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On the problem of non-smoothness of non-reflexive second conjugate spaces

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We prove that if E is a Banach space which has a subspace G such that the conjugate space G^* contains a proper norm closed linear subspace V of characteristic 1, then E^{**} is not smooth and there exist in $\pi_E(E)$ points of non-smoothness for E^{**}, where $\pi_E : E \to E^{**}$ is the canonical embedding. We show that the spaces E having such a subspace G constitute a large proper subfamily of the family of all non-reflexive Banach spaces.

1.

A Banach space E is called *smooth* if for every $x \in E$ with $\|x\| = 1$ there exists a unique $f \in E^*$ with $\|f\| = 1$ such that $f(x) = 1$. If E is not smooth, any $x \in E$ with $\|x\| = 1$ for which there are $f_1, f_2 \in E^*$ with $f_1 \neq f_2$, $\|f_1\| = \|f_2\| = 1$, satisfying $f_1(x) = f_2(x) = 1$, is called a *point of non-smoothness* for E.

Giles [*6*], Day ([*3*], p. 70), Pełczyński, Phelps and Rainwater [*12*] have proved that a non-reflexive Banach space E has non-smooth third conjugate space E^{***} and that for any $f \in E^*$ with $\|f\| = 1$ which does

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not attain its norm on the unit sphere of E, $\pi_{E^*}(f)$ is a point of non-smoothness for E^{***}, where $\pi_{E^*} : E^* \to E^{***}$ is the canonical embedding. In the present paper we shall study the problem whether the second conjugate E^{**} of a non-reflexive Banach space E is non-smooth, raised by Rainwater [12], and the problem whether one can find points of non-smoothness for E^{**} already in $\pi_E(E)$, where $\pi_E : E \to E^{**}$ is the canonical embedding. In §2 we shall prove that this is indeed the case for a large family of non-reflexive spaces E, including non-reflexive conjugate spaces (which yields again the result mentioned above) and spaces E for which $\operatorname{dens} E < \operatorname{dens} E^*$, where $\operatorname{dens} E$ denotes the density character of E (that is, the smallest cardinality of a dense set in E). In §3 we shall prove that there exist non-reflexive Banach spaces E which do not satisfy our sufficient condition for the non-smoothness of E^{**}.

The notations and terminology used here will be the standard ones (see, for example, [3], [13]). We recall (see [4]) that the characteristic of a subspace V (by subspace we shall always mean *norm closed linear* subspace) of a conjugate space E^* is the number

$$r(V) = \inf_{\substack{x\in E \\ x\neq 0}} \sup_{\substack{f\in V \\ \|f\|\leq 1}} \left| f\left(\frac{x}{\|x\|}\right) \right| = \inf_{\substack{x\in E, \|x\|=1 \\ \Phi\in V^{\perp}}} \|\pi_E(x)+\Phi\| ,$$

where $V^{\perp} = \{\Phi \in E^{**} \mid \Phi(f) = 0 \ (f \in V)\}$. Thus, $r(E^*) = 1$, $0 \leq r(V) \leq 1$ $(V \subset E^*)$, and we have $r(V) = 1$ if and only if

$$\|x\| = \sup_{\substack{f\in V \\ \|f\|=1}} |f(x)| , \quad (x \in E) ,$$

or equivalently, if and only if the projection p of $\pi_E(E) \oplus V^{\perp}$ onto $\pi_E(E)$ along $V^{\perp}$ has norm $\|p\| = 1$. Also, we recall that if the conjugate space E^* is separable, the norm of E is called [5] a *Kadec'-Klee norm* if the relations $\{f_n\} \subset E^*$, $f \in E^*$, $f_n \xrightarrow{w^*} f$, $\|f_n\| \to \|f\|$ imply $\|f_n-f\| \to 0$. In [5], Corollary 1, it was proved that every Banach space E with separable conjugate space E^* admits an equivalent norm such that in this new norm for every proper subspace V of E^* we have

$r(V) < 1$, namely, any equivalent Kadec'-Klee norm has this property (equivalent Kadec'-Klee norms exist by [8], [9]).

2.

DEFINITION 1. We shall say that a Banach space E has

(a) *property* (CH_1) , if the conjugate space E^* contains a proper subspace V (that is, such that $V \neq E^*$) of characteristic $r(V) = 1$,

(b) *property* (SCH_1) , if E contains a subspace G with property (CH_1) .

THEOREM 1. *If E is a Banach space with property (SCH_1) , then E^{**} is not smooth. Moreover, there exist in $\pi_E(E)$ points of non-smoothness for E^{**} .*

Proof. Let us first assume that the theorem holds for every Banach space with property (CH_1) and let G be a subspace, with property (CH_1) , of E . Then, by our assumption, $G^{**} \equiv G^{\perp\perp} \subset E^{**}$ is not smooth, hence E^{**} is not smooth. Also, by our assumption, let $x \in G$ be such that $\pi_G(x)$ is a point of non-smoothness for G^{**} , so there exist $\psi_1, \psi_2 \in G^{***}$ with $\psi_1 \neq \psi_2$ such that $\|\psi_1\| = \|\psi_2\| = 1$, $\psi_1(\pi_G(x)) = \psi_2(\pi_G(x)) = 1$. If u is the identical embedding of G into E , then $\pi_E u = u^{**}\pi_G$, u^{***} maps E^{***} onto G^{***} and for each $\psi \in G^{***}$ there exists $\Xi \in E^{***}$ with $u^{***}(\Xi) = \psi$, $\|\Xi\| = \|\psi\|$. Hence, if $\Xi_1, \Xi_2 \in E^{***}$ are such that $u^{***}(\Xi_j) = \psi_j$, $\|\Xi_j\| = \|\psi_j\| = 1$ $(j = 1, 2)$, then $\Xi_1 \neq \Xi_2$ and

$$\Xi_j(\pi_E u(x)) = \Xi_j(u^{**}\pi_G(x)) = u^{***}(\Xi_j)(\pi_G(x)) = \psi_j(\pi_G(x)) = 1 , \quad (j = 1, 2) ,$$

so $\pi_E u(x)$ is a point of non-smoothness for E^{**} . Thus, it is enough to prove Theorem 1 under the assumption that E has property (CH_1) .

Let V be a proper subspace of E^* with $r(V) = 1$. It will be

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enough to prove that there exist in $\pi_E(E)$ points of non-smoothness for the subspace $\pi_E(E) \oplus V^{\perp}$ of E^{**}, since then by the Hahn-Banach Theorem these will be points of non-smoothness also for E^{**}.

Since $V \neq E^*$ and V is norm-closed, there exists $f \in E^* \setminus V$ with $\|f\| = 1$ such that $f(x) = 1$ for some $x \in E$ with $\|x\| = 1$ (indeed, by the Bishop-Phelps Theorem [1] those $f \in E^*$ with $\|f\| = 1$ which admit such an x are norm dense in the unit sphere of E^*). We shall show that $\pi_E(x) \in \pi_E(E)$ is a point of non-smoothness for $\pi_E(E) \oplus V^{\perp}$.

Let us denote by ρ the restriction mapping

$$\rho(\Xi) = \Xi|_{\pi_E(E) \oplus V^{\perp}}, \quad (\Xi \in E^{***}),$$

and by p the projection of $\pi_E(E) \oplus V^{\perp}$ onto $\pi_E(E)$ along $V^{\perp}$ (hence, by $r(V) = 1$ we have $\|p\| = 1$). We shall show that the functionals

$$\varphi_1 = \rho\pi_{E^*}(f), \quad \varphi_2 = \left(\pi_E^{-1}p\right)^*(f) \in \left(\pi_E(E) \oplus V^{\perp}\right)^*$$

satisfy, for any $\Phi \in V^{\perp}$ with $\Phi \neq 0$ (such a Φ exists, as $f \in E^* \setminus V$),

$$\varphi_1(\Phi) \neq 0 = \varphi_2(\Phi), \quad \varphi_1\left(\pi_E(x)\right) = \varphi_2\left(\pi_E(x)\right) = 1, \quad \|\varphi_1\|, \|\varphi_2\| \leq 1,$$

which will complete the proof. Indeed, we have

$$\varphi_1(\Phi) = \rho\pi_{E^*}(f)(\Phi) = \pi_{E^*}(f)|_{\pi_E(E)\oplus V^{\perp}}(\Phi) = \pi_{E^*}(f)(\Phi) = \Phi(f) \neq 0,$$

$$\varphi_2(\Phi) = \left(\pi_E^{-1}p\right)^*(f)(\Phi) = f\left(\pi_E^{-1}p(\Phi)\right) = f\left(\pi_E^{-1}(0)\right) = f(0) = 0,$$

$$\varphi_1\left(\pi_E(x)\right) = \pi_{E^*}(f)|_{\pi_E(E)\oplus V^{\perp}}\left(\pi_E(x)\right) = \pi_{E^*}(f)\left(\pi_E(x)\right) = \pi_E(x)(f) = f(x) = 1,$$

$$\varphi_2\left(\pi_E(x)\right) = f\left(\pi_E^{-1}p\left(\pi_E(x)\right)\right) = f\left(\pi_E^{-1}\pi_E(x)\right) = f(x) = 1,$$

$$\|\varphi_1\| \leq \|\rho\|\|\pi_{E^*}\|\|f\| = 1, \quad \|\varphi_2\| \leq \|p^*\|\left\|\left(\pi_E^{-1}\right)^*\right\|\|f\| = 1,$$

which completes the proof of Theorem 1.

REMARK 1. Clearly, for every $\pi_E(x) \in \pi_E(E)$ with $\|\pi_E(x)\| = 1$, the

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unit ball $S_{E^{**}}$ admits a weak* closed support hyperplane at $\pi_E(x)$; that is , of the form $H_1 = \{\Psi \in E^{**} \mid \pi_{E^*}(f)(\Psi) = 1\}$ for some $f \in E^*$ with $\|f\| = 1$. The above proof shows that if E is smooth and has property (CH_1) , then *there exists a* $\pi_E(x) \in \pi_E(E)$ *with* $\|\pi_E(x)\| = 1$, *such that* $S_{E^{**}}$ *admits also a weak* dense support hyperplane at* $\pi_E(x)$; that is, of the form $H_2 = \{\Psi \in E^{**} \mid \Xi(\Psi) = 1\}$, for some $\Xi \in E^{***}\backslash\pi_{E^*}(E^*)$ with $\|\Xi\| = 1$ (containing the support line through $\pi_E(x)$ and $\pi_E(x) + \Phi$, where $\Phi \in V^{\perp}$ is as above).

REMARK 2. The conclusion of Theorem 1 remains valid, with a similar proof, *for any space* E *for which there exists a triple* (Φ, f, x) *with* $\Phi \in E^{**}$, $f \in E^*$, $x \in E$ *of norm* $\|\Phi\| = \|f\| = \|x\| = 1$, such that

$$\Phi(f) \neq 0 , \quad f(x) = 1 , \quad \|\pi_E(x)+\alpha\Phi\| \geq \|\pi_E(x)\| = 1 \quad \text{for all scalars } \alpha .$$

Indeed, it is enough to replace then $\pi_E(E) \oplus V^{\perp}$ by the two-dimensional subspace $[\pi_E(x)] \oplus [\Phi]$ of E^{**} and the functionals φ_1, φ_2 above by

$$\varphi_1' = \rho_{[\pi_E(x)]\oplus[\Phi]}\pi_{E^*}(f) , \quad \varphi_2' = p_0^*\rho_{[\pi_E(x)]}\left(\pi_E^{-1}\right)^*(f) \in \left([\pi_E(x)] \oplus [\Phi]\right)^* ,$$

where $\rho|_{\Gamma}(\Xi) = \Xi|_{\Gamma}$ for all $\Xi \in E^{***}$ and for any subspace Γ of E^{**} and where p_0 denotes the projection of $[\pi_E(x)] + [\Phi]$ onto $[\pi_E(x)]$ along $[\Phi]$.

REMARK 3. There may exist also other points of non-smoothness for E^{**} . For example, *if there exists a* $\Phi \in E^{**}\backslash\pi_E(E)$ *with* $\|\Phi\| = 1 = \operatorname{dist}(\Phi, \pi_E(E))$ *which attains its norm at some* $f \in E^*$ *with* $\|f\| = 1$, *then* Φ *is a point of non-smoothness for* E^{**} . Indeed, on the one hand, $\pi_{E^*}(f)(\Phi) = 1$, so the weak* closed hyperplane $H_1 = \{\Psi \in E^{**} \mid \pi_{E^*}(f)(\Psi) = 1\}$ supports $S_{E^{**}}$ at Φ and, on the other hand, by a corollary of the Hahn-Banach Theorem there exists $\Xi \in \pi_E(E)^{\perp} \subset E^{***}\backslash\pi_{E^*}(E^*)$ with $\|\Xi\| = 1$, $\Xi(\Phi) = 1$, so the weak* dense

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hyperplane $H_2 = \{\Psi \in E^{**} \mid \Xi(\Psi) = 1\}$ also supports $S_{E^{**}}$ at Φ. However, it is well known that there are Banach spaces for which there exists no $\Phi \in E^{**}\setminus\pi_E(E)$ such that $\|\Phi\| = 1 = \operatorname{dist}(\Phi, \pi_E(E))$.

Let us give now some corollaries of Theorem 1.

COROLLARY 1. *If* dens $E <$ dens E^*, *then* E^{**} *is not smooth and there exist in* $\pi_E(E)$ *points of non-smoothness of* E^{**}.

Proof. It is well known that E has property (CH_1) (it is enough to take a dense set $\{x_i\}_{i\in I}$ in S_E with card $I =$ dens E and then for each $i \in I$ a functional $f_i \in S_{E^*}$ with $f_i(x_i) = 1$ and to put $V = [f_i]_{i\in I}$, the closed linear span of $\{f_i\}_{i\in I}$).

We also obtain as a corollary the following slight improvement of the Giles-Rainwater result:

COROLLARY 2. *If* E *contains a subspace* G *isometric to a non-reflexive conjugate space* B^*, *then* E^{**} *is not smooth and there exist in* $\pi_E(E)$ *points of non-smoothness for* E^{**}.

Proof. It is well known that G has property (CH_1) (take $V \subset G^*$ to be the image of $\pi_B(B) \subset B^{**}$ under the isometry $B^{**} \equiv G^*$).

Finally, let us observe that in the above cases E cannot have any one of the properties implied by the smoothness of E^{**}, for example:

COROLLARY 3. *If* E *satisfies the condition of Theorem* 1 *(or of Corollary* 1 *or* 2*), then* E^{***} *is not strictly convex.*

Let us also observe that *if a Banach space* E *contains a subspace* G *isomorphic to* c_0, *then* E^{**} *is not smooth.* Indeed, $G^{**} \equiv G^{\perp\perp} \subset E^{**}$ is then isomorphic to l^∞ and hence is not smooth, by a result of Day [2], so E^{**} is not smooth.

If there exists a non-reflexive space E_0 with smooth second conjugate space E_0^{**}, then (by passing to a subspace, if necessary) we may assume that E_0 is separable (even that E_0 has a basis, by [10]) and

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then by the above results, E_0 must have the following properties:

(a) E_0^* is separable;

(b) E_0 contains no subspace isomorphic to c_0 ;

(c) E_0 contains no non-reflexive subspace isometric to a conjugate Banach space B^* .

Therefore it is natural to raise

PROBLEM 1. Does the quasi-reflexive space $E = J$ of James [7] admit an equivalent norm $|||\cdot|||$ such that $(E^{**}, |||\cdot|||)$ is smooth?

3.

Clearly, a Banach space with property (SCH_1) is non-reflexive. Unfortunately, there are non-reflexive spaces which do not have property (SCH_1) , as we shall show below (and therefore Theorem 1 alone does not imply that every non-reflexive Banach space E has non-smooth second conjugate space).

THEOREM 2. *Let E be a non-reflexive Banach space with separable conjugate space E^* . Then E admits an equivalent norm $|||\cdot|||$ such that $(E, |||\cdot|||)$ does not have property (SCH_1) . In fact, every equivalent Kadec'-Klee norm on E satisfies this condition.*

Proof. By §1, for any equivalent Kadec'-Klee norm $|||\cdot|||$, $(E, |||\cdot|||)$ does not have property (CH_1) . Therefore it will be sufficient to prove that *for any subspace $G \subset E$ the restriction to G of a Kadec'-Klee norm $|||\cdot|||$ on E is a Kadec'-Klee norm on G .*

Let $\{\varphi_n\} \subset G^*$, $\varphi \in G^*$, $\varphi_n \xrightarrow{w^*} \varphi$, $|||\varphi_n||| \to |||\varphi|||$. We shall show that every subsequence $\{\varphi_{n_k}\} \subset \{\varphi_n\}$ contains a subsequence $\{\varphi_{n_{k_m}}\}$ such that $\left|\left|\left|\varphi_{n_{k_m}} - \varphi\right|\right|\right| \to 0$, which will complete the proof (since then $|||\varphi_n - \varphi||| \to 0$).

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Let $\{f_n\} \subset E^*$, $f \in E^*$ be any Hahn-Banach extensions (that is, with the same norm) of $\{\varphi_n\}$ and φ respectively. Then

$$\sup_k \left|\left|\left| f_{n_k} \right|\right|\right| = \sup_k \left|\left|\left| \varphi_{n_k} \right|\right|\right| < \infty ,$$

whence, since E is separable, $\left\{f_{n_k}\right\}$ contains a subsequence $\left\{f_{n_{k_m}}\right\}$ such that $f_{n_{k_m}} \xrightarrow{w^*} h \in E^*$. Then, from $f_{n_{k_m}}\Big|_G = \varphi_{n_{k_m}} \xrightarrow{w^*} \varphi$, we obtain $h|_G = \varphi$, whence $|||\varphi||| \le |||h|||$. On the other hand,

$$|||h||| \le \varliminf_{m\to\infty} \left|\left|\left| f_{n_{k_m}} \right|\right|\right| = \varliminf_{m\to\infty} \left|\left|\left| \varphi_{n_{k_m}} \right|\right|\right| = |||\varphi||| ,$$

hence $\left|\left|\left| f_{n_{k_m}} \right|\right|\right| \to |||h|||$. Consequently, since the norm on E is a Kadec'-Klee norm, $\left|\left|\left| f_{n_{k_m}} - h \right|\right|\right| \to 0$, whence, by restriction to G , $\left|\left|\left| \varphi_{n_{k_m}} - \varphi \right|\right|\right| \to 0$, which completes the proof of Theorem 2.

REMARK 4. Theorem 2 shows, in particular, that there exist non-reflexive Banach spaces E in which there is no asymptotically monotone non-shrinking basic sequence (although it is well known that in every Banach space E there are asymptotically monotone basic sequences and in every non-reflexive space E there are non-shrinking basic sequences); indeed, for an asymptotically monotone basic sequence the coefficient functionals span a proper subspace of characteristic 1 . In [11] it was proved that for every non-reflexive Banach space E , there exists in $E^{**}\backslash\pi_E(E)$ an asymptotically monotone non-shrinking basic sequence.

REMARK 5. After this paper was completed, Dr J.R. Giles communicated to us that Corollary 1 can be also proved by using a result of Tacon, [14], Lemma 6, p. 420.

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