A DOUBLE-CENTRALIZER THEOREM FOR SIMPLE ASSOCIATIVE ALGEBRAS

W. L. WERNER

Consider the following result.

PROPOSITION. *Let D be a finite-dimensional central division algebra over a field F, and let D_n be the algebra (over F) of all $n \times n$ matrices with entries in D. Let A and B be in D_n, and suppose that $BX = XB$ for every X in D_n such that $XA = AX$. Then B is a polynomial in A with coefficients in F.*

The case $D = F$ is a well-known classical result. Recently, the particular case where D is the algebra of real quaternions was established by Cullen and Carlson (**2**). In this note, the general proposition is proved by reduction to the classical case by way of tensor products.

Proof of the proposition. Since D_n is a finite-dimensional central simple algebra over F, if D_n' is an algebra anti-isomorphic to D_n, then the tensor product $D_n \otimes D_n'$ is isomorphic to a total matrix algebra F_m over the field F (**1**, p. 42). Let us identify this tensor product with F_m, so that F_m is the product of subalgebras D_n and D_n', and every element of D_n commutes with every element of D_n'.

For A in D_n let $K(A) = \{X \in D_n \mid XA = AX\}$ and let $K^*(A) = \{Y \in F_m \mid YA = AY\}$. We first show that $K^*(A) \subseteq K(A)D_n'$. Indeed, let $\{V_1, V_2, \ldots, V_r\}$ be a basis for D_n' and let $Y = \sum_{i=1}^r X_i V_i$, where X_i is in D_n, $i = 1, \ldots, r$, be an element of $K^*(A)$. Then $(\sum_{i=1}^r X_i V_i)A = A(\sum_{i=1}^r X_i V_i)$. Since A commutes with each V_i, it follows that

$$\sum_{i=1}^r (X_i A - AX_i) V_i = 0.$$

Hence, $X_i A = AX_i$ and X_i is in $K(A)$, $i = 1, \ldots, r$. Thus, Y is in $K(A)D_n'$ and $K^*(A) \subseteq K(A)D_n'$.

Now, let A and B satisfy the hypothesis of the proposition. In other words, let B commute with every element of $K(A)$. Since B also commutes with every element of D_n', it follows that B commutes with every element of $K(A)D_n'$, and hence with every element of $K^*(A)$. That is, B commutes with every element of F_m that commutes with A. Consequently, by the classical theorem for matrices over a field, B is a polynomial in A over F.

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College of Southern Utah,
Cedar City, Utah