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ON THE DECOMPOSITION OF INTEGRAL LATTICES

THOMAS BIER

The purpose of this note is to record a few formulas relating the indices of various lattices and sublattices all arising from the decomposition of a euclidean space E into three mutually orthogonal subspaces $E = E_0 \perp E_1 \perp E_2$ which are rational with respect to a given lattice $\Lambda \subset E$. In the case that Λ is unimodular these formulas simplify to give very simple identities between various intersection lattices.

1.

We use the following notation: E denotes a euclidean space, $\Lambda \subset E$ any integral lattice with $\operatorname{rank}(\Lambda) = \dim E$. $E_i \subset E$ is a rational subspace if and only if $\operatorname{rank}(\Lambda \cap E_i) = \dim E$. Let $E = E_0 \perp E_1 \perp \ldots \perp E_n$ be a decomposition of E into mutually orthogonal euclidean rational subspaces. For any set $I \subset \{0,1,\ldots,n\}$ we denote by $E_I = \perp_{i \in I} E_i$ the orthogonal sum of the spaces E_i, $i \in I$. We let

(1) $$\Delta_I = \Lambda \cap E_I \ , \ \Gamma_I = P_I(\Lambda) \ ,$$

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be the lattice in E_I obtaining by intersecting Λ with E_I or projecting Λ orthogonally into E_I via the map $P_I : E \to E_I$ respectively. Obviously we have $P_I(\Delta_I) = \Delta_I$, and therefore $\Delta_I \subset \Gamma_I$. We consider only $n = 1$ and $n = 2$.

2.

At first we let $n = 1$ and recall some well-known results (for the proofs see [1]). Let $\Lambda' \subset \Lambda \subset E$ be sublattices with $\text{rank}(\Lambda') = \dim E$:

(2) $$\Gamma_i/\Gamma_i' \simeq \Lambda/\Lambda' + \Delta_j \quad \text{for} \quad \{i,j\} = \{0,1\} .$$

(3) $$\Gamma_0/\Delta_0 \simeq \Lambda/\Delta_0 \perp \Delta_1 \simeq \Gamma_1/\Delta_1 .$$

(4) $$[\Gamma_0:\Delta_0] = [\Lambda:\Delta_0 \perp \Delta_1] = [\Gamma_1:\Delta_1] .$$

(5) $$\det(\Gamma_0)\det(\Delta_1) = \det(\Lambda) = \det(\Gamma_1)\det(\Delta_0) .$$

(6) $$[\Lambda:\Lambda'] = [\Gamma_i:\Gamma_i'] \cdot [\Delta_j:\Delta_j'] \quad \text{for} \quad \{i,j\} = \{0,1\} .$$

For any lattice Λ in a fixed euclidean space we denote by $\Lambda^\#$ the reciprocal of Λ , that is $\Lambda^\# = \{e \in E \mid \langle e,x\rangle \in \mathbb{Z} \text{ for all } x \in \Lambda\}$. We have

(7) $$\Gamma_i^\# = \Lambda^\# \cap E_i \quad i = 0,\ 1 ,$$

(8) $$\Gamma_i \subseteq \Delta_i^\# \qquad i = 0,\ 1 .$$

Applying (7) in the reciprocal case we obtain

(9) $$P_i(\Lambda^\#)^\# = \Delta_i , \ P_i(\Lambda^\#) = \Delta_i^\# ,$$

and combining this with (6) in the case $\Lambda' = \Lambda$, $\Lambda = \Lambda^\#$ we get

(10) $$[\Delta_0^\#:\Gamma_0] \cdot [\Delta_1^\#:\Gamma_1] = [\Lambda^\#:\Lambda] .$$

In the case that Λ is unimodular, that is $\Lambda^\# = \Lambda$ we obtain from (10)

(11) $$\Gamma_i = \Delta_i^\# , \ \Delta_i = \Gamma_i^\# ,$$

(12) $$\det(\Gamma_1) = \det(\Gamma_2) , \ \det(\Delta_1) = \det(\Delta_2) .$$

3.

We now consider the case $n = 2$. Then we find

(13) $$P_j(\Gamma_{i,j}) = \Gamma_j \quad \text{for } \{i,j\} \subset \{0,1,2\} ,$$

(14) $$\Gamma_{i,j} \cap E_i = P_i(\Delta_{i,k}) \quad \text{for } \{i,j,k\} = \{0,1,2\} ,$$

(15) $$P_i(\Delta^{\#}_{i,j}) = \Delta^{\#}_i \quad \text{for } \{i,j\} \subset \{0,1,2\} .$$

LEMMA 1. *For any lattice* Λ *we have*

(16) $$P_j(\Delta_{i,j})^{\#} \supseteq P_j(\Delta_{j,k}) \quad \text{for } \{i,j,k\} = \{0,1,2\} .$$

In the case where Λ *is unimodular we have*

(17) $$P_j(\Delta_{i,j})^{\#} = P_j(\Delta_{j,k}) .$$

Proof. $P_j(\Delta_{i,j})^{\#} = \Delta^{\#}_{i,j} \cap E_j$ by (7) ,

$\supseteq \Gamma_{i,j} \cap E_j$ by (8) ,

with equality in the unimodular case by (11),

$= P_j(\Delta_{j,k})$ by (14). □

We now apply (6) in the case where $\Lambda' = \Delta_{j,k}$, $\Lambda = \Gamma_{j,k}$ and use (14) to get

(18) $$[\Gamma_{j,k}:\Delta_{j,k}] = [\Gamma_j:P_j(\Delta_{j,k})] \cdot [P_k(\Delta_{i,k}):\Delta_k] ,$$

$$\text{for } \{i,j,k\} = \{0,1,2\} .$$

We obviously have the following inclusions

(19) $$\Delta_j \subset P_j(\Delta_{j,k}) \subset P_j(\Delta_{i,j})^{\#} \subset \Delta^{\#}_j ,$$

which yield

$$[\Delta^{\#}_j:\Delta_j] = [\Delta^{\#}_j:P_j(\Delta_{i,j})^{\#}] \cdot [P_j(\Delta_{i,j})^{\#}:P_j(\Delta_{j,k})] \cdot [P_j(\Delta_{j,k}):\Delta_j]$$

$$= [P_j(\Delta_{i,j}):\Delta_j] \cdot [P_j(\Delta_{j,k}):\Delta_j] \cdot [P_j(\Delta_{i,j})^{\#}:P_j(\Delta_{j,k})]$$

$$= [P_i(\Delta_{i,j}):\Delta_i] \cdot [P_k(\Delta_{j,k}):\Delta_k] \cdot [P_j(\Delta_{i,j})^{\#}:P_j(\Delta_{j,k})]$$

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where the last equality follows from (4). This implies the following "product formula with correction term":

(20) $$[\Delta_j^{\#}:\Delta_j] = [P_i(\Delta_{i,j}):\Delta_i]\cdot[P_k(\Delta_{j,k}):\Delta_k]\cdot[P_j(\Delta_{i,j})^{\#}:P_j(\Delta_{j,k})],$$

where $\{i,j,k\} = \{0,1,2\}$. In the case that Λ is unimodular this implies by (17)

(21) $$[\Delta_j^{\#}:\Delta_j] = [P_i(\Delta_{i,j}):\Delta_i]\cdot[P_k(\Delta_{j,k}):\Delta_k],$$

for $\{i,j,k\} = \{0,1,2\}$.

On the other hand we have the inclusions

(22) $$\Delta_j \subset P_j(\Delta_{i,j}) \subset \Gamma_j ,$$

which can be used as above, together with (18), to give

(23) $$\frac{[\Gamma_i:\Delta_i]}{[\Gamma_j:\Delta_j]} = \frac{[P_i(\Delta_{i,k}):\Delta_i]}{[P_j(\Delta_{j,k}):\Delta_j]} \quad \text{for } \{i,j,k\} = \{0,1,2\} .$$

Using (10) with $\Lambda = \Delta_{i,j}$ we find

(24) $$[\Delta_{i,j}^{\#}:\Delta_{i,j}] = [\Delta_i^{\#}:P_i(\Delta_{i,j})]\cdot[\Delta_j^{\#}:P_j(\Delta_{i,j})] .$$

By using (24) and (23) we find

(25) $$[P_i(\Delta_{i,j})^{\#}:P_i(\Delta_{i,k})] = \frac{[\Delta_{i,j}^{\#}:\Delta_{i,j}]}{[\Delta_j^{\#}:\Delta_j]}\cdot\frac{[\Gamma_j:\Delta_j]}{[\Gamma_k:\Delta_k]},$$

which we can substitute into (20) to obtain

(26) $$[P_j(\Delta_{i,j}):\Delta_j]\cdot[P_k(\Delta_{i,k}):\Delta_k] = \frac{[\Delta_i^{\#}:\Delta_i][\Delta_j^{\#}:\Delta_j]}{[\Delta_{i,j}^{\#}:\Delta_{i,j}]}\cdot\frac{[\Gamma_k:\Delta_k]}{[\Gamma_j:\Delta_j]} .$$

Note that in the unimodular case (26) reduces to (21).

By (6) with $\Lambda = \Delta_{i,j}^{\#}$, $\Lambda' = \Delta_{i,j}$ we obtain

$$[\Delta_{i,j}^{\#}:\Delta_{i,j}] = [\Delta_{i,j}^{\#}\cap E_i:\Delta_{i,j}\cap E_i]\cdot[P_j(\Delta_{i,j}^{\#}):P_j(\Delta_{i,j})],$$

so we get

(27) $$[\Delta_{i,j}^{\#}:\Delta_{i,j}] = [P_i(\Delta_{i,j})^{\#}:\Delta_i]\cdot[\Delta_{i,j}^{\#}:\Delta_{i,j}+(E_i\cap\Delta_{i,j}^{\#})] \quad \text{by (7), (2)} .$$

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But for the expression $[\Delta_{i,j}^{\#}:\Delta_{i,j}+(E_i\cap\Delta_{i,j}^{\#})]$ we have the following product formula

$$(28)\qquad [\Delta_{i,j}^{\#}:\Delta_{i,j}] = [\Delta_{i,j}^{\#}:\Delta_{i,j}+(E_j\cap\Delta_{i,j}^{\#})][\Delta_{i,j}^{\#}:\Delta_{i,j}+(E_i\cap\Delta_{i,j}^{\#})] ,$$

which combined with (27) gives

$$(29)\qquad [\Delta_{i,j}^{\#}:\Delta_{i,j}+(E_j\cap\Delta_{i,j}^{\#})] = [P_i(\Delta_{i,j})^{\#}:P_i(\Delta_{i,k})][P_i(\Delta_{i,k}):\Delta_i] .$$

Equation (29) together with the following equation

$$(30)\qquad \frac{[P_i(\Delta_{i,j})^{\#}:P_i(\Delta_{i,k})]}{[P_j(\Delta_{i,j})^{\#}:P_j(\Delta_{j,k})]} = \frac{[\Delta_i^{\#}:\Gamma_i]}{[\Delta_j^{\#}:\Gamma_j]} ,$$

which easily follows from (25), implies

$$(31)\qquad [\Delta_k^{\#}:\Gamma_k]\cdot[\Delta_{i,j}^{\#}:\Delta_{i,j}+(E_j\cap\Delta_{i,j}^{\#})] = [\Delta_i^{\#}:\Gamma_i][\Delta_{j,k}^{\#}:\Delta_{j,k}+(E_j\cap\Delta_{j,k}^{\#})] .$$

Analogously to (23) we find

$$(32)\qquad \frac{[\Delta_i^{\#}:\Delta_i]}{[\Delta_j^{\#}:\Delta_j]} = \frac{[\Delta_{i,k}^{\#}:\Delta_{i,k}+(E_k\cap\Delta_{i,k}^{\#})]}{[\Delta_{j,k}^{\#}:\Delta_{j,k}+(E_k\cap\Delta_{j,k}^{\#})]} .$$

Finally we note

$$(33)\qquad \frac{[\Delta_i^{\#}:\Delta_i][\Delta_j^{\#}:\Delta_j]}{[\Delta_{i,j}^{\#}:\Delta_{i,j}]} = [P_i(\Delta_{i,j}):\Delta_i]^2 = [P_j(\Delta_{i,j}):\Delta_j]^2 .$$

4.

We now assume that Λ is unimodular. Then we have (11) and (17) and all the above computations simplify. We let

$$(34)\qquad \begin{aligned} \phi_0 &= [\Delta_{1,2}:\Delta_1\perp\Delta_2] ,\\ \phi_1 &= [\Delta_{0,2}:\Delta_0\perp\Delta_2] ,\\ \phi_2 &= [\Delta_{0,1}:\Delta_0\perp\Delta_1] ,\end{aligned}$$

and by (4), (11), (17), (29), (31) we find

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(35) $$\phi_i = [P_j(\Delta_{j,k}):\Delta_j] = [P_k(\Delta_{j,k}):\Delta_k] ,$$

(36) $$\phi_i = [\Gamma_j:P_j(\Delta_{i,j})] = [\Gamma_k:P_k(\Delta_{i,k})] ,$$

(37) $$\phi_i = [\Gamma_{i,j}:\Delta_{i,j}+(E_i\cap\Gamma_{i,j})] = [\Gamma_{i,k}:\Delta_{i,k}+(E_i\cap\Gamma_{i,k})] .$$

The product formulas all become equal to

(38) $$[\Delta_i^{\#}:\Delta_i] = \phi_j \cdot \phi_k \quad \text{for} \quad \{i,j,k\} = \{0,1,2\} .$$

In the case that E_0 is spanned by the vector $(1,\ldots,1) \in \mathbb{R}^n = E$, (38) gives a set of three formulas, one of which was previously proved in [1].

References

[1] Th. Bier, "A product formula for Euler's totient", *Bull. London Math. Soc.* (to appear).

Department of Mathematics
National University of Singapore
Kent Ridge
Singapore 0511