BULL. AUSTRAL. MATH. SOC. 17D15
VOL. 46 (1992) [81–90]

RIGHT ALTERNATIVE ALGEBRAS WITH COMMUTATORS IN A NUCLEUS

ERWIN KLEINFELD AND HARRY F. SMITH

Let A be a right alternative algebra, and $[A, A]$ be the linear span of all commutators in A. If $[A, A]$ is contained in the left nucleus of A, then left nilpotence implies nilpotence. If $[A, A]$ is contained in the right nucleus, then over a commutative-associative ring with $1/2$, right nilpotence implies nilpotence. If $[A, A]$ is contained in the alternative nucleus, then the following structure results hold: (1) If A is prime with characteristic $\neq 2$, then A is either alternative or strongly $(-1, 1)$. (2) If A is a finite-dimensional nil algebra, over a field of characteristic $\neq 2$, then A is nilpotent. (3) Let the algebra A be finite-dimensional over a field of characteristic $\neq 2, 3$. If A/K is separable, where K is the nil radical of A, then A has a Wedderburn decomposition

1. INTRODUCTION

Let A be a nonassociative algebra. As is customary, for $x, y, z \in A$ we denote by (x, y, z) the associator $(x, y, z) = (xy)z - x(yz)$ and by $[x, y]$ the commutator $[x, y] = xy - yx$. If the algebra A satisfies the identity

$$(y, x, x) = 0, \tag{1}$$

then it is called right alternative. A right alternative algebra which also satisfies the identity $(x, x, y) = 0$ is called alternative, and one which satisfies the identity $[[x, y], z] = 0$ is called strongly $(-1, 1)$.

In any nonassociative algebra A, the following are subalgebras:

$$N_\ell = \{n \in A \mid (n, x, y) = 0 \text{ for all } x, y \in A\} \text{ – left nucleus,}$$
$$N_r = \{n \in A \mid (x, y, n) = 0 \text{ for all } x, y \in A\} \text{ – right nucleus.}$$

For A a right alternative algebra with characteristic $\neq 2, 3$,

$$U = \{u \in A \mid [u, x] = 0 \text{ for all } x \in A\} \text{ – commutative centre}$$

Received 10 July 1991

is a subalgebra of A; and for characteristic $\neq 2$,

$$N_\beta = \{v \in A \mid (x, x, v) = 0 \text{ for all } x \in A\} \text{ – alternative nucleus}$$

is a subalgebra with both $N_r \subseteq N_\beta$ and $U \subseteq N_\beta$.

For A a nonassociative algebra, if for some positive integer n every product of n elements from A is zero, no matter how the elements are associated, then A is called nilpotent. Less restrictively, let $A_{[1]} = A$ and define inductively $A_{[k]} = AA_{[k-1]}$. If $A_{[n]} = 0$ for some n, then A is said to be left nilpotent. Analogously, setting $A^{[1]} = A$ and defining inductively $A^{[k]} = A^{[k-1]}A$, then A is right nilpotent if $A^{[n]} = 0$ for some n.

In Section 2 we consider left and right nilpotency in certain varieties of right alternative algebras. Let $[A, A]$ denote the linear span of all commutators in an algebra A. Then for A a right alternative algebra with $[A, A] \subseteq N_\ell$, we show that for each natural number n there exists a natural number $f(n)$ such that $A^{f(n)} \subseteq A_{[n]}$. In particular, if A is left nilpotent, then A is nilpotent. We next consider a right alternative algebra A, over a commutative-associative ring with $1/2$, such that $[A, A] \subseteq N_r$. We show that for such an algebra A right nilpotence implies nilpotence. In particular, if such an A satisfies the minimum condition on right ideals, then its quasi-regular radical $J(A)$ is nilpotent. We also note that existing examples [**2, 11, 16**] can be used to show that in these indicated varieties there are no other implications between left or right nilpotence and nilpotence.

Let A be right alternative algebra with characteristic $\neq 2$. It is known that if $[A, A] \subseteq N_\beta$, then A is alternative if A is either simple [**19**] or prime and finitely-generated [**9**]. In Section 3 we first extend these results by showing that if A is prime with $[A, A] \subseteq N_\beta$, then A is either alternative or strongly $(-1, 1)$. We then assume A is a finite-dimensional right alternative algebra with $[A, A] \subseteq N_\beta$, and prove the following: (1) If A is a nil algebra over a field of characteristic $\neq 2$, then A is nilpotent. (2) (Wedderburn Decomposition) Let the algebra A be over a field with characteristic $\neq 2, 3$, and A/K be separable, where K is the nil radical of A. Then there exists a subalgebra S of A such that $A = S \oplus K$ (vector space direct sum). It is known that neither of these results holds for finite-dimensional right alternative algebras in general [**2, 20**].

Finally, we note that in addition to (1) we shall also make use of the following identities:

$$(1') \qquad (x, y, z) + (x, z, y) = 0,$$

$$(2) \qquad [xy, z] - x[y, z] - [x, z]y = (x, y, z) - (x, z, y) + (z, x, y),$$

$$(3) \qquad (xy, z, w) + (x, y, [z, w]) = x(y, z, w) + (x, z, w)y,$$

$$(4) \qquad [[x, y], z] + [[y, z], x] + [[z, x], y] = 2\{(x, y, z) + (y, z, x) + (z, x, y)\}.$$

Identity $(1')$ is just the linearised form of (1). A straightforward verification shows that (2) holds in any algebra. Identities (3) and (4) hold in any right alternative algebra with characteristic $\neq 2$, for example see [**19**].

2. Nilpotency

We first consider the variety of right alternative algebras which satisfy the identity $([x, y], z, w) = 0$. As usual, for any algebra A we denote by L_a and R_a the operators of left and right multiplication by $a \in A$. Using an argument analogous to that used by Slin'ko for $(-1, 1)$ algebras [**12**], we prove:

Theorem 1. *Let A be a right alternative algebra such that $[A, A] \subseteq N_\ell$. For each natural number n there exists a natural number $f(n)$ such that $A^{f(n)} \subseteq A_{[n]}$.*

Proof: As noted in [**16**], if I is an ideal in a right alternative algebra A, then AI is also an ideal. In particular, $A_{[n]}$ is an ideal of A for each n.

Our proof will be by induction on n. Since $A = A_{[1]}$ and $A^2 = A_{[2]}$, we start with $f(1) = 1$ and $f(2) = 2$. Suppose then there exists a number $f(n-1) \geqslant 2$ such that $A^{f(n-1)} \subseteq A_{[n-1]}$. We first consider $A_{[n-1]}R_{x_1} \dots R_{x_k}$, where $k \geqslant 3$. The identity $([x, y], z, w) = 0$ written in operator form gives

$$R_y R_z R_w = R_y R_{zw} + L_y R_z R_w - L_y R_{zw}.$$

Using this to substitute for $R_{x_1}R_{x_2}R_{x_3}$, we see $A_{[n-1]}R_{x_1}R_{x_2}R_{x_3}\dots R_{x_k} \subseteq A_{[n-1]}R_{x_1}R_{x_2x_3}\dots R_{x_k} + A_{[n-1]}\big(L_{x_1}R_{x_2}R_{x_3}\dots R_{x_k} - L_{x_1}R_{x_2x_3}\dots R_{x_k}\big)$. Thus, since $A_{[n-1]}L_{x_1} \subseteq A_{[n]}$ and $A_{[n]}$ is an ideal, we have $A_{[n-1]}R_{x_1}\dots R_{x_k} \subseteq A_{[n-1]}R_{x_1}R_{x_2x_3}\dots R_{x_k} + A_{[n]}$ Applying this same argument to $A_{[n-1]}R_{x_1}R_{x_2x_3}\dots R_{x_k}$, after $k-2$ such procedures we arrive at $A_{[n-1]}R_{x_1}\dots R_{x_k} \subseteq A_{[n-1]}R_{x_1}R_{((x_2x_3)\dots)x_k} + A_{[n]}$. Now let $k - 1 = f(n-1)$. Then using $A_{[n-1]}$ is an ideal and our induction assumption, we see $A_{[n-1]}R_{x_1}R_{((x_2x_3)\dots)x_k} \subseteq A_{[n-1]}R_{A^{f(n-1)}} \subseteq A_{[n-1]}R_{A_{[n-1]}} \subseteq A_{[n-1]}A_{[n-1]} \subseteq A_{[n]}$. Thus we have $A_{[n-1]}R_{x_1}\dots R_{x_{f(n-1)+1}} \subseteq A_{[n]}$, and so it follows that

(*) $\qquad A_{[n-1]}S_1\dots S_{f(n-1)+1} \subseteq A_{[n]}$, where S_i is either L_{x_i} or R_{x_i}.

We now let $t > 1$ be an integer such that $2^{t-1} < f(n-1) + 1 \leqslant 2^t$. Then $A^{2^{t+f(n-1)+1}} \subseteq A^{2^{t+f(n-1)}}S_1 \subseteq \dots \subseteq A^{2^t}S_1\dots S_{f(n-1)+1} \subseteq A^{f(n-1)}S_1\dots S_{f(n-1)+1} \subseteq A_{[n-1]}S_1\dots S_{f(n-1)+1} \subseteq A_{[n]}$, using our induction assumption and $(*)$. Thus it suffices to take $f(n) = 2^{t+f(n-1)+1}$, which completes our induction and the proof of the theorem. $\square$

Corollary. *Let A be a right alternative algebra such that $[A, A] \subseteq N_\ell$. If A is left nilpotent, then A is nilpotent.*

In [**2**] Dorofeev constructed an example of a finite-dimensional right alternative algebra that is right nilpotent but not nilpotent. This algebra A has basis $\{a, b, c, d, e\}$,

https://doi.org/10.1017/S0004972700011692 Published online by Cambridge University Press

with the nonzero products of basis elements being $ab = -ba = ae = -ea = db = -bd = -c$, $ac = d$, $bc = e$. A straightforward computation shows that $[A, A]$ is contained in the subspace with basis $\{c, d, e\}$, and then that $[A, A] \subseteq N_\ell$. We also note that the subspaces with bases $\{a, c, d, e\}$ and $\{b, c, d, e\}$ are nilpotent ideals whose sum is A. Thus it follows that the locally nilpotent radical doesn't exist in the variety of right alternative algebras satisfying $([x, y], z, w) = 0$.

We next consider nilpotency in the variety of right alternative algebras which satisfy the identity $(x, y, [z, w]) = 0$.

THEOREM 2. *Let A be a right alternative algebra, over a commutative-associative ring with $1/2$, such that $[A, A] \subseteq N_r$. If A is right nilpotent, then A is nilpotent.*

PROOF: First, for any nonassociative algebra A, let $A^{(1)} = A$ and define inductively $A^{(n)} = \left(A^{(n-1)}\right)^2$. Then if $A^{(m)} = 0$, with m the least such integer, the algebra A is called solvable of index m. Now it is immediate that any right nilpotent algebra is solvable, and so to prove the theorem we induct on the index of solvability of A. For a start, it is clear A is nilpotent when $A = A^{(1)} = 0$ or $A^2 = A^{(2)} = 0$. Thus by induction we can assume A^2 is nilpotent, since A^2 is a right nilpotent right alternative algebra which satisfies $(x, y, [z, w]) = 0$ and has solvable index one less than that of A. In particular, let $\left(A^2\right)^n = 0$.

Now from the proof of Theorem 1 in [7], $\overline{N}_r = \{n \in N_r \mid nA \subseteq N_r\}$ is an ideal of A such that $[[A, A], A] \subseteq \overline{N}_r$. Thus $A/\overline{N}_r$ is a right nilpotent strongly $(-1, 1)$ algebra, over a commutative-associative ring with $1/2$, and so by Theorem 5 in [10] $A/\overline{N}_r$ is nilpotent. In particular, we must have $(A)L_{x_1} \ldots L_{x_m} \subseteq \overline{N}_r$ for some integer $m > 0$. Also, using that $\overline{N}_r$ is an ideal contained in N_r, for $2n$ factors of A we have $A(A(\ldots A(A\overline{N}_r))) = A^2(A(\ldots A(A\overline{N}_r))) = \ldots = A^2(A^2(\ldots A^2(A^2\overline{N}_r))) = (A^2)^2(A^2(\ldots A^2(A^2\overline{N}_r))) = \ldots = \left(\left(\left((A^2)^2A^2\right)A^2\ldots\right)A^2\right)\overline{N}_r \subseteq (A^2)^n\overline{N}_r = 0$, that is $(\overline{N}_r)L_{y_1} \ldots L_{y_{2n}} = 0$. Thus it now follows that $(A)L_{x_1} \ldots L_{x_m}L_{y_1} \ldots L_{y_{2n}} \subseteq (\overline{N}_r)L_{y_1} \ldots L_{y_{2n}} = 0$, and so A is left nilpotent. But by Lemma 1 in [16], a right alternative algebra that is both left and right nilpotent is nilpotent. This completes our induction, and so proves the theorem. ∎

COROLLARY. *Let A be a right alternative algebra, over a commutative-associative ring containing $1/2$, such that $[A, A] \subseteq N_r$. If A satisfies the minimum condition on right ideals, then the quasi-regular radical $\mathrm{J}(A)$ of A is nilpotent.*

PROOF: By [15] $\mathrm{J}(A)$ is right nilpotent, and so by Theorem 2 $\mathrm{J}(A)$ is in fact nilpotent. ∎

In [11] Pchelincev constructed an example of a right nilpotent right alternative algebra A that is not nilpotent. We note that a straightforward verification shows $[A, A] \subseteq N_\beta$, so Theorem 2 cannot be extended to the variety of right alternative algebras satisfying $(x, x, [y, z]) = 0$. Also, in [16] Slin'ko constructed an example of a left nilpotent right alternative algebra A that is not nilpotent. This example has the property $AA^2 = 0$, and so obviously satisfies the identity $(x, y, [z, w]) = 0$.

3. Alternative nucleus

In this section we consider the variety of right alternative algebras which satisfy the identity $(x, x, [y, z]) = 0$.

Proposition 1. *Let A be a right alternative algebra with characteristic $\neq 2$. If $[A, A] \subseteq N_\beta$, then $\overline{N}_\beta = \{v \in N_\beta \mid vA \subseteq N_\beta\}$ is an ideal of A such that $[N_\beta, A] \subseteq \overline{N}_\beta$.*

Proof: By Theorem 2 in [19], $\overline{N}_\beta$ is an ideal of A. Let $\nu \in N_\beta$ and $y, z \in A$. Using (2) and (1′), we see

$$\begin{aligned}[v, z]y &= [vy, z] - v[y, z] - (v, y, z) + (v, z, y) - (z, v, y) \\ &= [vy, z] - v[y, z] + 2(v, z, y) + (z, y, v).\end{aligned}$$

Now $[A, A] \subseteq N_\beta$ by assumption, and N_β is a subalgebra of A by Lemma 1 in [19]. Also, $(N_\beta, A, A) \subseteq N_\beta$ by the Corollary to Lemma 6 in [19]; and $(A, A, N_\beta) \subseteq N_\beta$ by Lemma 3.1 in [9]. Thus it follows $[v, z]y \in N_\beta$, that is, $[N_\beta, A] \subseteq \overline{N}_\beta$, which completes the proof. □

As usual, an algebra A is prime if $BC = 0$ for ideals B and C of A implies either $B = 0$ or $C = 0$.

Theorem 3. *Let A be a prime right alternative algebra with characteristic $\neq 2$. If $[A, A] \subseteq N_\beta$, then A is either alternative or strongly $(-1, 1)$.*

Proof: Let M be the submodule of A generated by all associators of the form (x, x, y). By Lemma 11 in [19], $M + MA$ is an ideal of A such that $(M + MA)\overline{N}_\beta = 0$. Since A is prime, either $M + MA = 0$, so A is alternative; or by Proposition 1, $[[A, A], A] \subseteq [N_\beta, A] \subseteq \overline{N}_\beta = 0$, so A is strongly $(-1, 1)$. □

Corollary. *Let A be a prime right alternative algebra with characteristic $\neq 2, 3$. If $[A, A] \subseteq N_\beta$, then A is alternative if A satisfies any of the following conditions:*

(i) *A is without nonzero locally nilpotent ideals,*
(ii) *A is finitely-generated,*
(iii) *A has an idempotent $e \neq 0, 1$,*
(iv) *A satisfies the minimum condition on right or left ideals.*

Proof: Let A be a strongly $(-1, 1)$ algebra. Then A is associative under condition (i) by Corollary 2 to Theorem 3 in [19]. If A is prime, then A is associative

https://doi.org/10.1017/S0004972700011692 Published online by Cambridge University Press

under condition (ii) by Theorem 5 in [4], and under condition (iii) by Theorem 2 in [**18**]. Under condition (iv), the locally nilpotent radical of A is nilpotent by Theorem 3 in [**12**]. Since if I is an ideal of A so is I^k, this means that if A is prime, then condition (iv) implies condition (i), that is, A is associative. ∎

Theorem 4. *Let A be a finite-dimensional right alternative nil algebra over a field of characteristic $\neq 2$. If $[A, A] \subseteq N_\beta$, then A is nilpotent.*

Proof: We first note that, over a field of characteristic $\neq 2$, any finite-dimensional right alternative nil algebra is right nilpotent, for example [**15**]. Our proof of the theorem will be by induction on the dimension of A, with $\dim(A) = 1$ being immediate. Now by Proposition 1 we have $[[A, A], A] \subseteq \overline{N}_\beta$. Thus if $\overline{N}_\beta = 0$, then A is a strongly $(-1, 1)$ nil algebra, and so A is nilpotent by Theorem 4 in [**3**]. We can therefore assume $\overline{N}_\beta \neq 0$, and then let I be a minimal nonzero ideal of A contained in $\overline{N}_\beta$. As noted in the proof of Theorem 1, AI is also an ideal of A; and so by the minimality of I we must have either $AI = 0$ or $AI = I$.

Suppose first that it is the case that $AI = 0$. Now by induction the algebra A/I is nilpotent, since $I \neq 0$ implies $\dim(A/I) < \dim(A)$. Thus $A^n \subseteq I$ for some integer n, whence $A_{[n+1]} = AA_{[n]} \subseteq AA^n \subseteq AI = 0$. This shows the right alternative algebra A is both left and right nilpotent, and so in this cas A is nilpotent by Lemma 1 in [**16**].

We suppose next that it's the case $AI = I$. By (1′) we have $(A^2I)A \subseteq (A^2A)I + A^2(IA) + A^2(AI) \subseteq A^2I$. Also, since $I \subseteq N_\beta$ implies

$$(**) \qquad (x, y, m) + (y, x, m) = 0 \text{ for all } x, y \in A \text{ and } m \in I,$$

we see $A(A^2I) \subseteq A^2(AI) + (AA^2)I + (A^2A)I \subseteq A^2I$. Thus A^2I is an ideal of A, and so by the minimality of I we have either $A^2I = 0$ or $A^2I = I$. We suppose first that $A^2I = I$. Since A is right nilpotent, we know $A^2 \neq A$. Thus by induction the ideal A^2 is nilpotent, say $(A^2)^k = 0$. Then for k factors of A^2, since $A^2I = I$ we have $I = A^2(A^2(\ldots(A^2(A^2I)))) \subseteq (A^2)^k = 0$, which is a contradiction. Suppose next that $A^2I = 0$. We let $\{x_1, \ldots, x_s\}$ be a basis for A and consider a product of the form $x_{i_{s+1}}(x_{i_s}(\ldots(x_{i_2}(x_{i_1}I))))$, where the $s+1$ factors x_{i_j} are any elements from this basis. Now from (**) and $A^2I = 0$, we see $x(yI) = -y(xI)$ for any $x, y \in A$. Then since I is an ideal contained in N_β, and since some basis element x_i must appear as a factor twice in the indicated product, we see $x_{i_{s+1}}(x_{i_s}(\ldots(x_{i_2}(x_{i_1}I)))) = \pm x_j(x_j(\ldots(x_{i_k}I))) \subseteq x_j^2I \subseteq A^2I = 0$. Thus for $s+1$ factors of A, since $AI = I$ it now follows that $I = A(A(\ldots(A(AI)))) = 0$, which again is a contradiction. This then shows the case $AI = I$ is impossible, which completes our induction and the proof of the theorem. ∎

We next let $e \neq 0, 1$ be an idempotent in a right alternative algebra A with characteristic $\neq 2$. With respect to e, one has the Albert decomposition $A = A_1 \oplus$

https://doi.org/10.1017/S0004972700011692 Published online by Cambridge University Press

$H_1 \oplus H_0 \oplus A_0$ (module direct sum), where $A_i = \{x \in A \mid ex = ix = xe\}$, $H_1 \oplus H_0 = \{x \in A \mid ex + xe = x\}$, $H_1 e \subseteq A_1$, and $eH_0 \subseteq A_0$ [1]. If $(e, e, A) = 0$, then this Albert decomposition can be refined to the Peirce decomposition $A = A_{11} \oplus A_{10} \oplus A_{01} \oplus A_{00}$ (module direct sum), where $A_{ij} = \{x \in A \mid ex = ix,\ xe = jx\}$ for $i, j = 0, 1$. In this latter case, one also has the following multiplication table for the submodules A_{ij} [5]:

	A_{11}	A_{10}	A_{01}	A_{00}
A_{11}	$A_{11} + A_{01}$	A_{10}	A_{10}	0
A_{10}	0	$A_{11} + A_{01}$	A_{11}	A_{10}
A_{01}	A_{01}	A_{00}	$A_{10} + A_{00}$	0
A_{00}	0	A_{01}	A_{01}	$A_{10} + A_{00}$

PROPOSITION 2. *Let A be a right alternative algebra, with characteristic $\neq 2$, such that $[A, A] \subseteq N_\beta$. If $e \neq 0, 1$ is an idempotent in A, then A permits a Peirce decomposition with respect to e, and the multiplication table is as follows:*

	A_{11}	A_{10}	A_{01}	A_{00}
A_{11}	$A_{11} + A_{01}$	A_{10}	0	0
A_{10}	0	A_{01}	A_{11}	A_{10}
A_{01}	A_{01}	A_{00}	A_{10}	0
A_{00}	0	0	A_{01}	$A_{10} + A_{00}$

Also, if x_{ij} denotes a generic element of A_{ij}, then $x_{ij}^2 = 0$ for $i \neq j$.

PROOF: First, setting $x = y = e$ in (3) and using $[A, A] \subseteq N_\beta$, we see $e(e, z, w) + (e, z, w)e = (e^2, z, w) + (e, e, [z, w]) = (e, z, w)$, that is $(e, A, A) \subseteq H_1 \oplus H_0$. In particular, this means $(e, e, H_i) \subseteq A_i \cap (H_1 \oplus H_0) = 0$. Thus $(e, e, A) = (e, e, A_1 + H_1 + H_0 + A_0) = (e, e, H_1) + (e, e, H_0) = 0$, and so A permits a Peirce decomposition with respect to e.

Next, since $[A, A] \subseteq N_\beta$, we have $(i - j)x_{ij} = [e, x_{ij}] \in N_\beta$ for $i \neq j$, that is $(y, z, x_{ij}) = -(z, y, x_{ij})$. Using this and the indicated multiplication table for a Peirce decomposition in any right alternative algebra, we can now compute as follows. First $(j - i)x_{ij}y_{ij} = (x_{ij}, e, y_{ij}) = -(e, x_{ij}, y_{ij}) = -ix_{ij}y_{ij} + e(x_{ij}y_{ij})$, whence $e(x_{ij}y_{ij}) = jx_{ij}y_{ij}$. Thus $A_{ij}A_{ij} \subseteq A_{ji}$. Next $(i - j)x_{ii}y_{ji} = (x_{ii}, e, y_{ji}) = -(e, x_{ii}, y_{ji}) = 0$, since $e(x_{ii}y_{ji}) = ix_{ii}y_{ji}$. Thus $A_{ii}A_{ji} = 0$. This then establishes the multiplication table as stated in the proposition, and from it we see that also $(i - j)x_{ij}^2 = (e, x_{ij}, x_{ij}) = 0$ by (1). ∎

COROLLARY. *If A is a right alternative algebra, with characteristic $\neq 2$, such that $[A, A] \subseteq N_\beta$, then any idempotent in A is in N_β.*

PROOF: Since it is clear we can assume the idempotent $e \neq 0, 1$, we let $x = x_{11} + x_{10} + x_{01} + x_{00}$. Now from just the definition of A_{ij}, we see $(x_{ij}, e, x_{jk}) = 0$.

https://doi.org/10.1017/S0004972700011692 Published online by Cambridge University Press

Also, from the multiplication table in Proposition 2 and the fact that $x_{ij}^2 = 0$ for $i \neq j$, we see $(x_{ii}, e, x_{ji}) = (x_{ii}, e, x_{jj}) = (x_{ij}, e, x_{ii}) = (x_{ij}, e, x_{ij}) = 0$. Thus by (1′) $(x, x, e) = -(x, e, x) = -(x_{11} + x_{10} + x_{01} + x_{00}, e, x_{11} + x_{10} + x_{01} + x_{00}) = 0$ for all $x \in A$, which proves the corollary. ∎

We note that the multiplication table in Proposition 2 cannot be reduced further to that for an alternative algebra. For let A be the finite-dimensional algebra with basis $\{1, e, z_{10}, x_{00}, y_{00}\}$, where 1 is a unity, $e^2 = e$, and the only other nonzero products of basis elements are $ez_{10} = x_{00}y_{00} = -y_{00}x_{00} = z_{10}$. A straightforward verification shows that over any field A is a right alternative algebra. Also, the subspace $[A, A]$ has basis $\{z_{10}\}$, whence it follows directly that $[A, A] \subseteq (N_\ell \cap N_r) \subseteq N_\beta$. However, $A_{00}^2 \not\subseteq A_{00}$ for the idempotent e, and $A_{11}^2 \not\subseteq A_{11}$ for the idempotent $1 - e$.

THEOREM 5. (Wedderburn Decomposition). *Let A be a finite-dimensional right alternative algebra, over a field F of characteristic $\neq 2, 3$, with $[A, A] \subseteq N_\beta$. If A/K is separable, where K is the nil radical of A, then there exists a subalgebra S of A such that $A = S \oplus K$ (vector space direct sum).*

PROOF: The proof is by induction on the dimension of A, with the initial case $\dim(A) = 1$ being immediate. Then as in [**8**], by induction one can assume the nil radical K of A does not properly contain any nonzero ideals of A. Let $\langle \mathrm{Alt} \rangle$ denote the ideal of A generated by all associators of the form (x, x, y). Then by [**14, 15**] we have $\langle \mathrm{Alt} \rangle \subseteq K$. Now if $\langle \mathrm{Alt} \rangle = 0$, then the algebra A is alternative; and so A has a Wedderburn decomposition by [**13**]. Thus we can assume $K = \langle \mathrm{Alt} \rangle$.

We next let $S(xy, x, y) = (xy, x, y) + (x, y, xy) + (y, xy, x)$. Now since the algebra $A/\langle \mathrm{Alt} \rangle$ is alternative, by the well-known Artin's theorem we must have $S(xy, x, y) \in \langle \mathrm{Alt} \rangle$. Also, by Proposition 1 the ideal $\overline{N}_\beta$ contains $[[xy, x], y] + [[x, y], xy] + [[y, xy], x]$. Thus by identity (4) we see $2S(xy, x, y) \in \langle \mathrm{Alt} \rangle \cap \overline{N}_\beta$. This means that if $\langle \mathrm{Alt} \rangle \cap \overline{N}_\beta = 0$, then the algebra A must satisfy the identity $S(xy, x, y) = 0$; and in this case A has a Wedderburn decomposition by Theorem 5 in [**17**]. Thus we can now assume $\langle \mathrm{Alt} \rangle = K \subseteq \overline{N}_\beta$. In particular, by Lemma 11 in [**19**] we now have $\langle \mathrm{Alt} \rangle^2 \subseteq \langle \mathrm{Alt} \rangle \overline{N}_\beta = 0$, and so as in [**8**] one can assume the base field F to be algebraically closed.

Now since $K = \langle \mathrm{Alt} \rangle$, by [**15**] we know $A/\langle \mathrm{Alt} \rangle \simeq B_1 \oplus \ldots \oplus B_t$, where each minimal ideal B_i is either an associative matrix algebra over a division ring or a Cayley-Dickson algebra. Since $\langle \mathrm{Alt} \rangle \subseteq \overline{N}_\beta$, we can thus take the ideal $\overline{N}_\beta / \langle \mathrm{Alt} \rangle \simeq B_{k+1} \oplus \ldots \oplus B_t$ (or 0), whence $A/\overline{N}_\beta \simeq (A/\langle \mathrm{Alt} \rangle)/(\overline{N}_\beta/\langle \mathrm{Alt} \rangle) \simeq B_1 \oplus \ldots \oplus B_k$ (where $k = t$ if $\overline{N}_\beta = \langle \mathrm{Alt} \rangle$). Now by Proposition 1 we have $[[A, A], A] \subseteq \overline{N}_\beta$, so $A/\overline{N}_\beta$ is a strongly $(-1, 1)$ algebra. Thus for $1 \leqslant i \leqslant k$ each B_i is a simple strongly $(-1, 1)$ algebra with idempotent. Since characteristic $F \neq 2, 3$, by [**6**] this means each of these B_i's is a field. But the field F is algebraically closed, so for $1 \leqslant i \leqslant k$ we must in fact have

$B_i \simeq F[u_i]$, where $[u_i] = u_i + \langle \mathrm{Alt} \rangle$ is idempotent.

Now $[u_i^m] = [u_i]^m = [u_i]$, so u_i cannot be nilpotent. Thus the finite-dimensional associative subalgebra generated by u_i in A must contain an idempotent $e_i = f(u_i)$, where $f(x)$ is some polynomial over F. Then $[e_i] = [f(u_i)] = \alpha[u_i]$, where $\alpha = f(1) \in F$; so $\alpha[u_i] = [e_i] = [e_i]^2 = \alpha^2[u_i]^2 = \alpha^2[u_i]$. Now the idempotent e_i cannot be in the nil radical $K = \langle \mathrm{Alt} \rangle$, so $\alpha[u_i] \neq 0$, that is $\alpha \neq 0$. Thus $\alpha = 1$, and so each $F[u_i] = F[e_i]$ where e_i is an idempotent in A. In particular, by the Corollary to Proposition 2, each $e_i \in N_\beta$.

We now take a basis for $K = \langle \mathrm{Alt} \rangle \subseteq \overline{N}_\beta$, and extend this to a basis $\{x_1, \ldots, x_s\}$ for $\overline{N}_\beta$. Then $\{x_1, \ldots, x_s, e_1, \ldots, e_k\} \subseteq N_\beta$ will be a basis for A. But this means the algebra A is alternative, and so as noted earlier A has a Wedderburn decomposition by [**13**]. This then completes our induction, and with it the proof of the theorem. ∎

References

[1] A.A. Albert, 'The structure of right alternative algebras', *Ann. of Math.* **59** (1954), 408–417.

[2] G.V. Dorofeev, 'The nilpotency of right alternative rings', (Russian), *Algebra i Logika* **9** (1970), 302–305.

[3] I.R. Hentzel, '(−1, 1) rings', *Proc. Amer. Math. Soc.* **22** (1969), 367–374.

[4] I.R. Hentzel, 'Nil semi-simple (−1, 1) rings', *J. Algebra* **22** (1972), 442–450.

[5] M.M. Humm, 'On a class of right alternative rings without nilpotent ideals', *J. Algebra* **5** (1967), 164–174.

[6] E. Kleinfeld, 'On a class of right alternative rings', *Math. Z.* **87** (1965), 12–16.

[7] E. Kleinfeld and H.F. Smith, 'On simple rings with commutators in the left nucleus', *Comm. Algebra* **19** (1991), 1593–1601.

[8] I.M. Miheev, 'The theorem of Wedderburn on the splitting of the radical for a (−1, 1) algebra', (Russian), *Algebra i Logika* **12** (1973), 298–304.

[9] Ng Seong Nam, 'Alternative nucleus of right alternative algebras', *Southeast Asian Bull. Math.* **10** (1986), 149–154.

[10] S.V. Pchelincev, 'Nilpotency of the associator ideal of a free finitely generated (−1, 1) ring', (Russian), *Algebra i Logika* **14** (1975), 543–572.

[11] S.V. Pchelincev, 'The locally nilpotent radical in certain classes of right alternative rings', (Russian), *Sibirsk. Mat. Zh.* **17** (1976), 340–360.

[12] R.E. Roomel'di, 'Nilpotency of ideals in a (−1, 1) ring with minimum condition', (Russian), *Algebra i Logiki* **12** (1973), 333–348.

[13] R.D. Schafer, 'The Wedderburn principal theorem for alternative algebras', *Bull. Amer. Math. Soc.* **55** (1949), 604–614.

[14] V.G. Skosyrskiĭ, 'Right alternative algebras', (Russian), *Algebra i Logika* **23** (1984), 185–192.

[**15**] V.G. Skosyrskiĭ, 'Right alternative algebras with minimality condition for right ideals', (Russian), *Algebra i Logika* **24** (1985), 205–210.

[**16**] A.M. Slin'ko, 'The equivalence of certain nilpotencies of right alternative rings', (Russian), *Algebra i Logika* **9** (1970), 342–348.

[**17**] H.F. Smith, 'Finite-dimensional locally $(-1, 1)$ algebras', *Comm. Algebra* **7** (1979), 177–191.

[**18**] N.J. Sterling, 'Prime $(-1, 1)$ rings with idempotent', *Proc. Amer. Math. Soc.* **18** (1967), 902–909.

[**19**] A. Thedy, 'Right alternative rings', *J. Algebra* **37** (1975), 1–43.

[**20**] A. Thedy, 'Right alternative algebras and Wedderburn principal theorem', *Proc. Amer. Math. Soc.* **72** (1978), 427–435.

Division of Mathematical
Sciences
University of Iowa
Iowa City, IA 52242
United States of America

Department of Mathematics
Statistics and Computing Science
University of New England
Armidale NSW 2351