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ON EXTENSIONS OF GROUPS OF FINITE EXPONENT

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Abstract. A well-known theorem of P. Hall says that if a group G contains a normal nilpotent subgroup N such that G/N' is nilpotent then G is nilpotent. We give a similar sufficient condition for a group G to be an extension of a group of finite exponent by a nilpotent group.

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1. Introduction. The following famous theorem is due to P. Hall [**2**].

THEOREM A. *Let N be a normal subgroup of a group G. If both G/N' and N are nilpotent, then so is G. Futhermore, the nilpotency class of G does not exceed $(c-1)\frac{k(k+1)}{2}+k$, where c and k are the classes of G/N' and N respectively.*

Later Steward showed in [**6**] that actually the class of G in Theorem A is bounded by $(c-1)(k-1)+ck$. In the present paper however we are not concerned with explicit expressions for functions whose existence we are going to prove.

Hall's result proved to be an extremely useful criterion for a soluble group to be nilpotent. In particular the next theorem can be easily deduced from Theorem A.

THEOREM B. *Let $\mathfrak{C}$ be a class of groups that is closed under taking subgroups and quotients. If all metabelian groups in $\mathfrak{C}$ are nilpotent, then so is any soluble group in $\mathfrak{C}$.*

In [**1**], Endimioni and Traustason considered the question whether the above results remain true with "nilpotent" replaced by "torsion-by-nilpotent". They obtained the following analogue to Theorem B.

THEOREM C. *Let $\mathfrak{C}$ be a class of groups that is closed under taking subgroups and quotients. If all metabelian groups in $\mathfrak{C}$ are torsion-by-nilpotent, then so is any soluble group in $\mathfrak{C}$.*

The purpose of the present paper is to establish yet another analogue to Theorem B.

THEOREM 1.1. *Let $\mathfrak{C}$ be a class of groups that is closed under taking subgroups and quotients. If any metabelian group in $\mathfrak{C}$ is an extension of a group of finite exponent by a nilpotent group, then so is any soluble group in $\mathfrak{C}$.*

The above theorem is an immediate consequence of the following quantitative result. We use the term "$\{a, b, c, \ldots\}$-bounded" to mean "bounded from above by some function of $a, b, c, \ldots$".

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THEOREM 1.2. *Let c, d, q be positive integers. Suppose G is a soluble group with derived length d. Assume that for any i the metabelian quotient $G^{(i)}/G^{(i+2)}$ is an extension of a group of finite exponent q by a nilpotent group of class c. Then there exist $\{c, d, q\}$-bounded numbers f and e such that G is an extension of a group of finite exponent e by a nilpotent group of class f.*

In turn, Theorem 1.2 is deduced from the following analogue to Theorem A for soluble groups. We write $f(c, k)$ for the expression $(c-1)\frac{k(k+1)}{2}+k$.

THEOREM 1.3. *Let G be a group having a normal nilpotent of class k subgroup N such that G/N' is an extension of a group of finite exponent q by a nilpotent group of class c. Assume that $\gamma_{c+1}(G)$ is soluble with derived length d. Then $\gamma_{f(k,c)+1}(G)$ has finite $\{c, d, k, q\}$-bounded exponent.*

2. Preliminaries. As usual, if M, N are subgroups of a group G, the subgroup $\langle [x, y] | x \in M, y \in N\rangle$ will be denoted by $[M, N]$. We use the left normed notation: thus if $N_1, N_2, \ldots, N_i$ are subgroups of G, then

$$[N_1, N_2, \ldots, N_i] = [\ldots[[N_1, N_2], N_3], \ldots, N_i].$$

The subgroup generated by all nth powers of elements of M will be denoted by M^n. It follows that $M^{nr} \leq (M^n)^r$ for any n, r. For positive integers d and q we define the function $e(d, q)$ by the rule $e(d, q) = q^d$, if q is odd and $e(d, q) = 2^d q^d$, if q is even.

LEMMA 2.1. *Let N be a soluble normal subgroup of a group G and assume that N has derived length d. Then for any q we have $[N, G]^{e(d,q)} \leq [N^q, G]$.*

Proof. Without any loss of generality we can assume that $[N^q, G] = 1$. The claim is true if N is abelian so we use induction on d. The induction hypothesis will be that $[N', G]^{e(d-1,q)} = 1$. Passing to the quotient $G/[N', G]$ we can assume that $N' \leq Z(G)$, in which case N is nilpotent of class at most 2 and so for any $a \in N$ the map from N to N' that takes an arbitrary element $x \in N$ to $[x, a]$ is a homomorphism whose kernel is $C_N(a)$. Because $[N^q, G] = 1$, it follows that N' has exponent dividing q. Now the Hall-Petrescu formula [**3**, III.9.4] (see also the proof of Lemma VIII.1.1 (b) in [**4**]) gives us $[N, G]^q \leq [N^q, G]N'^{\frac{q}{2}}$ so $[N, G]$ has exponent dividing q if q is odd and $2q$ otherwise. In either case the lemma follows. □

Having fixed d and q, for any $s \geq 1$ we set $e_s = e^{d^{s-1}}$, where $e = e(d, q)$.

COROLLARY 2.2. *Under the hypotheses of Lemma 2.1 for any s we have*

$$[N, \underbrace{G, \ldots, G}_{s}]^{e_s} \leq [N^q, \underbrace{G, \ldots, G}_{s}].$$

Proof. This is straightforward by induction on s. □

LEMMA 2.3. *Let N be a normal subgroup of a group G. Let c, k be positive integers. Set $s = s(c, k) = k(c-1)+1$. Then we have*

$$[\underbrace{N, \ldots, N}_{k} \underbrace{G, \ldots, G}_{s}] \leq [[N, \underbrace{G, \ldots, G}_{c}], \underbrace{N, \ldots, N}_{k-1}].$$

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Proof. We can assume that $[[N, \underbrace{G, \ldots, G}_{c}], \underbrace{N, \ldots, N}_{k-1}] = 1$. In that case, of course, $[\underbrace{N, \ldots, N}_{m}, [N, \underbrace{G, \ldots, G}_{c}], \underbrace{N, \ldots, N}_{k-1-m}] = 1$ for any $m = 0, 1, \ldots, k-1$. Now write

$$[\underbrace{N, \ldots, N}_{k}\underbrace{G, \ldots, G}_{s}] \leq \prod_{j_1+\cdots+j_k=s} [[N, \underbrace{G, \ldots, G}_{j_1}], \ldots, [N, \underbrace{G, \ldots, G}_{j_k}]].$$

This follows easily from equations (8) in [**5**, p. 117]. We notice that the number s is big enough to ensure that at least one of the j_i is bigger than or equal to c. Since $[\underbrace{N, \ldots, N}_{m}, [N, \underbrace{G, \ldots, G}_{c}], \underbrace{N, \ldots, N}_{k-1-m}] = 1$ for any $m = 0, 1, \ldots, k-1$, we derive that $[\underbrace{N, \ldots, N}_{k}\underbrace{G, \ldots, G}_{s}] = 1$, as required. □

3. Main Results. Now we are in a position to prove our main results. Since Theorem 1.1 is immediate from Theorem 1.2, it is sufficient to provide the proofs of Theorem 1.2 and Theorem 1.3. We let $f(c, k)$ stand for $(c-1)\frac{k(k+1)}{2} + k$ and $s = s(c, k)$ have the same meaning as in Lemma 2.3.

Proof of Theorem 1.3. We use induction on k. Set $H = \gamma_{f(k-1,c)+1}(G)$. The induction hypothesis is that the theorem holds if G is replaced by $G/\gamma_k(N)$. In other words, we assume that there exists a $\{c, d, k, q\}$-bounded number a such that $H^a \leq \gamma_k(N)$. Set $b = a^{d^{s(k,c)}}$. We will show that $\gamma_{f(k,c)+1}(G)$ has exponent dividing be_{k-1}. We have

$$(\gamma_{f(k,c)+1}(G))^{be_{k-1}} \leq ([H, \underbrace{G, \ldots, G}_{s}]^b)^{e_{k-1}}.$$

Notice that 2.2 shows that

$$[H, \underbrace{G, \ldots, G}_{s}]^b \leq [H^a, \underbrace{G, \ldots, G}_{s}]$$

so we write

$$([H, \underbrace{G, \ldots, G}_{s}]^b)^{e_{k-1}} \leq [H^a, \underbrace{G, \ldots, G}_{s}]^{e_{k-1}}.$$

Using that $H^a \leq \gamma_k(N)$, we obtain

$$[H^a, \underbrace{G, \ldots, G}_{s}]^{e_{k-1}} \leq [\gamma_k(N), \underbrace{G, \ldots, G}_{s}]^{e_{k-1}}.$$

Now Lemma 2.3 yields

$$[\gamma_k(N), \underbrace{G, \ldots, G}_{s}]^{e_{k-1}} \leq [\gamma_{c+1}(G), \underbrace{N, \ldots, N}_{k-1}]^{e_{k-1}}.$$

By 2.2 the latter expression is contained in $[(\gamma_{c+1}(G))^q, \underbrace{N, \ldots, N}_{k-1}]$ while $(\gamma_{c+1}(G))^q \leq [N, N]$ by the hypothesis. Finally we write

$$[(\gamma_{c+1}(G))^q, \underbrace{N, \ldots, N}_{k-1}] \leq [N, N, \underbrace{N, \ldots, N}_{k-1}] = \gamma_{k+1}(N) = 1.$$

Thus, we have shown that $(\gamma_{f(k,c)+1}(G))^{be_{k-1}} = 1$, as required. □

Proof of Theorem 1.2. Since G' has derived length $d-1$, we can assume by induction on d that there exist $\{c, d, q\}$-bounded numbers k_0 and e_0 such that G' is an extension of a group of finite exponent e_0 by a nilpotent group of class k_0. In particular, $\gamma_{k_0+1}(G')$ has exponent dividing e_0. Passing to the quotient-group $G/\gamma_{k_0+1}(G')$ we can assume without any loss of generality that G' is nilpotent of class at most k_0. By the hypothesis $\gamma_{c+1}(G/G'')$ has exponent dividing q. So, applying Theorem 1.3 with $N = G'$, we obtain that $\gamma_{f(k_0,c)+1}(G)$ has finite $\{c, d, q\}$-bounded exponent. □

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