SOME FRACTIONAL q-INTEGRALS AND q-DERIVATIVES †

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1. A q-analogue of the integral $\int_x^\infty f(t)dt$ is defined by means of

$$\int_x^\infty f(t)d(t;\, q) = x(1-q)\sum_{k=1}^{\infty} q^{-k}f(xq^{-k}) \tag{1.1}$$

which is an inverse of the q-derivative

$$D_q f(x) = \frac{f(xq)-f(x)}{x(q-1)}. \tag{1.2}$$

The present author (**2**) has recently obtained a q-analogue of a formula of Cauchy, namely,

$$\begin{aligned} K^{-N}f(x) &= \int_x^\infty \int_{x_{N-1}}^\infty \cdots \int_{x_1}^\infty f(t)d(t;\, q)d(x_1;\, q)\ldots d(x_{N-1};\, q) \\ &= \frac{q^{-N(N-1)/2}}{[N-1]!}\int_x^\infty (t-x)_{N-1}f(tq^{1-N})d(t;\, q) \end{aligned} \tag{1.3}$$

where, for real or complex α and N a positive integer,

$$[\alpha] = \frac{1-q^\alpha}{1-q},\ [0]! = 1,\ [N]! = [1][2]\ldots[N]$$

and

$$(t-x)_0 = 1,\ (t-x)_N = (t-x)(t-qx)\ldots(t-q^{N-1}x). \tag{1.4}$$

We shall also use the notation

$$(a)_0 = 1, \quad (a)_N = (1-a)(1-qa)\ldots(1-q^{N-1}a).$$

The q-binomial coefficient is defined by

$$\begin{bmatrix}\alpha\\0\end{bmatrix} = 1,\ \begin{bmatrix}\alpha\\k\end{bmatrix} = \frac{[\alpha][\alpha-1]\ldots[\alpha-k+1]}{[k]!},$$

so that we have

$$(t-x)_N = \sum_{k=0}^{N}(-1)^k\begin{bmatrix}N\\k\end{bmatrix} q^{\frac{1}{2}k(k-1)}t^{N-k}x^k. \tag{1.5}$$

† Dedicated to the memory of my friend E. L. Whitney.

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The purpose of this note is to obtain fractional generalisation of (1.3) as well as a q-analogue of the fractional operator (3.1) of Erdélyi and Sneddon. Elsewhere we shall give similar results for fractional q-integrals based on the q-analogue of $\int_a^x f(t)dt$.

We now give a few preliminary results.

An analogue of the exponential function is

$$e(u) = \prod_0^\infty (1-uq^n)^{-1} = \sum_{n=0}^\infty \frac{u^n}{(q)_n}. \tag{1.6}$$

The binomial (1.5) can be extended to non-integral values of N by means of

$$(x-y)_\nu = x^\nu \frac{e(q^\nu y/x)}{e(y/x)}. \tag{1.7}$$

In view of the known identity [(5), p. 92]

$$\frac{e(u)}{e(au)} = \sum_{k=0}^\infty \frac{(a)_k}{(q)_k} u^k$$

formula (1.7) can be written as

$$(x-y)_\nu = x^\nu \sum_{k=0}^\infty (-1)^k \begin{bmatrix} \nu \\ k \end{bmatrix} q^{\frac{1}{2}k(k-1)}(y/x)^k. \tag{1.8}$$

From (1.7) and (1.8) we see that

$$(1+x)_\alpha(1+xq^\alpha)_\beta = (1+x)_{\alpha+\beta} \tag{1.9}$$

so that, upon equating coefficients of x^n, we get

$$\sum_{s=0}^n \begin{bmatrix} \alpha \\ n-s \end{bmatrix} \begin{bmatrix} \beta \\ s \end{bmatrix} q^{s^2-ns+s\alpha} = \begin{bmatrix} \alpha+\beta \\ n \end{bmatrix}. \tag{1.10}$$

As a q-analogue of the gamma function we define

$$\Gamma_q(\alpha) = \frac{e(q^\alpha)}{e(q)(1-q)^{\alpha-1}} \quad (\alpha \neq 0, -1, -2, -3, \ldots). \tag{1.11}$$

This function satisfies the functional equation $\Gamma_q(\alpha+1) = [\alpha]\Gamma_q(\alpha)$ and if $\alpha = n$, a positive integer, we have $\Gamma_q(n+1) = [n]!$.

An analogue of Gauss' theorem for the sum of hypergeometric functions is [(5), p. 97]

$${}_2\phi_1[b, q^{-n}; d; q] = \sum_{k=0}^n \frac{(q^{-n})_k(b)_k}{(q)_k(d)_k} q^k = \frac{(d/b)_n}{(d)_n} b^n. \tag{1.12}$$

In the following we shall assume that $0<q<1$ and that the functions under consideration are such that the series in (1.1) is absolutely convergent. In particular this implies that $\lim_{x=\infty} f(x) = 0$.

2. We are now in position to define the fractional generalisation of (1.3). We put, for arbitrary $\nu \neq 0, -1, -2, \ldots,$

$$K_q^{-\nu}f(x) = \frac{q^{-\frac{1}{2}\nu(\nu-1)}}{\Gamma_q(\nu)}\int_x^{\infty}(t-x)_{\nu-1}f(tq^{1-\nu})d(t;q),$$
$$K_q^0 f(x) = f(x). \tag{2.1}$$

This is a q-analogue of the Weyl fractional integral

$$K^{-\nu}f(x) = \frac{1}{\Gamma(\nu)}\int_x^{\infty}(t-x)^{\nu-1}f(t)dt.$$

When $\nu = n$, a positive integer, formula (2.1) reduces to (1.3). On the other hand formula (2.1) can be written as

$$K_q^{-\nu}f(x) = q^{-\frac{1}{2}\nu(\nu+1)}x^{\nu}(1-q)^{\nu}\sum_{k=0}^{\infty}(-1)^k\begin{bmatrix}-\nu\\k\end{bmatrix}q^{\frac{1}{2}k(k-1)}f(xq^{-\nu-k}). \tag{2.3}$$

This formula is now valid for all ν and, in fact, when $\nu = -n$ a negative integer, (2.3) reduces to

$$K_q^n f(x) = x^{-n}(1-q)^{-n}\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}q^{\frac{1}{2}k(k-1)-\frac{1}{2}n(n-1)}f(xq^{n-k}) \tag{2.4}$$

which is a well-known formula (4) for $(-1)^n D_q^n f(x)$.

It follows immediately from the definition (2.1) that

$$K_q^{\alpha}(c_1f_1+c_2f_2) = c_1K_q^{\alpha}f_1+c_2K_q^{\alpha}f_2.$$

We now proceed to prove that

$$K_q^{\alpha}.K_q^{\beta}f(x) = K_q^{\alpha+\beta}f(x) \tag{2.5}$$

valid for all α and β.

To prove (2.5) we have by means of (2.3)

$$\begin{aligned}K_q^{\alpha}.K_q^{\beta}f(x) &= K_q^{\alpha}\left\{q^{-\frac{1}{2}\beta(\beta-1)}x^{-\beta}(1-q)^{-\beta}\sum_{k=0}^{\infty}(-1)^k\begin{bmatrix}\beta\\k\end{bmatrix}q^{\frac{1}{2}k(k-1)}f(xq^{\beta-k})\right\}\\
&= q^{-\frac{1}{2}\alpha(\alpha-1)-\frac{1}{2}\beta(\beta-1)}(1-q)^{-\alpha-\beta}x^{-\alpha}\sum_{r=0}^{\infty}(-1)^r\begin{bmatrix}\alpha\\r\end{bmatrix}q^{\frac{1}{2}r(r-1)}\\
&\quad .\sum_{k=0}^{\infty}(-1)^k\begin{bmatrix}\beta\\k\end{bmatrix}q^{\frac{1}{2}k(k-1)}(xq^{\alpha-r})^{-\beta}f(xq^{\alpha+\beta-k-r})\\
&= q^{-\frac{1}{2}\alpha(\alpha-1)-\frac{1}{2}\beta(\beta-1)-\alpha\beta}(1-q)^{-\alpha-\beta}\sum_{n=0}^{\infty}(-1)^n q^{\frac{1}{2}n(n-1)}f(xq^{\alpha+\beta-n})\\
&\quad .x^{-\alpha-\beta}.\sum_{r=0}^{n}\begin{bmatrix}\alpha\\r\end{bmatrix}\begin{bmatrix}\beta\\n-r\end{bmatrix}q^{r^2-nr+r\beta}.\end{aligned}$$

The inner sum in the right-hand side can be evaluated by means of (1.10). We get

$$\begin{aligned}K_q^{\alpha}K_q^{\beta}f(x) &= q^{-\frac{1}{2}(\alpha+\beta)(\alpha+\beta-1)}(1-q)^{-\alpha-\beta}x^{-\alpha-\beta}\\
&\qquad \sum_{n=0}^{\infty}(-1)^n\begin{bmatrix}\alpha+\beta\\n\end{bmatrix}q^{\frac{1}{2}n(n-1)}f(xq^{\alpha+\beta-n})\\
&= K_q^{\alpha+\beta}f(x).\end{aligned}$$

3. Erdélyi and Sneddon (3) defined the fractional operator

$$K^{\eta,\alpha}f(x) = \frac{x^{\eta}}{\Gamma(\alpha)} \int_x^{\infty} (y-x)^{\alpha-1} y^{-\alpha-\eta} f(y)dy. \tag{3.1}$$

A q-analogue of this is

$$K_q^{\eta,\alpha}f(x) = \frac{q^{-\eta}x^{\eta}}{\Gamma_q(\alpha)} \int_x^{\infty} (y-x)_{\alpha-1} y^{-\eta-\alpha} f(yq^{1-\alpha})d(y;q) \tag{3.2}$$

where $\alpha \neq 0, -1, -2, \ldots$. This formula can also be written as

$$K_q^{\eta,\alpha}f(x) = \frac{1-q}{\Gamma_q(\alpha)} \sum_{k=0}^{\infty} q^{k\eta}(1-q^{k+1})_{\alpha-1} f(xq^{-\alpha-k}). \tag{3.3}$$

By means of (1.7) and (1.10) we can write

$$K_q^{\eta,\alpha}f(x) = (1-q)^{\alpha} \sum_{k=0}^{\infty} (-1)^k q^{k(\eta+\alpha)+\frac{1}{2}k(k+1)} \begin{bmatrix} -\alpha \\ k \end{bmatrix} f(xq^{-\alpha-k}). \tag{3.4}$$

Formula (3.4) is valid for all α and may be taken as a continuation of (3.1).

If $\alpha = -N$, a negative integer, we get

$$K_q^{\eta,-N}f(x) = (1-q)^{-N} \sum_{k=0}^{N} (-1)^k q^{k(\eta-N)+\frac{1}{2}k(k-1)} \begin{bmatrix} N \\ k \end{bmatrix} f(xq^{N-k}).$$

Comparing this with (2.5) we see that

$$K_q^{\eta,-N}f(x) = (-1)^N q^{-\frac{1}{2}N(N-\eta+1)} x^{\eta} D_q^N \{x^{-\eta+N}f(x)\}. \tag{3.5}$$

Let us consider the expression

$$D_q^N\{x^{-\eta-\alpha}K_q^{\eta,\alpha+N}f(x)\}.$$

If we substitute for $K_q^{\eta,\alpha+N}f(x)$ from (3.4) and then q-differentiate the resulting expression N times by means of (2.5) we obtain, using formula (1.10),

$$K_q^{\eta,\alpha}f(x) = (-1)^N q^{\frac{1}{2}N(N+\eta+\alpha-1)} x^{\eta+N+\alpha} D_q^N\{x^{-\eta-\alpha}K_q^{\eta,\alpha+N}\}. \tag{3.6}$$

We now prove that

$$K_q^{\eta,\alpha}K_q^{\eta+\alpha,\beta}f(x) = K_q^{\eta,\alpha+\beta}f(x), \tag{3.7}$$

valid for all η, α, β.

The left-hand side is

$$\frac{e(q)e(q)(1-q)^{\alpha+\beta}}{e(q^{\alpha})e(q^{\beta})} \sum_{s=0}^{\infty} q^{s\eta} f(xq^{-\alpha-\beta-s}) \sum_{k=0}^{s} q^{k\alpha}(1-q^{k+1})_{\beta-1}(1-q^{s+1-k})_{\alpha-1}.$$

The inside sum is equal to

$$q^{\alpha}\frac{e(q^{\beta})e(q^{s+\alpha-1})}{e(q)e(q^s)}\,{}_2\phi_1(q^{1-s}, q;\, q^{2-s-\alpha};\, q).$$

This expression, by (1.12), is equal to

$$q^{\alpha}\frac{e(q^{\beta})e(q^{\alpha})e(q^{\alpha+\beta+s-1})}{e(q)e(q^{\alpha+\beta})e(q^{s})}=q^{\alpha}\frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)}(1-q^{s})_{\alpha+\beta-1}.$$

We thus get

$$K_q^{\eta,\alpha}K_q^{\eta+\alpha,\beta}f(x)=\frac{1-q}{\Gamma_q(\alpha+\beta)}\sum_{s=0}^{\infty}q^{s\eta}(1-q^{s+1})_{\alpha+\beta-1}f(xq^{-\alpha-\beta-s})$$
$$=K_q^{\eta,\alpha+\beta}f(x).$$

This completes the proof of (3.7).

From (3.7) it follows that

$$\{K_q^{\eta,\alpha}\}^{-1}f(x)=K_q^{\eta+\alpha,-\alpha}f(x). \tag{3.8}$$

It is easy to see that

$$K_q^{\eta,\alpha}\{x^{\beta}f(x)\}=x^{\beta}q^{(1-\alpha)\beta}K_q^{\eta-\beta,\alpha}f(x). \tag{3.9}$$

The relationship between the two fractional operators we defined above is

$$K_q^{\eta,\alpha}f(x)=x^{\eta}q^{-\frac{1}{2}\alpha(\alpha-1)-\alpha\eta}K_q^{-\alpha}\{x^{-\eta-\alpha}f(x)\}. \tag{3.10}$$

4. We give now a short table of transform pairs. Because of (3.10) we shall give only those involving the operator $K_q^{\eta,\alpha}$.

We first recall that

$${}_r\phi_s(a_1,a_2,\dots,a_r;\,b_1,b_2,\dots,b_s;\,x)=\sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k\dots(a_r)_k}{(q)_k(b_1)_k(b_2)_k\dots(b_s)_k}x^k$$

and that

$$\frac{\Gamma_q(\mu)}{\Gamma_q(\mu+\alpha)}=(1-q)^{\alpha}\prod_{s=0}^{\infty}\frac{1-q^{\mu+\alpha+s}}{1-q^{\mu+s}}.$$

For brevity we shall write the left-hand side. We shall also write $f(x)\to g(x)$ instead of $K_q^{\eta,\alpha}f(x)=g(x)$.

$$x^{\lambda}\to q^{-\alpha\lambda}\frac{\Gamma_q(\eta-\lambda)}{\Gamma_q(\eta-\lambda+\alpha)}x^{\lambda} \tag{4.1}$$

$$x^{\lambda+N}\to q^{-\alpha\lambda}\frac{\Gamma_q(\eta-\lambda)}{\Gamma_q(\eta-\lambda+\alpha)}\frac{(q^{1-\eta-\alpha+\lambda})_N}{(q^{1-\eta+\lambda})_N}x^{\lambda+N}\ (N\text{, a positive integer}). \tag{4.2}$$

$$x^{\eta+\alpha}(x+b)_{\lambda}\to x^{\eta}q^{-\alpha(\eta+\lambda+\alpha)}\frac{\Gamma_q(-\alpha-\lambda)}{\Gamma_q(-\lambda)}(x+b)_{\lambda+\alpha} \tag{4.3}$$

$$x^{-\lambda+\eta}e(c/x)\to x^{(\eta-\lambda)}q^{\alpha(\lambda-\eta)}\frac{\Gamma_q(\lambda)}{\Gamma_q(\lambda+\alpha)}\,{}_1\phi_1(q^{\lambda};\,q^{\lambda+\alpha};\,cq^{\alpha}/x). \tag{4.4}$$

$$x^{\mu+\eta-1}e(x)\to x^{\mu+\eta-1}q^{-\alpha(\mu+\eta-1)}\frac{\Gamma_q(1-\mu)}{\Gamma_q(1-\mu+\alpha)}\,{}_1\phi_1(q^{\mu-\alpha};\,q^{\mu};\,x). \tag{4.5}$$

$$x^{\mu+\eta-1}(b+x)_\nu \to b^\nu x^{\mu+\eta-1} \frac{\Gamma_q(1-\mu)}{\Gamma_q(1+\alpha-\mu)} {}_2\phi_1(q^{-\nu}, q^{\mu-\alpha}; q^\mu; xq^\nu/b). \quad (4.6)$$

$$x^{\eta-\nu-1}(b+x)_\nu \to b^{-\alpha} x^{\eta-\nu-1} q^{\alpha(1-\eta-\nu)} \frac{\Gamma_q(\nu+1)}{\Gamma_q(\nu+\alpha+1)} (bq^\alpha+x)_{\alpha+\nu} \quad (\nu \neq 0). \quad (4.7)$$

$$x^{\lambda+\eta-1} {}_2\phi_1(q^a, q^b; q^c; x)$$
$$\to x^{\lambda+\eta-1} q^{-\alpha(\lambda+\eta-1)} \frac{\Gamma_q(1-\lambda)}{\Gamma_q(1+\alpha-\lambda)} {}_3\phi_2(q^a, q^b, q^{\lambda-\alpha}; q^\lambda, q^c; x) \quad (4.8)$$

Formula (4.8) reduces, when $\lambda = a$, $\alpha = -n$ or when $\lambda = c+\alpha$, $\alpha = -n$, to formulae of Agarwal (**1**). Formula (4.8) can be extended further so that the left-hand side involves a ${}_s\phi_{s-1}$ and the right-hand side a ${}_{s+1}\phi_s$.

We now illustrate an application of the above formulae. We have from (1.9) and (1.8)

$$\sum_{k=0}^{\infty} \frac{(q^{-\nu})_k}{(q)_k} q^{k(\nu+\lambda)} x^{\mu+k}(1+x)_\lambda = x^\mu(1+x)_{\lambda+\nu}.$$

Now applying (4.6) we get

$${}_2\phi_1(q^{-\lambda-\nu}, q^\alpha; q^\mu; xq^{\lambda+\nu}) = \sum_{k=0}^{\infty} \frac{(q^{-\nu})_k(q^\alpha)_k}{(q)_k(q^\mu)_k} q^{k(\lambda+\nu)} \cdot {}_2\phi_1(q^{-\lambda}, q^{\alpha+k}, q^{\mu+k}; xq^{\mu+k}).$$

REFERENCES

(**1**) R. P. Agarwal, Associated basic hypergeometric series, *Proc. Glasgow Math. Assoc.* **1** (1952-53), 182-184.

(**2**) W. A. Al-Salam, q-analogues of Cauchy's formula, *Proc. Amer. Math. Soc.* **17** (1966), 616-621.

(**3**) A. Erdélyi and I. N. Sneddon, Fractional integration and dual integral equations, *Canad. J. Math.* **14** (1962), 685-693.

(**4**) W. Hahn, Uber die höheren Heineschen Reihen und eine einheitliche Theorie der sogenannten speziellen Funktionen, *Math. Nachr.* **3** (1950), 257-294.

(**5**) L. J. Slater, *Generalized Hypergeometric Functions* (Cambridge University Press, 1966).

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