UNIT-REGULAR ORTHODOX SEMIGROUPS

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Introduction. Unit-regular rings were introduced by Ehrlich [**4**]. They arose in the search for conditions on a regular ring that are weaker than the ACC, DCC, or finite Goldie dimension, which with von Neumann regularity imply semisimplicity. An account of unit-regular rings, together with a good bibliography, is given by Goodearl [**5**].

The basic definition of unit-regularity is purely multiplicative; it is simply that for each element x of a monoid S (initially a ring R with identity) there is a unit u of S for which $x = xux$. The concept of a unit-regular semigroup is a natural one; for example, the full transformation semigroup on a finite set, and the semigroup of endomorphisms of a finite-dimensional vector space, are unit-regular semigroups [**1**]. Unit-regularity has been studied by Chen and Hsieh [**2**], by Tirasupa [**9**], and by McAlister [**6**]. The connection between unit-regularity and finiteness conditions has been considered by D'Alarcao [**3**].

The problem of describing the structure of an arbitrary unit-regular semigroup S is difficult. It appears reasonable to attempt to provide such a description in terms of the group of units of S and the set of idempotents of S, and in this direction Blyth and McFadden did determine the structure of a narrow class of unit-regular semigroups. Calling a semigroup S uniquely unit orthodox if it is orthodox and, for each x in S, there exists a unique unit u of S for which $x = xux$, they proved that every such semigroup is a semidirect product of a group (the group of units of S) and a band (the band of idempotents of S).

In the present paper we shall show first that every unit-regular orthodox semigroup S is an idempotent-separating homomorphic image of a uniquely unit orthodox semigroup (determined by the idempotents of S and the units of S). The main result here is then the determination of all idempotent-separating congruences on uniquely unit orthodox semigroups.

1. Unit-regular semigroups. Much of what follows depends on the following elementary fact. If S is a monoid and g is a unit in S, then for each idempotent e of S, geg^{-1} is an idempotent, and if f in S is also an idempotent then

$$g(ef)g^{-1} = (geg^{-1})(gfg^{-1}).$$

Definition 1.1. A monoid S is said to be *unit-regular* if for each x in S there exists a unit u of S for which $x = xux$.

For the rest of this paper we shall deal with unit-regular semigroups; each such is, of course, regular, though it is certainly not the case that $x = xux$ implies that $uxu = u$. We shall denote by $E(=E(S))$ the set of idempotents of S and by $G(=G(S))$ the group of units of S.

Clearly $x = xux$ implies that xu and ux are idempotent, and each element of S may be written in the form eg for e in E and g in G. This factorization is not unique, though e is unique for unit-regular inverse semigroups [**2**].

If eg and fh, for e, f in E and g, h in G, are elements of S then

$$egfh = e(gfg^{-1})gh = jkgh$$

where e, gfg^{-1}, j are idempotent and k, g, h are units. Here

$$j = j(e, gfg^{-1}) \in E \quad \text{and} \quad k = k(e, gfg^{-1}) \in G.$$

Any useful description of the structure of S in terms of E and G would need to provide more information about j and k than just the fact of their existence.

But there is one case in which j and k are obvious; if S is orthodox (E is a subsemigroup of S) we can take $j = e(gfg^{-1})$ and $k = 1$, so that

$$egfh = e(gfg^{-1})gh$$

with $e(gfg^{-1})$ in E and gh in G.

DEFINITION 1.2. A monoid S is said to be *unit orthodox* if S is unit-regular and orthodox [**1**].

When S is unit orthodox it is clear that G acts automorphically (in the sense that there is a homomorphism from G into the group of automorphisms of E) on the band E of idempotents of S under the action

$$g \,.\, e = geg^{-1} \quad \text{for } g \text{ in } G \text{ and } e \text{ in } E.$$

As yet there is no method for describing the structure of a unit orthodox semigroup S directly in terms of $E(S)$ and $G(S)$, but a method does exist for a sub-class of unit orthodox semigroups. Calling a monoid S *uniquely unit orthodox* if for each x in S there is a unique unit u in S for which $x = xux$, Blyth and McFadden proved the following theorem [**1**].

THEOREM 1.3. *Let E be a band with identity* 1 *and let G be a group which acts automorphically on E. Denoting by $g \,.\, e$ the action of $g \in G$ on $e \in E$, define a product on $T = E \times G$ as follows:*

$$(a, g)(b, h) = (a(g \,.\, b), gh).$$

Then T is a uniquely unit orthodox semigroup, which we shall denote by $E|\times|G$. Denoting also by 1 *the identity of G,*

$$E(T) = \{(e, 1) \mid e \in E\} \sim E,$$

$$G(T) = \{(1, g) \mid g \in G\} \sim G.$$

If $x = (e, g)$ is an element of T then the unique unit u of T for which $x = xux$ is $u = (1, g^{-1})$.

Conversely, every uniquely unit orthodox semigroup arises in this way.

It is worth noting, as pointed out in the proof of Theorem 1.3, that $1 \,.\, e = e$ for each e in E, and $g \,.\, 1 = 1$ for each g in G.

While the class of uniquely unit orthodox semigroups is not a large one, it plays a decisive role in determining the structure of unit orthodox semigroups in general, as the following theorem shows.

THEOREM 1.4. *Let S be a unit orthodox semigroup with group of units G. For e in $E = E(S)$ and g in $G = G(S)$ define $g\,.\,e = geg^{-1}$. This action defines an automorphism of E, and the map which assigns this automorphism to g is a homomorphism from G into the group of automorphisms of E. If the uniquely unit orthodox semigroup $T = E|\times|G$ is defined as in Theorem 1.3, then the map $\theta: T \to S$ defined by $(e, g)\theta = eg$ is an idempotent-separating epimorphism.*

Proof. As noted above, it is obvious that G acts automorphically on E under $g\,.\,e = geg^{-1}$. For a in S choose a unit g in G such that $a = ag^{-1}a$, so that $e = ag^{-1} \in E$; then

$$(e, g)\theta = eg = agg^{-1} = a,$$

so θ maps T onto S.

If (e, g) and (f, h) are elements of T then

$$(e, g)\theta(f, h)\theta = egfh = e(gfg^{-1})gh = e(g\,.\,f)gh = ((e, g)(f, h))\theta,$$

so θ is a homomorphism.

Finally, each idempotent of T is of the form $(e, 1)$ for some e in E, so if $(e, 1)$ and $(f, 1)$ are idempotents of T satisfying $(e, 1)\theta = (f, 1)\theta$ then $e = e1 = f1 = f$; therefore θ is idempotent-separating, and the proof is complete.

Now that we know that every unit orthodox semigroup is an idempotent-separating homomorphic image of a uniquely unit orthodox semigroup, the question arises: What are the congruence relations on a uniquely unit orthodox semigroup which are contained in $\mathcal{H}$? We shall answer this question in Section 2.

2. The $\mathcal{D}$-class structure of $E|\times|G$. For any orthodox semigroup S the finest inverse semigroup congruence $\mathcal{Y}$ on S is given by $x\,\mathcal{Y}\,w$ if and only if $V(x) = V(w)$ where $V(x)$ denotes the set of inverses of x. Further, on any band E the $\mathcal{Y}$-classes coincide with the $\mathcal{D}$-classes, and E is a semilattice Y of rectangular bands; in particular, if x, y, and z are $\mathcal{D}$-equivalent elements of E then $xyz = xz$. We shall write D_x for the $\mathcal{D} = \mathcal{Y}$-class of x in E, and similarly for $\mathcal{R}$ and $\mathcal{L}$.

Now let T be a uniquely unit orthodox semigroup. Since we need to know about congruence relations on T contained in $\mathcal{H}$, we proceed to determine Green's relations on T. But while $\mathcal{H}$ determines the "local" properties it is $\mathcal{D}$ which determines the "global" properties of the congruences we are seeking.

By Theorem 1.3 we may assume that $T = E|\times|G$ under the operation

$$(e, g)(f, h) = (e(g\,.\,f), gh),$$

where $E = E^1$ is a band with identity and G is a group which acts automorphically on E. Recall that $1\,.\,e = e$ and $g\,.\,1 = 1$ for each e in E and each g in G.

LEMMA 2.1. *Let (a, g) be an element of T. Then (x, u) is an inverse of (a, g) in T if and only if $u = g^{-1}$ and $g \,.\, x \in D$ where D is the $\mathcal{D}$-class of a in E.*

Proof. First,

$$(a, g)(x, u)(a, g) = (a(g \,.\, (x(u \,.\, a))), gug) = (a, g)$$

if and only if $u = g^{-1}$ and $a(g \,.\, x)a = a$, while

$$(x, g^{-1})(a, g)(x, g^{-1}) = (x(g^{-1} \,.\, (a(g \,.\, x))), g^{-1}) = (x, g^{-1})$$

if and only if $x(g^{-1} \,.\, a)x = x$. Therefore (x, g^{-1}) and (a, g) are mutually inverse if and only if $a(g \,.\, x)a = a$ and $x(g^{-1} \,.\, a)x = x$. Since g acts as an automorphism, this is true if and only if $g \,.\, x$ is in D.

LEMMA 2.2. *For each e in E and each g in G,*

$$g \,.\, L_e = L_{g.e}, \qquad g \,.\, R_e = R_{g.e}, \qquad g \,.\, D_e = D_{g.e}.$$

Proof. This follows directly from the definition of Green's relations and the fact that G acts automorphically.

LEMMA 2.3. *Let (a, g), (b, h) be elements of T. Then*
(i) *$(a, g)\, \mathcal{R}\, (b, h)$ if and only if $a \,\mathcal{R}\, b$ in E.*
(ii) *$(a, g)\, \mathcal{L}\, (b, h)$ if and only if $g^{-1} \,.\, a \,\mathcal{L}\, h^{-1} \,.\, b$ in E.*
(iii) *$(a, g)\, \mathcal{H}\, (b, h)$ if and only if $hg^{-1} \,.\, a \,\mathcal{D}\, a$ and $b = a(hg^{-1} \,.\, a)$.*

Proof. (i) $(a, g)\, \mathcal{R}\, (b, h)$ if and only if there exist inverses $(a, g)'$ of (a, g) and $(b, h)'$ of (b, h) respectively for which $(a, g)(a, g)' = (b, h)(b, h)'$. By Lemma 2.1 this is true if and only if there exist a' in $V(a)$ and b' in $V(b)$ for which

$$(a, g)(g^{-1} \,.\, a', g^{-1}) = (aa', 1) = (b, h)(h^{-1} \,.\, b', h^{-1}) = (bb', 1);$$

in other words, if and only if $a \,\mathcal{R}\, b$ in E.

(ii) As in (i), $(a, g)\, \mathcal{L}\, (b, h)$ if and only if $g^{-1} \,.\, a'a = h^{-1} \,.\, b'b$ for some a' in $V(a)$ and some b' in $V(b)$. But $a'a \,\mathcal{L}\, a$ and $b'b \,\mathcal{L}\, b$, while g^{-1} and h^{-1} preserve $\mathcal{L}$-classes by Lemma 2.2, and therefore $(a, g)\, \mathcal{L}\, (b, h)$ implies

$$g^{-1} \,.\, a \,\mathcal{L}\, g^{-1} \,.\, a'a = h^{-1} \,.\, b'b \,\mathcal{L}\, h^{-1} \,.\, b.$$

Conversely, $g^{-1} \,.\, a \,\mathcal{L}\, h^{-1} \,.\, b$ implies

$$g^{-1} \,.\, ((gh^{-1} \,.\, b)a) = (h^{-1} \,.\, b)(g^{-1} \,.\, a) = h^{-1} \,.\, b,$$

and certainly $h^{-1} \,.\, bb = h^{-1} \,.\, b$, so with $a' = gh^{-1} \,.\, b$ and $b' = b$ we have $g^{-1} \,.\, a'a = h^{-1} \,.\, b'b$, so that $(a, g)\, \mathcal{L}\, (b, h)$.

(iii) By parts (i) and (ii) we have

$$\begin{aligned}(a, g)\, \mathcal{H}\, (b, h) \quad &\text{if and only if} \quad a \,\mathcal{R}\, b \quad \text{and} \quad g^{-1} \,.\, a \,\mathcal{L}\, h^{-1} \,.\, b \text{ in } E\\ &\text{if and only if} \quad a \,\mathcal{R}\, b \quad \text{and} \quad b \,\mathcal{L}\, hg^{-1} \,.\, a \text{ in } E\\ &\text{if and only if} \quad hg^{-1} \,.\, a \,\mathcal{D}\, a \quad \text{and} \quad b = a(hg^{-1} \,.\, a),\end{aligned}$$

COROLLARY. *Let* $(a, 1)$ *be an idempotent in* T. *Then* $(b, h)\,\mathscr{H}\,(a, 1)$ *if and only if* $b = a(h\,.\,a)$ *for* $h\,.\,a \in D_a$.

The egg-box picture in E when $(a, g)\,\mathscr{H}\,(b, h)$ in T is as follows.

	a		b	
	$gh^{-1}\,.\,b$			
			$hg^{-1}\,.\,a$	

LEMMA 2.4. *Let* D *be a* $\mathscr{D}$*-class of* E, *let* $a \in D$, *and let* $h \in G$. *Then*

$$h\,.\,a \in D \quad \text{if and only if} \quad h\,.\,D = D.$$

Proof. Suppose $h\,.\,a\,\mathscr{D}\,a$ and let $b\,\mathscr{D}\,a$. Then since D is a rectangular band, $b = bab$ and $a = aba$. Applying h to these equations yields $h\,.\,b\,\mathscr{D}\,h\,.\,a\,\mathscr{D}\,a$; that is, $h\,.\,D \subseteq D$.

Also, $a\,\mathscr{D}\,h\,.\,a$ implies that for some c in D we have $a\,\mathscr{R}\,c\,\mathscr{L}\,h\,.\,a$, so $h^{-1}\,.\,a\,\mathscr{R}\,h^{-1}\,.\,c\,\mathscr{L}\,a$; that is, $h^{-1}\,.\,a\,\mathscr{D}\,a$. By the first part of the proof, $h^{-1}\,.\,D \subseteq D$ and therefore $D = 1\,.\,D \subseteq h\,.\,D$. Combining these inclusions yields the result.

DEFINITION 2.5. For a $\mathscr{D}$-class D of E define

$$S_D = \{g \in G \mid g\,.\,D = D\}.$$

Clearly S is a subgroup of G, and by Lemma 2.4,

$$g \in S_D \quad \text{if and only if} \quad D \cap g\,.\,D \neq \varnothing.$$

For each element D of the semilattice of $\mathscr{D}$-classes of E we now have a subgroup of G which stabilizes D. These stabilizers are in fact intimately connected with the $\mathscr{H}$-classes, in particular with the maximal subgroups, of T.

LEMMA 2.6. *Let* (a, g) *be an element of* T *and let* D *be the* $\mathscr{D}$*-class of* a *in* E. *Then the* $\mathscr{H}$*-class of* (a, g) *in* T *is* $\{(a(k\,.\,a), kg) \mid k \in S_D\}$.

Proof. By Lemma 2.3 we have

$$\begin{aligned}(a, g)\,\mathscr{H}\,(b, h) \quad &\text{if and only if} \quad a\,\mathscr{R}\,b \quad \text{and} \quad g^{-1}\,.\,a\,\mathscr{L}\,h^{-1}\,.\,b \\ &\text{if and only if} \quad hg^{-1}\,.\,a\,\mathscr{L}\,b\,\mathscr{R}\,a \quad \text{and} \quad hg^{-1} = k \in S_D \\ &\text{if and only if} \quad b = a(k\,.\,a) \quad \text{and} \quad k \in S_D.\end{aligned}$$

Therefore (b, h) is in the $\mathscr{H}$-class of (a, g) precisely when $(b, h) = (a(k\,.\,a), kg)$ for some k in S_D.

LEMMA 2.7. *Let* $(a, 1)$ *be an idempotent in* T. *Then the maximal subgroup of* T *containing* $(a, 1)$ *is isomorphic to* S_D *where* D *is the* $\mathscr{D}$*-class of* a *in* E.

Proof. Let H denote the $\mathcal{H}$-class of $(a, 1)$. Define the mapping $\theta: H \to S$ by $(b, k)\theta = k$, for $(b, k) \in H$. Clearly θ is injective; it is also surjective, because if $k \in S$ then $(a(k \,.\, a), k) \in H$ by Lemma 2.6. If $(b, k), (c, h) \in H$ then $b = a(k \,.\, a)$, $c = a(h \,.\, a)$, again by Lemma 2.6, and so

$$\begin{aligned}(b, k)(c, h) &= (a(k \,.\, a), k)(a(h \,.\, a), h) = (a(k \,.\, a)k \,.\, (a(h \,.\, a)), kh) \\ &= (a(k \,.\, a)(k \,.\, a)(kh \,.\, a), kh) = (a(kh \,.\, a), kh),\end{aligned}$$

the last equality following from the fact that D is a rectangular band of which a, $k \,.\, a$ and $kh \,.\, a$ are each elements. Therefore θ is a homomorphism, and the proof is complete.

3. Idempotent-separating congruences on $E|\times|G$. Consider an arbitrary orthodox semigroup S, and let ρ be an idempotent-separating congruence on S. If a and b are elements of S for which $a \,\rho\, b$ then for any inverse a' of a there is an inverse b' of b for which $a' \,\rho\, b'$. For certainly $aa' \,\mathcal{R}\, b \,\mathcal{L}\, a'a$, so there exists an inverse b' of b in the $\mathcal{H}$-class of a' and since $a \,\rho\, b$ then $aa' = bb' \,\rho\, ab'$, whence $a' \,\rho\, a'ab' = b'bb' = b'$. This is, of course, almost exactly the treatment provided by Meakin in **[7]**, the only difference being that the choice of a' is at our disposal, a fact we shall use below to some advantage. Precisely as Meakin shows, the fact that ρ is idempotent-separating implies that:

$$\text{for each } x \text{ in } E, \quad a'xa = b'xb \quad \text{and} \quad axa' = bxb'.$$

Let us apply this to the uniquely unit orthodox semigroup $T = E|\times|G$ defined as in Section 1. First, it is easy to verify that $(g^{-1} \,.\, a, g^{-1})$ is an inverse of the element (a, g) of T. Therefore if ρ is an idempotent-separating congruence on T and $(a, g) \,\rho\, (b, h)$ then there exists an inverse $(b, h)'$ of (b, h) such that $(b, h)' \,\rho\, (g^{-1} \,.\, a, g^{-1})$. By Lemma 2.1 we can take $(b, h)' = (h^{-1} \,.\, b', h^{-1})$ for some $b' \,\mathcal{D}\, b$. In fact, using Lemma 2.3(iii) and $\rho \subseteq H$ we have the $\mathcal{D}$-class picture

	a		b	
	b'		$hg^{-1} \,.\, a$	

and so $b' = (hg^{-1} \,.\, a)a$.

Applying to T the property of ρ noted above, it follows that for each $(x, 1)$ in $E(T)$, or equivalently, for each element x of E,

$$\begin{aligned}(g^{-1} \,.\, a, g^{-1})(x, 1)(a, g) &= (h^{-1} \,.\, b', h^{-1})(x, 1)(b, h) \\ (a, g)(x, 1)(g^{-1} \,.\, a, g^{-1}) &= (b, h)(x, 1)(h^{-1} \,.\, b', h^{-1}).\end{aligned}$$

Evaluating these products and equating their first components yields:

If $(a, g) \,\rho\, (b, h)$ then for each x in E, $axa = gh^{-1} \,.\, (b'xb)$ and $a(g \,.\, x)a = b(h \,.\, x)b'$ for some inverse b' of b.

There is a dual condition based on the fact that $(h^{-1}.b, h^{-1})$ is an inverse of (b, h); it is that:

If $(a, g)\,\rho\,(b, h)$ then for each x in E, $bxb = hg^{-1}.(a'xa)$ and $b(h.x)b = a(g.x)a'$ for some inverse a' of a.

Any congruence relation ρ on T, when restricted to a group $\mathcal{H}$-class of T, determines a normal subgroup of that maximal subgroup. Let $(a, 1) \in E(T)$ and denote by D the $\mathcal{D}$-class of E containing a. By Lemma 2.7 the $\mathcal{H}$-class of $(a, 1)$ is isomorphic to the subgroup S_D of G which stabilizes D, and when ρ is restricted to this maximal subgroup of T it determines, by the second projection mapping, a normal subgroup N_a of S_D, namely

$$N_a = \{k \in S_D \mid (a(k.a), k)\rho(a, 1)\}.$$

Given any two idempotents $(a, 1)$ and $(b, 1)$ of T, then since ρ is a congruence and $(a, 1)(b, 1) = (ab, 1)$, it follows that

$$N_aN_b \subseteq N_{ab}.$$

Since each N_x is a subgroup of G it is then the case that:

$$\text{for each } a, x \text{ in } E, \quad N_a \subseteq N_{ax} \quad \text{and} \quad N_a \subseteq N_{xa}.$$

In particular, when $a\,\mathcal{D}\,b$ in E, so that $a = aba$ and $b = bab$, we can conclude that $N_a \subseteq N_{ab} \subseteq N_{bab} = N_b$, and similarly that $N_b \subseteq N_a$. Therefore $a\,\mathcal{D}\,b$ implies that $N_a = N_b = N_D$, say, where D is the $\mathcal{D}$-class of a and b. We have therefore:

(1) for any two elements D, D' of the semilattice $\mathcal{Y}$ of $\mathcal{D}$-classes of E,

$$N_D \subseteq N_{DD'} \qquad (= N_{D'D}).$$

Further, if D is any $\mathcal{D}$-class of E, if $a \in D$, if k is an arbitrary element of G, and $n \in N_D$, then $(n(n.a), n)\,\rho\,(a, 1)$ implies

$$(1, k)(n(n.a), n)(1, k^{-1}) = (\ldots, knk^{-1})\,\rho\,(1, k)(a, 1)(1, k^{-1}) = (k.a, 1);$$

that is,

(2) for each k in G and for each D in $\mathcal{Y}$,

$$kN_Dk^{-1} \subseteq N_{D'} \quad \text{where } D' \text{ is the } \mathcal{D}\text{-class of } k.a.$$

We are now in a position to state and prove the main result of the paper.

THEOREM 3.1. *Let T be a uniquely unit orthodox semigroup, say $T = E|\times|G$ as defined in Section* 1. *Suppose that for each element D of the semilattice $\mathcal{Y}$ of $\mathcal{D}$-classes of E we have a normal subgroup N_D of the stabilizer S_D of D, and that the collection of these normal subgroups satisfies* (1) *and* (2) *above. Define the relation σ on T by*:

$$(a, g)\sigma(b, h) \quad \textit{if and only if} \quad hg^{-1} \in N_D,$$

where D is the $\mathcal{D}$-class of a (and of b), and there exist $a' \in V(a)$, $b' \in V(b)$ such that for *each x in E,*

(3)

$$axa = gh^{-1}\,.\,(b'xb), \qquad a(g\,.\,x)a = b(h\,.\,x)b',$$
$$bxb = hg^{-1}\,.\,(a'xa), \qquad b(h\,.\,x)b = a(g\,.\,x)a'.$$

Then σ is an idempotent-separating congruence on T.

Conversely, every idempotent-separating congruence on a uniquely unit orthodox semigroup arises in this way.

Proof. We begin by showing that $\sigma \subseteq \mathcal{H}$. Suppose that $(a, g)\sigma(b, h)$ and that $a' \in V(a)$ and $b' \in V(b)$ satisfy (3). Setting $x = 1$ in $a(g\,.\,x)a = b(h\,.\,x)b'$ yields $a = bb'\,\mathcal{R}\,b \in D$; setting $x = 1$ in $bxb = hg^{-1}\,.\,(a'xa)$ yields $b = hg^{-1}\,.\,(a'a)\,\mathcal{L}\,hg^{-1}\,.\,a$, and so $h^{-1}\,.\,b\,\mathcal{L}\,g^{-1}\,.\,a$. It follows from Lemma 2.3(iii) that $\sigma \subseteq H$.

It is obvious that σ is reflexive and symmetric. For transitivity, suppose that $(a, g)\sigma(b, h)$ and $(b, h)\sigma(c, k)$. Then there exist $a' \in V(a)$, $b', b'' \in V(b)$, $c' \in V(c)$ such that for each x in E we have (3) and

$$bxb = hk^{-1}\,.\,(c'xc), \qquad b(h\,.\,x)b = c(k\,.\,x)c',$$
$$cxc = kh^{-1}\,.\,(b''xb), \qquad c(k\,.\,x)c = b(h\,.\,x)b''.$$

First, $k^{-1}g = (k^{-1}h)(h^{-1}g) \in N_D N_D = N_D$. Next, we require the existence of a'' in $V(a)$, c'' in $V(c)$ such that for each x in E,

$$axa = gk^{-1}\,.\,(c''xc), \qquad a(g\,.\,x)a = c(k\,.\,x)c'',$$
$$cxc = kg^{-1}\,.\,(a''xa), \qquad c(k\,.\,x)c = a(g\,.\,x)a''.$$

The following configuration holds in D.

	a		b		c	
	$gh^{-1}\,.\,b$		a'			
	b'		$hg^{-1}\,.\,a$			
			c'		$kh^{-1}\,.\,b$	
			$hk^{-1}\,.\,c$		b''	
	$gk^{-1}\,.\,c$				a''	
	c''				$kg^{-1}\,.\,a$	

Defining a'' and c'' by this configuration, for each x in E,

$$\begin{aligned} c(k\,.\,x)c'' &= c(k\,.\,x)(kg^{-1}\,.\,a)a \\ &= c(k\,.\,(x(g^{-1}\,.\,a)))ca \quad \text{since} \quad a\,\mathcal{R}\,c \\ &= b(h\,.\,(x(g^{-1}\,.\,a))b''a \quad \text{using} \quad (b,h)\sigma(c,k) \\ &= b(h\,.\,x)(hg^{-1}\,.\,a)a \quad \text{since} \quad hg^{-1}\,.\,a,\ b'',\ a \text{ are all in } D \\ &= b(h\,.\,x)b' = a(g\,.\,x)a \quad \text{using} \quad (a,g)\sigma(b,h). \end{aligned}$$

Similarly, $a(g\,.\,x)a'' = c(k\,.\,x)c$. Also,

$$\begin{aligned} c''xc &= c''axac \quad \text{since} \quad c''\,\mathcal{L}\,a\,\mathcal{R}\,c \\ &= c''(a(g\,.\,(g^{-1}\,.\,x))a)c = c''(c(k\,.\,(g^{-1}\,.\,x))c'')c \\ &= c''c(kg^{-1}\,.\,x)c''c = (kg^{-1}\,.\,a)(kg^{-1}\,.\,x)(kg^{-1}\,.\,a) = kg^{-1}\,.\,(axa), \end{aligned}$$

so $axa = gk^{-1}\,.\,(c''xc)$ and, similarly, $cxc = kg^{-1}\,.\,(a''xa)$. Therefore σ is an equivalence.

To show that σ is compatible we show first that if $(a,g)\sigma(b,h)$ and $(c,k)\in T$ then $(c(k\,.\,a), kg)\,\mathcal{H}\,(c(k\,.\,b), kh)$, and dually for products on the right. Since $a\,\mathcal{R}\,b$ implies $k\,.\,a\,\mathcal{R}\,k\,.\,b$, and $\mathcal{R}$ is a left congruence, $c(k\,.\,a)\,\mathcal{R}\,c(k\,.\,b)$; by Lemma 2.3(i)

$$(c(k\,.\,a), kg)\,\mathcal{R}\,(c(k\,.\,b), kh).$$

For $\mathcal{L}$-equivalence, note that for each x in E,

$$\begin{aligned} xa = xaxa &= xgh^{-1}\,.\,(b'xb) \quad \text{using} \quad (a,g)\sigma(b,h) \\ &= (x(gh^{-1}\,.\,b'))(gh^{-1}\,.\,xb)\,\mathcal{L}\,gh^{-1}\,.\,xb \end{aligned}$$

in D_{bx}, because $\mathcal{D}$ is a congruence and, using (1), $gh^{-1}\in N_D \subseteq N_{D'} \subseteq S_{D'}$ where D' is the $\mathcal{D}$-class of xb. Therefore $g^{-1}\,.\,(xa)\,\mathcal{L}\,h^{-1}\,.\,(xb)$, and in particular, setting $x = k^{-1}\,.\,c$, we have $g^{-1}\,.\,((k^{-1}\,.\,c)a)\,\mathcal{L}\,h^{-1}\,.\,((k^{-1}\,.\,c)b)$. By Lemma 2.3(iii) again, we obtain

$$(c,k)(a,g)\,\mathcal{H}\,(c,k)(b,h).$$

For the products on the right by (c, k),

$$\begin{aligned} (a(g\,.\,c))(b(h\,.\,c)) &= (a(g\,.\,c)a)(b(h\,.\,c)) \quad \text{since} \quad a\,\mathcal{R}\,b \\ &= b(h\,.\,c)b'b(h\,.\,c) \text{ using } (a,g)\sigma(b,h) \\ &= (b(h\,.\,c))((h\,.\,c)b')(b(h\,.\,c)) \\ &= b(h\,.\,c), \end{aligned}$$

and similarly $b(h\,.\,c)a(g\,.\,c) = a(g\,.\,c)$, so $(a,g)(c,k)\,\mathcal{R}\,(b,h)(c,k)$. Finally,

$$\begin{aligned} g^{-1}\,.\,a\,\mathcal{L}\,h^{-1}\,.\,b &\Rightarrow a(g\,.\,c)\,\mathcal{L}\,(gh^{-1}\,.\,b)(g\,.\,c) = gh^{-1}\,.\,(b(h\,.\,c)) \\ &\Rightarrow (gk)^{-1}\,.\,(a(g\,.\,c))\,\mathcal{L}\,(hk)^{-1}\,.\,(b(h\,.\,c)), \end{aligned}$$

and so $(a,g)(c,k)$ and $(b,h)(c,k)$ are $\mathcal{L}$-equivalent, therefore $\mathcal{H}$-equivalent.

To prove that σ is compatible with multiplication in T, let us continue to suppose that

$(a, g)\sigma(b, h)$ and $(c, k) \in T$. We have the following D-class configurations.

	a		b	
	$gh^{-1} . b$		a'	
	b'		$hg^{-1} . a$	

	$a(g . c)$		$b(h . c)$	
	$gh^{-1} . (b(h . c))$		$(a(g . c))'$	
	$(b(h . c))'$		$hg^{-1} . (a(g . c))$	

Define $(a(g . c))'$ and $(b(h . c))'$ by the configuration on the right. Then

$$\begin{aligned}(a(g . c))' &= (gh^{-1} . (b(h . c)))b(h . c)\\ &= (gh^{-1} . b)a(g . c)ab(h . c) \quad \text{since} \quad gh^{-1} . b\,\mathscr{L}\,a\,\mathscr{R}\,b\\ &= (gh^{-1} . b)b(h . c)b'b(h . c) \quad \text{using} \quad (a, g)\sigma(b, h)\\ &= (gh^{-1} . b)b(h . c)(h . c)b'b(h . c)\\ &= (gh^{-1} . b)b(h . c) = a'(h . c)\end{aligned}$$

using $b(h . c)\,\mathscr{D}\,(h . c)b'$. In the same way, $(b(h . c))' = b'(g . c)$. Now for each x in E,

$$\begin{aligned}(hk)(gk)^{-1} . (a(g . c))'xa(g . c) &= (hg^{-1} . (a'((h . c)x)a))(h . c)\\ &= b(h . c)xbb(h . c),\end{aligned}$$

and similarly $(gk)(hk)^{-1} . ((b(h . c))'xb(h . c)) = a(g . c)xa(g . c)$. Further,

$$\begin{aligned}a(g . c)(gk . x)a(g . c) = a(g . (c(k . x)))a(g . c) &= b(h . (c(k . x)))b'(g . c)\\ &= b(h . c)(hk . x)b(h . c))'\end{aligned}$$

and $b(h . c)(hk . x)b(h . c) = a(g . c)(gk . x)(a(g . c))'$.

This completes the proof of compatibility on the right. For left compatibility, we note first that because $kN_D k^{-1} \subseteq N_{D'}$ where $k . a \in D'$, we can multiply on the left by the units $(1, k)$ or $(1, k^{-1})$ respectively and observe that it is enough to prove

$$((k^{-1} . c)a, g)\sigma(k^{-1} . c)b, h).$$

Write $d = k^{-1} . c$ and note that $hg^{-1} \in N_{D_{da}}$ because $N_D \subseteq N_{D_{da}}$ by (1). Consider the configurations below.

	a		b	
	$gh^{-1} . b$		a'	
	b'		$gh^{-1} . a$	

	da		db	
	$gh^{-1} . (db)$		$(da)'$	
	$(db)'$		$hg^{-1} . (da)$	

Using these,

$$(da)' = (gh^{-1} . d)(gh^{-1} . b)(ada)b \quad \text{since} \quad db = dab, \qquad ada\,\mathcal{L}\,da\,\mathcal{L}\,(gh^{-1} . d)(gh^{-1} . b)$$
$$= (gh^{-1} . (dbb'ddb))b \quad \text{using} \quad (a, g)\sigma(b . h)$$
$$= (gh^{-1} . (db))b \quad \text{calculating in } D \text{ and in } D_{da}$$
$$= (gh^{-1} . d)a'.$$

Similarly $(db)' = (hg^{-1} . d)b'$. It follows that for each x in E,

$$hg^{-1} . ((da)'xda) = hg^{-1} . ((gh^{-1} . d)a'xda) = dbxdb,$$

and that $hg^{-1} . ((db)'xdb) = daxda$. The other two equations in (3) of Theorem 3.1 also follow from $(da)' = (gh^{-1} . d)a'$ and $(db)' = (hg^{-1} . d)b'$. Therefore σ is compatible, so is an idempotent-separating congruence on T.

Conversely, suppose that ρ is an idempotent-separating congruence on T. We saw above that ρ determines a collection of normal subgroups N_D, for D in $\mathcal{Y}$, satisfying (1) and (2); let σ denote the congruence determined by this collection as in the first part of the proof. If $(a, g)\sigma(b, h)$ then $hg^{-1} \in N_D$ implies that $(a(hg^{-1} . a), hg^{-1})\rho(a, 1)$, and therefore $(b, hg^{-1})\rho(a, 1)$ or, equivalently, $(b, h)\rho(a, g)$; that is $\sigma \subseteq \rho$. The reverse inclusion is obvious, so $\sigma = \rho$. This completes the proof of the theorem.

4. Examples. Theorem 1.3 enables us to construct all uniquely unit orthodox semigroups, Theorem 1.4 shows that every unit-regular orthodox semigroup is an idempotent-separating homomorphic image of one of the former, and Theorem 3.1 provides a description of the appropriate congruences. In practice, starting with a given group G that acts automorphically on a band $E = E^1$, one would construct as above a unit-regular orthodox semigroup T whose band is necessarily isomorphic to E, and usually one would like the group of units of T to be isomorphic to G. To ensure isomorphism between the two groups one only has to take $N_G = \{1\}$ (denoting by G the $\mathcal{D}$-class of 1 in E) in (1) and (2), and to notice for (1) that G is the identity element of $\mathcal{Y}$, for (2) that $k . 1 = 1$ for each k in G.

A unit-regular orthodox semigroup S constructed as in Section 3 by factoring via an idempotent-separating congruence on $T = E(S)|\times|G(S)$ will be an inverse semigroup precisely when $E = E(S)$ is a semilattice. In this case the inverse semigroup T is the semi-direct product of a semilattice and a group. Every factorizable inverse semigroup [**2**] is of this sort [**6**], and may therefore be obtained by factoring T by a congruence of the type described in Theorem 3.1. And when E is a semilattice its $\mathcal{D}$-classes are singletons and the groups S_D are the stabilizers of individual elements of E. The idempotent-separating congruences ρ in this case may be defined much more simply than in Theorem 3.1; it is easy to see that:

$$(a, g)\rho(b, h) \quad \text{if and only if} \quad a = b \quad \text{and} \quad hg^{-1} \in N_b.$$

Chen and Hsieh [**2**] proved that each element of a factorizable inverse semigroup S may be written in the form $x = eg$ for a unique e in $E(S)$ and g in G. This does not hold in general for unit-regular orthodox semigroups, as the example below shows. So while it is still true that each element is under a unit in the natural ordering, the uniqueness of e in $x=eg$ has gone. Further, the natural ordering on a regular monoid S is compatible with multiplication only if S is an inverse semigroup [**8**].

EXAMPLE. Let S be the semigroup of all 3×3 real matrices of the form

$$\begin{bmatrix} 1 & p & q \\ 0 & 1 & r \\ 0 & 0 & s \end{bmatrix}$$

Then S is a unit-regular orthodox semigroup with $E=E(S)$ consisting of the identity matrix and those matrices of S for which $p=s=0$, and $G=G(S)$ those for which $s\neq 0$. The band of S consists of two $\mathcal{D}$-classes, namely $\{1\}$ and one $\mathcal{R}$-class D, say, consisting of all the non-identity idempotents of S. By straightforward calculation, for each e in D the subgroup N of units g satisfying $ege=e$ [**1**] consists of those units whose $(1,2)$-entry is zero, and N is a normal subgroup of G. Each eSe except S itself is isomorphic to the additive group of real numbers, and S is the union of its group of units and its kernel, a single $\mathcal{R}$-class consisting of the eSe, $e\in E-\{1\}$. There is no uniqueness of e in $x=eg$ because $(E-\{1\})N\subseteq E-\{1\}$.

By Theorem 1.4 the semigroup S is the idempotent-separating homomorphic image of $T=E|\times|G$ under the mapping $(e,g)\mapsto eg$. Since the $\mathcal{D}$-class D is a rectangular band $eTe=\{(a(g\,.\,a),g)\mid g\in G\}\cong G$ for each idempotent $e=(a,1)$ with $a\in D$. An element $(a(g\,.\,a),g)$ maps to

$$a(g\,.\,a)g=a(gag^{-1})g=aga.$$

Therefore $(a(g\,.\,a),g)$ is congruent to e if and only if the $(1,2)$-entry of g is zero; that is, $N_D=N$. Therefore the congruence σ defined in Theorem 3.1 is determined by just two normal subgroups of G, namely $N_G=\{1\}$ and $N_D=N$.

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