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# TRIVIALIZING RIBBON LINKS BY KIRBY MOVES

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In this note it is shown that any ribbon link is a sublink of a ribbon link for which surgery on the longitudes gives a connected sum of copies of $S^1 \times S^2$ . In particular there are many links for which the analogue of the knot theoretic Property R fails, and sublinks of homology boundary links need not be homology boundary links. Higher dimensional analogues of these results are also given and it is shown that if $n \geq 2$ the group of a $\mu$-component ribbon $n$-link has a presentation of deficiency $\mu$ . Hence there are high dimensional slice knots which are not ribbon knots.

DEFINITION. A $\mu$-component $n$-link is a locally flat embedding $L : \mu S^n \to S^{n+2}$ . It is a ribbon link if it extends to an immersion $R : \mu D^{n+1} \to S^{n+2}$ with no triple points and such that the components of the singular set are $n$-discs whose boundary $(n-1)$-spheres either lie on $\mu S^n = \partial(\mu D^{n+1})$ ("throughcut") or are disjoint from $\mu S^n$ ("slit").

It is well known and easy to see that ribbon links are null concordant [3]. The converse remains an open conjecture when $n = 1$ , even for knots $(\mu = 1)$ [3], but is false in higher dimensions [8], [19], as will be shown below.

DEFINITION. A $\mu$-component $n$-link $L$ is an homology boundary link if there an epimorphism $\pi : G(L) \to F(\mu)$ , where $G(L) = \pi_1(S^{n+2}-\mathrm{im}L)$ is

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the link group and $F(\mu)$ is the free group of rank $\mu$ . It is a boundary link if it extends to an embedding of $\mu$ disjoint orientable $(n+1)$-manifolds, each with one boundary component.

Smythe had conjectured that if the first Alexander ideal of a 2-component 1-link were zero, then the link would have to be an homology boundary link [15]. In [6] I gave an example of a 2-component ribbon 1-link which was not an homology boundary link, thus refuting this conjecture (for the first $\mu - 1$ Alexander ideals of a null concordant $\mu$-component link must vanish [7]). This example and the following theorem show that a sublink of an homology boundary link need not be an homology boundary link, although a sublink of a boundary link is clearly a boundary link.

THEOREM 1. *Let* $L$ *be a* $\mu$*-component ribbon* $n$*-link. Then* $L$ *is a sublink of a* $\nu$*-component ribbon* $n$*-link* $\hat{L}$ *for which surgery on the longitudes gives* $\overset{\nu}{\#} S^1 \times S^{n+1}$ . *In particular* $\hat{L}$ *is an homology boundary link.*

Proof. Let $R : \mu D^{n+1} \to S^{n+2}$ be a ribbon extending $L$ . Let $S_i$ , $1 \leq i \leq \sigma$ , be the slits of $R$ and for each slit choose a regular neighbourhood $N_i$ contained in the interior of the corresponding disc and such that $N_i \cap N_j = \emptyset$ for $i \neq j$ . Let $\nu = \mu + \sigma$ and let

$\hat{L} = R \mid \left\{\mu S^1 \cup \bigcup_{i=1}^{\sigma} \partial N_i\right\}$ . Clearly $\hat{L}$ is a $\mu$ -component ribbon $n$-link with $L$ as a sublink. If $n > 1$ the normal bundle of $\hat{L}$ in $S^{n+2}$ has an essentially unique framing; if $n = 1$ give each component of $\hat{L}$ the 0-framing. Let $W(\hat{L}) = D^{n+3} \cup_T \nu D^{n+1} \times D^2$ where $T : \nu S^n \times D^2 \to S^{n+2}$ is an embedding of a regular neighbourhood of $\hat{L} = T \mid \nu S^n \times \{0\}$ determined by this framing. Then $\partial W(\hat{L})$ is the result of surgery on $S^{n+2}$ along the longitudes of $\hat{L}$ .

Now by adding a pushoff of $\hat{L}|\partial N_i$ to the component of $L$ bounding the $(n+1)$-disc containing $N_i$ , $\hat{L}$ may be replaced by a ribbon link with

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one less singularity; moreover if $n = 1$ each component of the newlink still has the 0-framing. Continuing thus $\hat{L}$ may be replaced by a ribbon link $\tilde{L}$ for which the only singularities are those corresponding to the components $\partial N_i$ . Clearly these components may be slipped off the ends of the other components of the new ribbon and so $\tilde{L}$ is a trivial $\nu$-component link. Now adding pushoffs of link components to one another (a Kirby move of type 2 [*10*]) corresponds to sliding $(n+1)$-handles of $W(\hat{L})$ across one another, which leaves unchanged the topological type of $W(\hat{L})$ and hence of $\partial M(\hat{L})$ . Thus $\partial W(\hat{L})$ is homeomorphic to $\partial W(\tilde{L})$ , $\partial W(\tilde{L}) = \overset{\nu}{\#} S^1 \times S^{n+1}$ and $G(\hat{L}) = \pi_1(S^{n+2}-\mathrm{im}\hat{L})$ maps onto $\pi_1(\partial W(\hat{L})) \approx F(\nu)$ . //

If $n = 1$ the kernel of the map $G(\hat{L}) \to F(\nu)$ is necessarily $G(\hat{L})_\omega = \bigcap_{n\geq 1} G(\hat{L})_n$ , the intersection of the terms of the lower central series for $G(L)$ [*17*], and is trivial if and only if $\hat{L}$ is trivial, in which case $L$ is also trivial. If $n > 1$ the map $G(\hat{L}) \to F(\nu)$ is an isomorphism, but need not carry meridians to a generating set. (Poenaru gave the first examples of this phenomenon in [*13*].) This is the case if and only if $\hat{L}$ is a boundary link [*4*]. In the latter case, since moreover $\pi_i(S^{n+2}-\mathrm{im}\hat{L}) \approx \pi_i(S^{n+2}-\mathrm{im}\tilde{L}) = 0$ for $2 \leq i < n$ , so if $n > 2$ , $\hat{L}$ (and hence $L$ ) is trivial, by Gutiérrez' unlinking theorem [*4*]. The above theorem is not the best possible, in that fewer new components may suffice to trivialize $L$ thus. For instance, if $L$ is the square knot, it is a component of a 2-component homology boundary link with the above property. Recalling that a 1-knot is said to have Property R if surgery on a longitude of the knot does *not* give $S^1 \times S^2$ , and that is has been conjectured that all nontrivial 1-knots have Property R ([*11*], Problem 1.16), this example shows that the most direct analogue of Property R for links fails already for a 2-component 1-link. However I know of no example of a *boundary* 1-link for which 0-framed surgery gives a connected sum of copies of $S^1 \times S^2$ . If such a link exists, the subgroup $G_\omega$ of the link group must be perfect $(G_\omega = G'_\omega)$ for the longitudes of a boundary link always lie in $G'_\omega$ , as the $\omega$-cover $X^\omega$ of the link complement $X$

may be constructed by splitting along Seifert surfaces [4]. In general, call an homology boundary link with group $G$ intractable if $G_\omega$ is perfect. For an intractable homology boundary link the longitudes are trivially in $G'_\omega$. If $L$ is an intractable boundary link, then all sublinks of $L$ are intractable, and in particular all component knots have Alexander polynomial 1. Conversely if all the component knots of a boundary link have Alexander polynomial 1, and if all its 2-component sublinks are 2-split [16] then it is intractable. In particular the links denoted $k \cup k_n$ in [16], which are iterated doubles of the Whitehead link for $n \geq 1$, are intractable if $n \geq 3$. For a presentation matrix for $G_\omega/G'_\omega = H_1(X^\omega; \mathbb{Z})$ as a module over $\mathbb{Z}[G/G_\omega]$ may be determined from the linking numbers of cycles on the Seifert surfaces as in [5]; the above assumption then implies that these linking numbers are like those of a trivial link. (However for the iterated doubles of the Whitehead link the normal closure of the longitudes in $G$ may be a *proper* subgroup of $G_\omega$.)

Kirby and Melvin showed that any knot which does not have property R is (TOP)·null concordant [12], and this suggests the following complement to the above result.

THEOREM 2. *If $n \geq 2$ and $L$ is a $\nu$-component $n$-link such that surgery on the longitudes of $L$ gives $\#^{\nu} (S^1 \times S^{n+1})$, then $L$ is an homology boundary link and is null concordant. (Hence also any sublink of $L$ is null concordant.)*

Proof. That $L$ is an homology boundary link is clear. Let $U(L)$ be the trace of the surgeries on $L$, so $\partial U(L) = S^{n+2} \coprod \#^{\nu} (S^1 \times S^{n+1})$. Then $D(L) = U(L) \cup (\natural^{\nu} D^2 \times S^{n+1})$ is a contractible $(n+3)$-manifold with boundary $S^{n+2}$, and so is an $(n+3)$-disc. The link $L$ clearly bounds $\nu$ disjoint $(n+1)$-discs in $D(L)$. //

REMARK. If $n = 1$ it can be proven that $L$ bounds $\nu$ embedded discs in a contractible 4-manifold $W_0$, by imitating the first part of the theorem of Kirby and Melvin [12]. Whether the Mazur trick may be used to show that $W_0$ is $D^4$ may be related to the Andrews-Curtis conjecture ([11], Problem 5.7). For this comment I am indebted to Rubinstein, who has also recently proven that if 0-framed surgery on the longitudes of the first $\rho$ components of $L$ gives $\#^{\rho} (S^1 \times S^2)$ , for each $\rho \in \{1, \ldots, \nu\}$ , then $L$ is TOP null concordant [14].

In higher dimensions links which are not homology boundary links but which are sublinks of homology boundary links may be constructed as a consequence of the following theorem.

THEOREM 3. *A finitely presentable group* $G$ *is the group of a* $\mu$*-component sublink of a locally flat* $m$*-link* $L : \nu S^m \to S^{m+2}$ *(for some* $\nu$ *) with group free, if and only if deficiency* $G$ = weight $G = \mu$ . *(If* $m = 2$ *the ambient space may be merely a* homotopy *4-sphere.)*

Proof. The necessity of the condition is obvious. Suppose that $G$ has a presentation $\langle X_i,\ 1 \le i \le \nu | r_j,\ 1 \le j \le \nu-\mu \rangle^{\phi}$ and that $S_k$ , $1 \le k \le \mu$ , are words in $F(\nu)$ such that the normal closure of $\{\phi(S_i) \mid 1 \le i \le \mu\}$ in $G$ is $G$ . The fundamental group of $\#^{\nu} (S^1 \times S^{m+1})$ is naturally isomorphic to $F(\nu)$ , and the words $r_j$ and $S_k$ may be represented by embeddings $\rho_j : S^1 \to \#^{\nu} (S^1 \times S^{m+1})$ and $\sigma_k : S^1 \to \#^{\nu} (S^1 \times S^{m+1})$ respectively. If surgery is performed on all the $\rho_j$ , $1 \le j \le \nu-\mu$ and $\sigma_k$ , $1 \le k \le \mu$ , then the resulting manifold is a homotopy $(m+2)$-sphere, and

$$\#^{\nu} (S^1 \times S^{m+1}) - \bigcup^{\nu-\mu} \rho_j(S^1 \times D^{m+1}) - \bigcup^{\mu} \sigma_k(S^1 \times D^{m+1})$$

is the complement of a $\nu$-component $m$-link in this homotopy sphere with fundamental group $F(\nu)$ (if $m \ge 2$ ) [9]. Therefore if surgery is performed only on the $\rho_j$ , $1 \le j \le \nu-\mu$ , the space

$$\left[\left(\#^{\nu} (S^1 \times S^{m+1}) - \bigcup^{\nu-\mu} \rho_j(S^1 \times D^{m+1})\right) \cup \bigcup^{\nu-\mu} (D^2 \times S^m)\right] - \bigcup^{\mu} \sigma_k(S^1 \times D^{m+1})$$

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is the complement of a $\mu$-component sublink with group presented by $\langle X_i,\ 1 \le i \le \nu \mid r_j,\ 1 \le i \le \nu-\mu \rangle$ , that is, with group $G$ . //

If for instance $G$ has a presentation

$$\left\langle X_1,\ X_2,\ X_3 \mid X_1^{-1}\left[X_3^i,\ X_1\right]\left[X_3^j,\ X_1\right]\right\rangle$$

where $ij \ne 0$ , then according to Baumslag [1], $G$ is parafree but not free, and so cannot map onto $F(2)$ . Thus the link constructed as above from this presentation is not an homology boundary link, although it is a sublink of a 3-component homology boundary link.

An immediate consequence of Theorems 1 and 3 is that if $n > 1$ the group of a $\mu$-component ribbon $n$-link has a presentation of deficiency $\mu$ . Therefore for instance Fox's 2-knot with nonprincipal Alexander ideal [2] is slice [9] but not ribbon. (This was shown earlier by Hitt [8] and Yanagawa [19].)

It is not hard to see that a group $G$ is the group of a $\mu$-component ribbon $n$-link for $n \ge 2$ if and only if $G$ has a Wirtinger presentation of deficiency $\mu$ and $G/G' = \mathbb{Z}^\mu$ . (This was proven for $n = 2$ , $\mu = 1$ by Yajima [18].) The argument is similar to that of the related result characterizing certain quotients of the groups of ribbon 1-links given in [6]: the generators correspond to meridional loops transverse to the components of the complements of the throughcuts, and there is one relation for each throughcut.

If $L$ is a $\nu$-component sublink of an homology boundary link (in any dimension) then the inclusion of the meridians into the link group $G$ induces isomorphisms on all the nilpotent quotients $F(\nu)/F(\nu)_n \to G/G_n$ [17]. It would be interesting to know whether this condition were sufficient for $L$ to be a sublink of an homology boundary link, in particular whether every high dimensional link be a sublink of an homology boundary link.

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