqslant \phi_1 - \epsilon$. The proof for the range $\phi_1 + \epsilon \leqslant \theta \leqslant \pi$ is similarly obtained.

THEOREM 2.5. *Let us assume that the set of maximal asymptotic angles $\phi_1 < \phi_2 < \ldots < \phi_t$ consists of at least two angles. Further let us assume that ϕ_i and ϕ_{i+1} are two consecutive maximal asymptotic angles. If ϵ is given by* (2.19) *then one of the following inequalities must hold*:

https://doi.org/10.4153/CJM-1957-023-3 Published online by Cambridge University Press

(2.28) $$S(R, \theta) \leqslant S(R, \phi_i + \epsilon)$$
or
(2.29) $$S(R, \theta) \leqslant S(R, \phi_{i+1} - \epsilon)$$

for all values of θ in the range $\phi_i + \epsilon \leqslant \theta \leqslant \phi_{i+1} - \epsilon$.

We omit the proof of this theorem because of its similarity to the proof used in Theorem 2.4.

THEOREM 2.6. *Let us assume that the conditions of Theorem* 2.4 *are satisfied. Further let* I_1, I_2 *be defined by*

(2.30) $$I_1 = \int_0^{\phi_1-\epsilon} \exp[P(Re^{i\theta}) - P(Re^{i\phi_1}) - in\theta]\, d\theta,$$

(2.31) $$I_2 = \int_{\phi_1+\epsilon}^{\pi} \exp[P(Re^{i\theta}) - P(Re^{i\phi_1}) - in\theta]\, d\theta,$$

where n is any real number. Then a number $k > 0$ *exists such that*

(2.32) $$|I_1| \leqslant \exp(-kR^{1/4})$$
(2.33) $$|I_2| \leqslant \exp(-kR^{1/4}).$$

Proof.

$$|I_1| \leqslant \int_0^{\phi_1-\epsilon} \exp[S(R, \theta) - S(R, \phi_1)]\, d\theta.$$

By Theorem (2.4), $S(R, \theta) \leqslant S(R, \phi_1 - \epsilon)$. Hence

(2.28) $$|I_1| \leqslant \pi \exp\{S(R, \phi_1 - \epsilon) - S(R, \phi_1)\}.$$

However, using (2.23) we have

(2.29) $$|I_1| \leqslant \pi \exp(-kR^{1/4}). \qquad k > 0.$$

We may absorb the π into the exponent and write

(2.30) $$|I_1| \leqslant \exp(-kR^{1/4}). \qquad k > 0.$$

The proof for I_2 is similar.

THEOREM 2.7. *Let us assume that the conditions of Theorem* (2.5) *are satisfied. Under these conditions the absolute value of each of the integrals*:

(2.31) $$I_0 = \int_0^{\phi_1-\epsilon} \exp[P(Re^{i\theta}) - P(Re^{i\phi_j}) - in\theta]\, d\theta,$$

(2.32) $$I_i = \int_{\phi_i+\epsilon}^{\phi_{i+1}-\epsilon} \exp[P(Re^{i\theta}) - P(Re^{i\phi_j}) - in\theta]\, d\theta, \quad i = 1, 2, \ldots, t-1,$$

(2.33) $$I_t = \int_{\phi_t+\epsilon}^{\pi} \exp[P(Re^{i\theta}) - P(Re^{i\phi_j}) - in\theta]\, d\theta,$$

is at most of order $\exp\{-o(R^{1/4})\}$ *no matter which maximal asymptotic angle* ϕ_j *is chosen.*

The proof is essentially the same as that used in Theorem 2.6.

THEOREM 2.8. *Let $P(x)$ be given by (1.4) and as usual $S(R, \theta)$ denotes the dominant function of $P(x)$, given by (2.1). We shall denote by τ the complex number*

(2.34) $$\tau = Re^{i\theta}$$

and by Θ the operator

(2.35) $$\Theta = \tau \frac{d}{d\tau}.$$

If n is a sufficiently large real number then the equation

(2.36) $$\Theta P(\tau) = n$$

has a unique solution for τ corresponding to each asymptotic angle of $S(R, \theta)$. Further the solution $\tau_r(n)$ corresponding to the asymptotic angle θ_r, given by (2.3), satisfies the equations

(2.37) $$\arg \tau_r = \theta_r$$

and

(2.38) $$|\tau_r(n)| = R_r(n) \sim (n/m|a_m|)^{1/m}.$$

Proof. From (1.4)

(2.39) $$\Theta P(\tau) = \sum_{k=1}^{m} k\, a_k\, \tau^k = \sum_{k=1}^{m} ka_k\, R^k \cos(k\,\theta) - i\, S'(R, \theta).$$

Hence equating the real and imaginary parts of (2.36) we find

(2.40) $$\sum_{k=1}^{m} k\, a_k\, R^k \cos k\,\theta = n$$

and

(2.41) $$S'(R, \theta) = 0.$$

Equation (2.41) is of course the same as (2.2) hence all of the solutions of (2.41) for θ are given by (2.3). Let us choose a particular θ_r which is also an asymptotic angle of $S(R, \theta)$. We have already seen that this implies $a_m \cos r\pi = |a_m|$. If we substitute (2.3) into (2.40) and expand into powers of R we see that (2.40) can be written

(2.42) $$m|a_m|\, R^m + O(R^{m-1}) = n.$$

Although the first term is independent of r the order term will in general be dependent on r. From (2.42) it is easily seen that for large values of n the solution of (2.42) for a real positive value of R is unique for each fixed value of r. Denoting this solution by $R_r(n)$ it is also easily seen

(2.43) $$R_r(n) \sim (n/m|a_m|)^{1/m}.$$

Since $\theta_r = \arg \tau_r(n)$ and $R_r(n) = |\tau_r(n)|$ the theorem is proven. One may also show without too much difficulty that if $R_r(n)$ and $R_j(n)$ are two different solutions for $|\tau|$ corresponding to different asymptotic angles that

$$\lim_{n\to\infty} [R_r(n) - R_j(n)] = 0.$$

In concluding this section we might mention that only the maximal asymptotic angles are necessary in developing our asymptotic formula for B_n.

3. An example of equivalent maxima. By Theorem 2.3 we know that if $\theta = 0$ is a maximal asymptotic angle that $S(R, \theta)$ has no other maximal asymptotic angles. This result is, of course, only true when we assume that the greatest common divisor of the values of k, for which $a_k \neq 0$, is one. For a long time the authors conjectured that under the above assumption any dominant function $S(R, \theta)$ would have only one maximal asymptotic angle. Since the conjecture proved to be false it is of interest to give an example in which the dominant function has two such angles.

Let us take $P(x)$ to be given by

$$P(x) = x^{90} + x^{30} - x^{24} + x^{15} - x^{10}. \tag{3.1}$$

Clearly the G.C.D. (90, 30, 24, 15, 10) = 1. The dominant of (3.1) is given by

$$S(R, \theta) = R^{90}\cos(90\theta) + R^{30}\cos(30\theta) - R^{24}\cos(24\theta) + R^{15}\cos(15\theta) - R^{10}\cos(10\theta). \tag{3.2}$$

(3.3) Clearly the solutions of $S'(R, \theta) = 0$ which correspond to maxima of $S(R, \theta)$ are of the form

$$\theta_r = \frac{r\pi}{45} + O(1/R^{60}), \qquad r = 0, 1, 2, \dots, 45. \tag{3.4}$$

Hence

$$S(R, \theta_r) = S(R, r\pi/45) + O(1/R^{30}). \tag{3.5}$$

This means that the maximal asymptotic angles can be obtained by choosing those values of r which will make $S(R, r\pi/45)$ as large as possible. In each case the coefficient of R^{90} is one. All the other coefficients of powers of R depend on r. In order to make $S(R, r\pi/45)$ as large as possible we start with coefficient of R^{30} and make this coefficient as large as possible. Then, in turn, we deal with each of the coefficients of the lower powers.

$$S(R, r\pi/45) = R^{90} + R^{30}\cos(2r\pi/3) - R^{24}\cos(8r\pi/15) + R^{15}\cos(r\pi/3) - R^{10}\cos(2r\pi/9) \tag{3.6}$$

Step A. $\cos(2r\pi/3)$. The values of r in the set $0, 1, 2, \dots, 45$ which make $\cos(2r\pi/3) = 1$ are $r = 0, 3, 6, \ \ , 45$.

Step B. $-\cos(8r\pi/15)$. The values of r in the set $0, 3, 6, \dots, 45$ which make $-\cos(8r\pi/15)$ as large as possible are 6, 9, 21, 24, 36, 39.

For these values $-\cos(8r\pi/15) = \cos(\pi/5)$.

Step C. $\cos(r\pi/3)$. The values of r of the set 6, 9, 21, 24, 36, 39 which make $\cos(r\pi/3) = 1$ are $r = 6, 24, 36$.

Step D. $-\cos(2r\pi/9)$. The values of r of the set 6, 24, 36 which make $-\cos(2r\pi/9)$ as large as possible are $r = 6, 24$.

Hence there will be two maximal asymptotic angles corresponding to the values $r = 6$ and $r = 24$.

It is interesting to note that if the sign of x^{10} in $P(x)$ is changed to positive then we would again have had a unique maximal asymptotic angle. Namely, the one corresponding to $r = 36$.

4. Asymptotic formula for unique maximal asymptotic angles. When $\theta = 0$ is a maximal asymptotic angle of the dominant function we have seen that there are no other maximal asymptotic angles. For this case the derivation of an asymptotic formula for B_n, as given by (1.3), can be obtained by using the same procedure as was used in our previous paper **(1)**. The proofs and final formulae are identical. For this reason we state, without proof, that the first term of the asymptotic formula is

$$B_n \sim \frac{n!\exp(P(R))}{R^n}(2\pi A^2P(R))^{-\frac{1}{2}}, \tag{4.1}$$

where A is the operator

$$A = R\frac{d}{dR} \tag{4.2}$$

and R as a function of n is given by

$$A\,P(R) = n. \tag{4.3}$$

Other terms of the asymptotic formula can be obtained from the general formula given in **(1)**.

Further the fact that

$$A^2P(R) = \sum_{k=1}^{m} k^2 a_k R^k \tag{4.4}$$

allows us to reduce (4.1) to

$$B_n \sim \frac{n!\exp(P(R))}{(2\pi)^{\frac{1}{2}}m(a_m)^{\frac{1}{2}}R^{n+\frac{1}{2}m}}, \tag{4.5}$$

with R again given by (4.3).

The case $\theta = \pi$ can be reduced to the case $\theta = 0$ by replacing θ by $\pi - \theta$. This is equivalent to replacing R by $-R$. For this case the first term of the asymptotic formula for B_n is given by

$$B_n \sim \frac{(-1)^n\exp(P(-R))}{R^n}(2\pi A^2P(-R))^{-\frac{1}{2}} \tag{4.6}$$

and R as a function of n is given by

$$A\,P(-R) = n. \tag{4.7}$$

The formula that corresponds to (4.5) is

$$B_n \sim \frac{(-1)^n\exp(P(-R))}{(2\pi|a_m|)^{\frac{1}{2}}mR^{n+\frac{1}{2}m}}, \tag{4.8}$$

and R is given by (4.7).

We now proceed to discuss the case when the dominant function has a unique maximal asymptotic angle say at

$$\theta = a(R), \quad a(R) \neq 0, \quad a(R) \neq \pi. \tag{4.9}$$

Throughout we shall simply write

$$a(R) = a. \tag{4.10}$$

From (1.3) we have

$$B_n = \frac{d^n}{dx^n}(\exp(P(x)))\Big]_{x=0}. \tag{4.11}$$

By Cauchy's Theorem

$$B_n = \frac{n!}{2\pi i}\int_C z^{-(n+1)} \exp(P(z))\, dz, \tag{4.12}$$

where C is chosen as the circle $z = Re^{i\theta}$. At this stage the radius of the circle R is an arbitrary positive number. Equation (4.12) can be written

$$\begin{aligned} B_n &= \frac{n!}{2\pi R^n}\int_{-\pi}^{\pi} \exp[P(Re^{i\theta}) - in\theta]\, d\theta \\ &= \frac{n!}{\pi R^n}\Re\int_0^{\pi} \exp[P(Re^{i\theta}) - in\theta]\, d\theta. \end{aligned} \tag{4.13}$$

If we define the complex number τ by

$$\tau = Re^{ia} \tag{4.14}$$

then (4.13) can be put into the form

$$B_n = \frac{n!}{\pi}\Re\left[\tau^{-n}\exp(P(\tau))\int_0^{\pi}\exp[P(\tau\exp(i(\theta-a)))-P(\tau)-in(\theta-a)]d\theta\right]. \tag{4.15}$$

We shall show that the integral of (4.15) has an asymptotic expansion in terms of powers of $1/\tau$. However by using Theorem (2.6) we can easily show that the integrals

$$\int_0^{a-\epsilon} \exp[P(\tau\exp(i(\theta - a))) - P(\tau) - in(\theta - a)]\, d\theta \tag{4.16}$$

and

$$\int_{a+\epsilon}^{\pi} \exp[P(\tau\exp(i(\theta - a))) - P(\tau) - in(\theta - a)]\, d\theta \tag{4.17}$$

are both of order $\exp(-O(|\tau|^{1/4}))$ when ϵ is given by (2.19). Anticipating this result we may write

$$B_n \sim \frac{n!}{\pi}\Re\left[\tau^{-n}\exp P(\tau)\int_{a-\epsilon}^{a+\epsilon}\exp\{P(\tau\exp i(\theta-a))-P(\tau)-in(\theta-a)\}d\theta\right]. \tag{4.18}$$

We define I to be the integral contained in (4.18) and replace $\theta - a$ by θ.

Hence

(4.19) $$I = \int_{-\epsilon}^{\epsilon} \exp(F(\theta))\, d\theta,$$

where

(4.20) $$F(\theta) = P(\tau e^{i\theta}) - P(\tau) - in\theta.$$

Remembering that Θ is the operator given by

(4.21) $$\Theta = \tau d/d\tau$$

we have on expanding $F(\theta)$ about $\theta = 0$ that

(4.22) $$F(\theta) = (\Theta\, P(\tau) - n)\, i\theta - \Theta^2\, P(\tau)\,\frac{\theta^2}{2} + \sum_{k=3}^{\infty} \Theta^k\, P(\tau)\,\frac{(i\theta)^k}{k!}\,.$$

We have assumed that there exists a unique maximal asymptotic angle $\theta = a$ and by Theorem (2.8) there corresponds a unique solution of the equations

(4.23) $$\Theta\, P(\tau) = n$$

(4.24) $$\arg \tau = a.$$

By choosing τ to be this unique solution we may write (4.22) as

(4.25) $$F(\theta) = -\,\Theta^2\, P(\tau)\,\frac{\theta^2}{2} + \sum_{k=3}^{\infty} \Theta^k\, P(\tau)\,\frac{(i\theta)^k}{k!}\,.$$

Since $0 < \arg \tau < \pi$ there is no loss in generality in considering the complex plane of τ as cut along the negative real axis. For this reason there is no ambiguity in finding the square root of τ. Similarly we shall find that there is no ambiguity in finding the square root of certain polynomials in τ which later enter into the discussion.

For simplicity we introduce the following notation:

(4.26) $$\tau = Z^{-2},\ Z = \tau^{-\frac{1}{2}};$$

(4.27) $$\phi = \theta(\tfrac{1}{2}\Theta^2\, P(\tau))^{\frac{1}{2}};$$

(4.28) $$h = \epsilon(\tfrac{1}{2}\Theta^2\, P(\tau))^{\frac{1}{2}};$$

(4.29) $$C_k(Z) = \sum_{s=1}^{m} s^k\, a_s\, Z^{2m-2s};$$

(4.30) $$f_k(Z) = C_k(Z)\, Z^{m(k-2)}\, (\tfrac{1}{2}C_2(Z))^{-k/2};$$

(4.31) $$\psi(Z,\,\phi) = \sum_{k=3}^{\infty} f_k(Z)\,\frac{(i\phi)^{\frac{1}{2}k}}{k!}.$$

In this notation the substitution (4.27) reduces the integral (4.19) to the form

(4.32) $$I = \left(\frac{2}{\Theta^2\, P(\tau)}\right)^{\frac{1}{2}} \int_{-h}^{h} \exp[-\,\phi^2 + \psi(Z,\,\phi)]\, d\phi.$$

https://doi.org/10.4153/CJM-1957-023-3 Published online by Cambridge University Press

We note that although θ is a real variable that ϕ as given by (4.27) is, in general, complex. Similarly h is, in general, a complex variable. From Theorem (2.8) we know that

(4.33) $$\lim_{n\to\infty} |\tau| = \lim_{n\to\infty} R = \infty.$$

It is easily shown that $|h| = O(|\tau|^{1/8})$ and hence

(4.34) $$\lim_{n\to\infty} |h| = \infty.$$

Finally one can show without too much difficulty that

(4.35) $$\lim_{n\to\infty} \arg h = 0.$$

Our next step is to expand $\exp(\psi(Z, \phi))$ into a Maclaurin expansion about $Z = 0$, of the form

(4.36) $$\exp(\psi(Z, \phi)) = \sum_{k=0}^{\infty} \Psi_k(\phi) Z^k, \qquad \Psi_0 = 1,$$

where the $\Psi_k(\phi)$ are polynomials in ϕ. Quite formally one would have on integrating term by term and replacing h by ∞ that

(4.37) $$I \sim \left(\frac{2}{\Theta^2 P(\tau)}\right)^{\frac{1}{2}} \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} e^{-\phi^2} \Psi_k(\phi)\, d\phi\, Z^k.$$

The rigorous justification of (4.37) requires only a slight modification of the procedure that was used in our previous paper **(1)**. For this reason we omit this justification. Defining b_k to be given by

(4.38) $$b_k = \int_{-\infty}^{\infty} e^{-\phi^2} \Psi_k(\phi)\, d\phi,$$

we can obtain from (4.18) and (4.37) that

(4.39) $$B_n \sim \pi^{-1} 2^{\frac{1}{2}} n!\, \Re\left[\tau^{-n}(\Theta^2 P(\tau))^{-\frac{1}{2}} \exp(P(\tau)) \sum_{k=0}^{\infty} b_k \tau^{-\frac{1}{2}k}\right]$$

and τ is given as the solution of

(4.40) $$\Theta P(\tau) = n,$$

which corresponds to the unique (by assumption) maximal asymptotic angle of the dominant function of $P(x)$.

Since $b_0 = (\pi)^{\frac{1}{2}}$ the first term of (4.39) is

(4.41) $$B_n \sim 2^{\frac{1}{2}}(n!)\, \Re[\tau^{-n}(\pi\, \Theta^2 P(\tau))^{-\frac{1}{2}} \exp(P(\tau))].$$

We can, if we so desire, replace $(\Theta^2(P(\tau)))^{\frac{1}{2}}$ by

(4.42) $$(\Theta^2 P(\tau))^{\frac{1}{2}} \sim m|a_m|^{\frac{1}{2}}|\tau|^{\frac{1}{2}m};$$

hence

(4.42) $$B_n \sim n!\, \Re[\tau^{-n}(2\pi^{-1})^{\frac{1}{2}} \exp(P(\tau))/m|a_m|^{\frac{1}{2}}\, |\tau|^{\frac{1}{2}m}].$$

In concluding this section we would like to point out that formula (4.39) does not hold for $a = 0$ or $a = \pi$. The reason being that a maximal asymptotic angle internal to the range $0 < \theta < \pi$ gives a contribution to the asymptotic formula from both sides of the maximal asymptotic angle. However at the boundary points of the interval a contribution to the asymptotic formula is obtained from one side only. For this reason when $a = 0$ or π formula (4.39) and the subsequent formulas are out by a factor of $\frac{1}{2}$.

5. The general asymptotic formula. In the general case we have seen that the dominant function can have more than one maximal asymptotic angle. We assume that there is at least two such angles and denote the set of maximal asymptotic angles by ϕ_j $(j = 1, 2, \ldots, t;\ t \geqslant 2)$. Under these assumptions neither of the angles 0 or π can be a member of the set.

From Theorem (2.8) we have seen that corresponding to a fixed value of j there corresponds a unique solution for τ of the equations

$$\Theta P(\tau) = n, \tag{5.1}$$

$$\arg \tau = \phi_j. \tag{5.2}$$

Let us denote this solution by

$$\tau_j = R_i e^{i\phi_j}. \tag{5.3}$$

Without difficulty one can show that R_j, R_k are two different solutions for $|\tau|$ such that

$$\lim_{n\to\infty} (R_j - R_k) = 0. \tag{5.4}$$

Now B_n is given by

$$B_n = \frac{n!}{2\pi i}\int_C z^{-(n+1)} \exp(P(z))\, dz, \tag{5.5}$$

where C is any closed contour enclosing the origin. Instead of a circular contour we choose C to be given as follows:

A. *Range* $0 \leqslant \theta \leqslant \phi_1 + \epsilon_1$, where $\epsilon_1 = R_1^{(1-4m)/8}$. In this range C is the circular arc $z = R_1 e^{i\theta}$.

B. *Range*

$$\phi_j + \epsilon_j \leqslant \theta \leqslant \phi_{j+1} + \epsilon_{j+1},\ 1 \leqslant j \leqslant t-1, \quad \epsilon_j = R_j^{(1-4m)/8}.$$

For this range C is the circular arc $z = R_{j+1} e^{i\theta}$.

C. *Range*

$$\phi_t + \epsilon_t \leqslant \theta \leqslant \pi, \quad \epsilon_t = R_t^{(1-4m)/8}.$$

Here C is the circular arc $z = R_t e^{i\theta}$.

D. In order to make a closed contour we join all the circular arcs by radial lines at the end points of each arc.

E. *Range* $-\pi \leqslant \theta \leqslant 0$. The contour C for this range is taken to be the mirror image of the contour for $0 \leqslant \theta \leqslant \pi$.

By using Theorems (2.5) and (2.7) we can show that only the portions of the contour $\phi_j - \epsilon_j \leqslant \theta \leqslant \phi_j + \epsilon_j$, $j = 1, 2, \ldots, t$ and their mirror images contribute to the asymptotic formula. Hence

$$B_n \sim \frac{n!}{\pi} \Re\left[\sum_{j=1}^{t} \int_{\phi_j - \epsilon_j}^{\phi_j + \epsilon_j} \frac{\exp[P(R_j e^{i\theta}) - in\theta]\, d\theta}{R_j^n} \right]. \tag{5.6}$$

Each of the maximal asymptotic angles can then be treated individually by the method of the previous section. This leads to the asymptotic formula

$$B_n \sim \frac{n!}{\pi} \Re\left[\sum_{k=0}^{\infty} \sum_{j=1}^{t} b_{kj}\, \tau_j^{-n} \exp(P(\tau_j))\, \tau_j^{-\frac{1}{2}k} \left(\tfrac{1}{2}\Theta^2 P(\tau_j)\right)^{-\frac{1}{2}} \right], \tag{5.7}$$

where $\Theta P(\tau_j)$ means $[\Theta P(\tau)]_{\tau=\tau_j}$. Similarly for $\Theta^2 P(\tau_j)$. Further the τ_j are given as the solutions of the equations

$$\Theta P(\tau) = n \tag{5.8}$$

and

$$\arg \tau = \phi_j, \tag{5.9}$$

where ϕ_j is a maximal asymptotic angle of the dominant function of $P(x)$. Finally the b_{kj} will have formulae analogous to (4.38). To obtain explicit formulae, expansions about each maximal asymptotic angle are involved.

6. Example. We have chosen as an example to illustrate our method the Hermite polynomials $H_n(t)$. Szegö (**3**; p. 194) gives an asymptotic expansion of these polynomials and the method of proof divides the expansion formula into two cases according as n is even or odd. Our method makes no such separation and the two cases are treated as one.

The generating function of $H_n(t)$ is

$$\exp(2tx - x^2) = \sum_{n=0}^{\infty} H_n(t) \frac{x^n}{n!}. \tag{6.1}$$

In the notation of the present paper:

$$P(x) = 2tx - x^2, \tag{6.2}$$

$$S(R, \theta) = 2tR \cos\theta - R^2 \cos 2\theta. \tag{6.3}$$

It is easily seen that the dominant function has a single maximum at $\theta = \phi_1$ where ϕ_1 is given by

$$\phi_1 = \arccos\,(t/2R). \tag{6.4}$$

The value of τ is determined by

$$2t\tau - 2\tau^2 = n, \tag{6.5}$$

$$\arg \tau = \phi_1. \tag{6.6}$$

https://doi.org/10.4153/CJM-1957-023-3 Published online by Cambridge University Press

Hence

(6.7) $$R = (n/2)^{\frac{1}{2}},$$

(6.8) $$\phi_1 = \text{arc cos } (t/(2n)^{\frac{1}{2}}),$$

(6.9) $$\tau = Re^{i\phi_1} = \tfrac{1}{2}(t + i(2n - t^2)^{\frac{1}{2}}).$$

From (6.8) for all real t and large n

(6.10) $$\phi_1 = \tfrac{1}{2}\pi - t\,(2n)^{-\frac{1}{2}} - (t^3(2n)^{-3/2}/6)$$

and

(6.11) $$n\,\phi_1 = \tfrac{1}{2}\,n\pi - (\tfrac{1}{2}n)^{\frac{1}{2}}t + \text{terms of order } n^{-\frac{1}{2}}.$$

We shall drop terms of the order $n^{-\frac{1}{2}}$ in our final formula to get an expression for the first term in the asymptotic expansion. At certain stages one must retain such terms. For example to obtain (6.11) from (6.10).

From (6.7), (6.9) and (6.11)

(6.12) $$\tau^n = (\tfrac{1}{2}n)^{\frac{1}{2}n}\, e^{i(\frac{1}{2}n\pi - (\frac{1}{2}n)^{\frac{1}{2}}t + \ldots)}.$$

Similarly

(6.13) $$\exp(P(\tau)) = \exp[\tfrac{1}{2}\,(t^2 + n) + i((\tfrac{1}{2}n)^{\frac{1}{2}}t\ldots)],$$

(6.14) $$[\Theta^2\,P(\tau)]^{\frac{1}{2}} = (2n)^{\frac{1}{2}} + \ldots.$$

Hence by (4.41)

(6.15) $$H_n(t) \sim (2/\pi)^{\frac{1}{2}}\,n!\,\Re[(\tfrac{1}{2}n)^{-\frac{1}{2}n}\,(\exp \tfrac{1}{2}(t^2 + n) + i((2n)^{\frac{1}{2}}\,t - \tfrac{1}{2}n\pi + \ldots)](2n)^{-\frac{1}{2}}.$$

Using Stirling's formula for $n!$

(6.16) $$H_n(t) \sim 2^{\frac{1}{2}(n+1)}(n/e)^{\frac{1}{2}n}\,e^{\frac{1}{2}t^2}\cos((2n)^{\frac{1}{2}}\,t - \tfrac{1}{2}n\pi).$$

This term agrees to the proper order with Szegö's formula. Other terms can easily be calculated and it can be seen that our method does not distinguish between even and odd n.

We hope in subsequent papers to apply the method to other problems involving asymptotic expansions.

7. Conclusion. In concluding this paper we would like to point out several possibilities for further generalizations. The most obvious generalization would be to obtain an asymptotic formula when the coefficients a_k of $P(x)$ are allowed to be complex functions of a complex parameter t. We have carried this problem far enough to be reasonably certain that our method will generalize to this case with a minimum of modification. A second type of generalization would be to allow the a_k to be functions of n and obtain formulas of the Plancherel-Rotach type. We have found examples of this type of problem for which our method will apply but it does not seem likely that the method can be used to solve the general problem of this type. A final type of generalization we have considered is the replacement of the polynomial $P(x)$ by a function of $f(x)$. We have found that the method can be applied to the special case of the Bell numbers **(2)**, in which $f(x) = e^x - 1$, but fairly stringent conditions would have to be placed on $f(x)$ in order for our method to apply to an arbitrary class of functions.

References

1. L. Moser and M. Wyman, *Asymptotic expansions*, Can. J. Math., *8* (1956), 225–233.

2. ———, *An asymptotic formula for the Bell Numbers*, Trans. Roy. Soc. Can., Series III, *49* (1955), Sec. III, 49–54.

3. G. Szegö, *Orthogonal Polynomials*, Amer. Math. Soc. Coll. Publ. *23* (New York, 1939).

University of Alberta

https://doi.org/10.4153/CJM-1957-023-3 Published online by Cambridge University Press