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# κ-FOLD SYMMETRIC STARLIKE UNIVALENT FUNCTIONS

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This paper establishes the radius of convexity, distortion and covering theorems for the class

$$S_k^*(A,B) = \{f(z) = z + a_{k+1}z^{k+1} + a_{2k+1}\, z^{2k+1}+\ldots;\ \frac{zf'(z)}{f(z)} \in P_k(A,B)\},$$

where

$$P_k(A,B) = \{p(z) = 1 + p_k\, z^k + p_{2k}\, z^{2k}+\ldots; p(z) = \frac{1 + Aw(z)}{1 + Bw(z)}\},$$

$-1 \leq B < A \leq 1$, $w(0) = 0$, $|w(z)| < 1$ in the unit disc. Coefficient bounds for functions in $S_k^*(A,B)$ are also derived.

## 1. Introduction

Let $\mathcal{B}$ be the class of functions $w(z)$ regular in the unit disc $\Delta = \{z; |z| < 1\}$ and satisfying the conditions $w(0) = 0$, $|w(z)| < 1$ for $z \in \Delta$. We denote by $P(A,B)$, $-1 \leq B < A \leq 1$, the class of functions $p(z) = 1 + p_1\, z + p_2\, z^2 + \ldots$ defined by

$$p(z) = \frac{1 + Aw(z)}{1 + Bw(z)},\ w(z) \in \mathcal{B},\ z \in \Delta .$$

The definition of $P(A,B)$ is suggested by the classical result (see Nehari [10, p. 169]) that any regular function $p(z) = 1 + p_1 z + p_2 z^2+\ldots$

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such that $\mathrm{Re}\{p(z)\} > 0$ in $\Delta$ can be written in the form

$$p(z) = \frac{1 + w(z)}{1 - w(z)}, \; w(z) \in \mathcal{B}.$$

As is well-known, a necessary and sufficient condition for a function $f(z) = z + a_2z^2 + \ldots$ to be univalent starlike in $\Delta$ is

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \; z \in \Delta .$$

This condition suggests that starlike functions may be defined in terms of functions of positive real part in the unit disc. In fact, Janowski [6] defined a general class of starlike functions as

$$S^*(A,B) = \{f(z) = z + a_2z^2 + \ldots; \; \frac{zf'(z)}{f(z)} \in P(A,B)\}, \; z \in \Delta.$$

The following special cases of $S^*(A,B)$ are of interest:

$$S^*(1-2\alpha,-1)=\{f(z)=z+a_2z^2+\ldots; \; \mathrm{Re}\{zf'(z)/f(z)\} > \alpha, \; 0 \leqslant \alpha < 1\},$$

$$S^*(1,1/M-1)=\{f(z)=z+a_2z^2+\ldots; \; |zf'(z)/f(z) - M| < M, \; M > \tfrac{1}{2}\},$$

$$S^*(\alpha,0) = \{f(z) = z + a_2z^2+\ldots; \; |zf'(z)/f(z) - 1| < \alpha, \; 0 < \alpha \leqslant 1\},$$

$$S^*(\alpha,-\alpha)=\{f(z)=z+a_2z^2+\ldots; \; |zf'(z)/f(z)-1|/|zf'(z)/f(z)+1|<\alpha, 0 \leqslant \alpha < 1\}.$$

Several results on these subclasses of starlike functions may be found in Robertson [13], Janowski [5], McCarty [8] and Padmanabhan [12] respectively. It is seen that a study of $S^*(A,B)$ leads to unified results of properties of various subclasses of starlike functions.

In this paper, we pay attention to the class $S_k^*(A,B)$ of functions in $S^*(A,B)$ with $k$-fold symmetric expansion:

$$S_k^*(A,B) = \{f(z) = z + a_{k+1} z^{k+1} + a_{2k+1} z^{2k+1} + \ldots; \; \frac{zf'(z)}{f(z)} \in P_k(A,B)\},$$

where

$$P_k(A,B) = \{p(z) = 1 + p_k z^k + p_{2k} z^{2k} + \ldots \in P(A,B), \; k = 1,2,3,\ldots\}.$$

The functions $f \in S_k^*(A,B)$ are the $k$-th root transforms

$$f(z) = [g(z^k)]^{1/k}$$

of functions $g \in S^*(A,B)$. In particular, the square-root transformation of $S^*(A,B)$ yields the class of odd functions in $S_2^*(A,B)$.

The study of $k$-fold symmetric starlike functions was initiated in the early 1930's with the work of Golusin [4], Robertson [13] and Noshiro [11], each of whom established coefficient bounds for these functions. Noshiro [11] investigated in detail geometric properties, including bounds for $|f(z)|$, $|f'(z)|$, of the class $S_k^* \equiv S_k^*(1,-1)$.

This paper will establish distortion and covering theorems and the radius of convexity for $S_k^*(A,B)$. Coefficient bounds for functions in $S_k^*(A,B)$ are also derived. The results are sharp and extend the previously known results for starlike functions, particularly those of the classes listed above.

## 2. Extremal Problems over $P_k(A,B)$

By definition the radius of convexity of $S_k^*(A,B)$ is the smallest root in $(0,1]$ of the equation $\Omega(r) = 0$, where

$$\Omega(r) = \min\{\mathrm{Re}\{1 + \frac{zf''(z)}{f'(z)}\};\ |z| = r < 1,\ f \in S_k^*(A,B)\}.$$

From the definition of $S_k^*(A,B)$, we derive that

$$1 + \frac{zf''(z)}{f'(z)} = p(z) + \frac{zp'(z)}{p(z)},\ p(z) \in P_k(A,B).$$

Thus, the radius of convexity of $S_k^*(A,B)$ is obtained if we can determine the value of

$$\min_{|z| = r < 1} \mathrm{Re}\{p(z) + \frac{zp'(z)}{p(z)}\} \tag{2.1}$$

over $P_k(A,B)$.

Various methods have been developed to deal with extremal problems of the form (2.1), or more generally

$$\min_{|z| = r < 1} \mathrm{Re}\{F(p(z),\ zp'(z))\}$$

over $P \equiv P(1,-1)$. Based upon a variational formula for functions in $P$, Robertson [14] proved

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THEOREM 2.1. [14] *Let $F(u,v)$ be regular in the $v$ - plane and in the half - plane $\mathrm{Re}\ u > 0$; then for every $r$, $0 < r < 1$, the value of*

$$\min_{p(z)\in P}\ \min_{|z|=r}\ \mathrm{Re}\{F(p(z),\ zp'(z))\}$$

*occurs only for a function of the form*

$$p(z) = \frac{1+\alpha}{2}\ \frac{1+ze^{i\theta}}{1-ze^{i\theta}} + \frac{1-\alpha}{2}\ \frac{1+ze^{-i\theta}}{1-ze^{-i\theta}}, \tag{2.2}$$

where $-1 \leqslant \alpha \leqslant 1$, $0 \leqslant \theta \leqslant 2\pi$.

Thus, to solve an extremal problem such as (2.1) over $P$, we only have to substitute into (2.1) the function $p(z)$ defined by (2.2) and to find the minimum of the resulting function of three variables. However, this is precisely where the remaining difficulties lie (see Robertson [15, Theorem 3] and Libera [7, Theorem 1]). Zmorovic [18] developed a useful result to overcome these difficulties. This is described in the following theorem.

THEOREM 2.2. [18] *Let $p(z)$ be as given by (2.2); then $zp'(z)$ can be written in the form*

$$zp'(z) = \tfrac{1}{2}(p(z)^2 - 1) + \tfrac{1}{2}(\rho^2 - \rho_0^2)\ e^{2i\psi}, \tag{2.3}$$

*where* $(1+\epsilon_k z)/(1-\epsilon_k z) = a+\rho e^{i\psi_k}, k=1,2, \epsilon_1=e^{i\theta}, \epsilon_2=e^{-i\theta}, p(z)=a+\rho_0 e^{i\psi_0}$,

$0 \leqslant \rho_0 \leqslant \rho, a=(1+r^2)/(1-r^2), \rho=2r/(1-r^2),\ e^{i\psi} = e^{i(\psi_1+\psi_2)/2}$.

If we put $F(u,v) = M(u) + N(u)\ .v$, where $M(u)$, $N(u)$ are regular in the half - plane $\mathrm{Re}\ u > 0$, $u = p(z)$, $v = zp'(z)$ as given by (2.2), then it follows from (2.3) that

$$\min\ \mathrm{Re}\ \{F(u,v)\} = \mathrm{Re}\{M(u) + \tfrac{1}{2}(u^2 - 1)N(u)\} - \tfrac{1}{2}|N(u)|(\rho^2 - \rho_0^2). \tag{2.4}$$

In view of Robertson's Theorem 2.1 and equation (2.4), problem (2.1 ) is reduced to finding the minimum of a function of $u$ in the disc $|u - a| \leqslant \rho$. This is a significant simplification. Employing this technique, Zmorovic [18] found the radius of convexity for $S^*(1 - 2\alpha, - 1)$.

For the general class $P(A,B)$, it can be shown that $q(z)$ is in $P(A,B)$ if and only if

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$$q(z) = \frac{(1 + A)p(z) + 1 - A}{(1 + B)p(z) + 1 - B} \tag{2.5}$$

for some $p(z) \in P$. Robertson's result then implies that the functions which minimise the functional $\operatorname{Re}\{F(p(z), zp'(z))\}$ over $P(A,B)$ must be of the form (2.5) where $p(z)$ is now given by (2.2). Using this result, Janowski [6] extended Zmorovic's technique and solved the problems $\min \operatorname{Re}\{p(z) + zp'(z)/p(z)\}$ and $\min \operatorname{Re}\{zp'(z)/p(z)\}$ over $P(A,B)$. The analysis is, however, lengthy and extremely complicated.

For $k$-fold symmetric functions, Zawadzki [17] extended Robertson - Zmorovic's techniques and derived the radius of convexity for the class $S_k^*(\alpha, 0)$. Again, the development is rather involved.

In this paper, we employ classical tools to solve the following more general problem:

$$\min_{p(z) \in P_k(A,B)} \ \min_{|z| = r < 1} \operatorname{Re}\left\{\alpha\, p(z) + \beta \frac{zp'(z)}{p(z)}\right\}, \ \alpha \geqslant 0, \beta \geqslant 0. \tag{2.6}$$

The results by Zmorovic [18], Janowski [6], Zawadzki [17] are several special cases of (2.6).

Let $B_k$ denote the class of functions $w(z)$ in $B$ with the expansion

$$w(z) = b_k z^k + b_{2k} z^{2k} + \ldots$$

Then, for every $p(z) \in P_k(A,B)$, we have that

$$p(z) = H(w(z)), \ z \in \Delta \tag{2.7}$$

for some $w(z) \in B_k$, where $H(z) = (1 + Az)/(1 + Bz)$. Consequently, an application of the Subordination Principle (see Duren [3, p. 190-191]) yields that the image of $|z| \leqslant r$ under every $p(z) \in P_k(A,B)$ is contained in the disc

$$|p(z) - a_k| \leqslant d_k, a_k = \frac{1 - ABr^{2k}}{1 - B^2r^{2k}}, \ d_k = \frac{(A - B)r^k}{1 - B^2r^{2k}}. \tag{2.8}$$

It follows immediately from (2.8) that if $p(z) \in P_k(A,B)$, then on $|z| = r < 1$,

$$\frac{1 - Ar^k}{1 - Br^k} \leqslant \operatorname{Re}\{p(z)\} \leqslant |p(z)| \leqslant \frac{1 + Ar^k}{1 - Br^k}. \tag{2.9}$$

The inequalities are sharp for the function

$$p_0(z) = \frac{1 + Az^k}{1 + Bz^k}. \tag{2.10}$$

For the solution of (2.6), we require the following lemma.

LEMMA 2.3. *If* $w(z) \in B_k$, *then for* $z \in \Delta$,

$$|zw'(z) - kw(z)| \leq \frac{k(|z|^{2k} - |w(z)|^2)}{1 - |z|^{2k}}. \tag{2.11}$$

Proof. In view of the general Schwarz lemma, we have for $w(z) \in B_k$ that $|w(z)| \leq |z|^k$. Therefore, we may write

$$w(z) = z^k \psi(z^k), \; z \in \Delta,$$

where $\psi(z)$ is regular and $|\psi(z)| \leq 1$ in $\Delta$. An application of Carathéodory's inequality

$$|\psi'(z)| \leq \frac{1 - |\psi(z)|^2}{1 - |z|^2}, \; z \in \Delta$$

now yields

$$|zw'(z) - kw(z)| \leq k|z|^{2k} \frac{1 - |\psi(z^k)|^2}{1 - |z|^{2k}}$$

$$= \frac{k(|z|^{2k} - |w(z)|^2)}{1 - |z|^{2k}}.$$

Equality in (2.11) occurs for functions of the form $z^k(z^k - c)/(1 - cz^k)$, $|c| \leq 1$.

Going back to the expression $\alpha p(z) + \beta zp'(z)/p(z)$, we see from the representation (2.7) that

$$\alpha p(z) + \beta\frac{zp'(z)}{p(z)} = \alpha\frac{1 + Aw(z)}{1 + Bw(z)} + \beta\frac{(A - B)zw'(z)}{(1+Aw(z))(1+Bw(z))}, w(z) \in B_k.$$

Applying (2.11) to the second term of the right-hand side, we find

$$\text{Re}\left\{\alpha p(z) + \beta\frac{zp'(z)}{p(z)}\right\} \geq \text{Re}\left\{\alpha\frac{1 + Aw(z)}{1 + Bw(z)} + \frac{\beta(A - B)kw(z)}{(1 + Aw(z))(1 + Bw(z))}\right\}$$

$$- \frac{k\beta(A - B)(|z|^{2k} - |w(z)|^2)}{(1 - |z|^{2k})|1 + Aw(z)||1 + Bw(z)|}.$$

From (2.7), we also have for $w(z) \in B_k$ that

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$$w(z) = \frac{p(z) - 1}{A - Bp(z)}, \quad p(z) \in P_k(A,B).$$

Hence, in terms of $p(z)$, the above inequality becomes

$$\text{(2.12)} \quad \text{Re}\left\{\alpha p(z) + \beta\frac{zp'(z)}{p(z)}\right\} \geqslant \beta k\frac{A+B}{A-B} + \frac{1}{A-B}\,\text{Re}\left\{[\alpha(A-B)-\beta kB]p(z) - \frac{\beta kA}{p(z)}\right\}$$

$$- \frac{k\beta(r^{2k}|A - Bp(z)|^2 - |p(z) - 1|^2)}{(A - B)(1 - r^{2k})\,|p(z)|}$$

At this point, we see that the solution to (2.6) may be obtained by minimising the right-hand side of (2.12) where $p(z)$ takes its values in the disc $|p(z) - a_k| \le d_k$ as defined by (2.8). It can be shown that the minimum is reached on the diameter of this disc. In fact, using the same argument as in Theorem 1 of Anh and Tuan [1] with $r$ replaced by $r^k$ and $\beta$ replaced by $\beta k$, we can establish the following result.

THEOREM 2.4. *If* $p(z) \in P_k(A,B)$, $\alpha \geqslant 0$, $\beta \geqslant 0$, *then on* $|z| = r < 1$,

$$Re\left\{\alpha p(z) + \beta\frac{zp'(z)}{p(z)}\right\} \geqslant \begin{cases} \dfrac{\alpha - [\beta k(A - B) + 2\alpha A]r^k + \alpha A^2r^{2k}}{(1 - Ar^k)(1 - Br^k)}, & R_1 \leqslant R_2, \\[2ex] \beta k\,\dfrac{A + B}{A - B} + \dfrac{2}{(A - B)(1-r^{2k})}[(LK)^{\frac{1}{2}}-\beta k(1-ABr^{2k})], & R_2 \leqslant R_1, \end{cases}$$

*where* $R_1 = (L/K)^{\frac{1}{2}}$, $R_2 = (1 - Ar^k)/(1 - Br^k)$, $L = \beta k(1 - A)(1 + Ar^{2k})$, $K = \alpha(A - B)(1 - r^{2k}) + \beta k(1 - B)(1 + Br^{2k})$.

The result is sharp for the functions

$$p_0(z) = \frac{1 + Az^k}{1 + Bz^k} \quad \text{for} \quad R_1 \leqslant R_2$$

and

$$p_1(z) = \frac{1 + Aw_1(z)}{1 + Bw_1(z)} \quad \text{for} \quad R_2 \leqslant R_1$$

where $w_1(z) = z^k(z^k - c)/(1 - cz^k)$ is extremal for (2.11) with $c$ now defined by the condition $\text{Re}\{(1 + Aw_1(z))/(1 + Bw_1(z))\} = R_1$ at $z = -r$.

REMARK 2.5. It should be observed that a function $q(z)$ is in $P_k(A,B)$ if $q(z) = p(z^k)$ for some $p(z) \in P(A,B)$. In this representation,

$$\alpha q(z) + \beta \frac{zq'(z)}{q(z)} = \alpha p(z^k) + \beta k \frac{z^k p'(z^k)}{p(z^k)}, \quad z \in \Delta.$$

It therefore follows that the lower bound for $\mathrm{Re}\{\alpha q(z) + \beta zq'(z)/q(z)\}$ over $P_k(A,B)$ can be derived immediately from Theorem 1 of Anh and Tuan [1] with $r$ replaced by $r^k$ and $\beta$ replaced by $\beta k$. The argument leading to Theorem 2.4 of this section is presented to highlight the power and simplicity of the classical method compared to the variational method as employed by Zawadzki [17].

## 3. Some Geometric Properties of $S_k^*(A,B)$

As noted at the beginning of Section 2, the radius of convexity of $S_k^*(A,B)$ is given by the smallest root in $(0,1]$ of the equation $\Omega(r) = 0$, where

$$\Omega(r) = \min \left\{ \mathrm{Re}\left\{ p(z) + \frac{zp'(z)}{p(z)} \right\} ; \ |z| = r < 1, \ p(z) \in P_k(A,B) \right\}.$$

An application of Theorem 2.4 with $\alpha = 1$, $\beta = 1$ gives $\Omega(r)$, and solving $\Omega(r) = 0$ we obtain

THEOREM 3.1. *The radius of convexity of* $S_k^*(A,B)$ *is given by the smallest root in* $(0,1]$ *of*

(i) $A^2 r^{2k} - [(2+k)A - kB]r^k + 1 = 0$, *if* $R_1 \leqslant R_2$,

(ii) $[k(A-B) + 4A(1-A)]r^{4k} + 2[k(A-B) + 2(1-A)^2]r^{2k} + k(A-B) - 4(1-A) = 0$, *if* $R_2 \leqslant R_1$,

*where* $R_1$, $R_2$ *are as given in Theorem 2.4.*

The result previously obtained by Zmorovic [18] corresponds to the case $k = 1$, $A = 1 - 2\alpha$, $B = -1$.

We next derive sharp bounds for $|f(z)|$, $|f'(z)|$ in the class $S_k^*(A,B)$. Letting $r \to 1$ in the lower bound for $|f(z)|$ we obtain the disc which is covered by the image of the unit disc under every $f(z)$ in $S_k^*(A,B)$.

THEOREM 3.2. *Let* $f(z) \in S_k^*(A,B)$; *then on* $|z| = r < 1$,

(i) $r(1-Br^k)^{(A-B)/kB} \leq |f(z)| \leq r(1+Br^k)^{(A-B)/kB}$ , *if* $B \neq 0$ ,

$$r\ exp(-\frac{Ar^k}{k}) \leq |f(z)| \leq r\ exp(\frac{Ar^k}{k}) \ , \quad if \quad B = 0 \ ;$$

(ii) $(1-Ar^k)(1-Br^k)^{[A-(1+k)B]/B} \leq |f'(z)| \leq (1+Ar^k)(1+Br^k)^{[A-(1+k)B]/B}$ ,

*if* $B \neq 0$ ,

$$(1-Ar^k)exp(-\frac{Ar^k}{k}) \leq |f'(z)| \leq (1+Ar^k)exp(\frac{Ar^k}{k}) \ , \ if \quad B = 0 \ .$$

Proof. Write $z\ f'(z)/f(z) = p(z)$, $p(z) \in P_k(A,B)$; then

$$\frac{f'(z)}{f(z)} - \frac{1}{z} = \frac{1}{z}[p(z) - 1].$$

Hence, on integrating both sides, we get

$$\log \frac{f(z)}{z} = \int_0^z [p(\xi)-1] \frac{d\xi}{\xi} \ ,$$

that is,

$$\frac{f(z)}{z} = \exp \int_0^z \frac{p(\xi)-1}{\xi}\, d\xi \ , \quad p(z) \in P_k(A,B) \ .$$

Therefore,

$$\left|\frac{f(z)}{z}\right| = \exp[\mathrm{Re}\{\int_0^z \frac{p(\xi)-1}{\xi}\, d\xi\}] \ .$$

Substituting $\xi$ by $zt$ in the integral we have

$$\left|\frac{f(z)}{z}\right| = \exp \int_0^1 \mathrm{Re}\{\frac{p(zt)-1}{t}\}\, dt \ .$$

It follows from (2.9) that, on $|zt| = rt$ ,

$$\mathrm{Re}\{\frac{p(zt)-1}{t}\} \leq \frac{(A-B)r^k t^{k-1}}{1+Br^k t^k} \ .$$

Hence, for $B \neq 0$ ,

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$$\frac{f(z)}{z} \leqslant \exp\int_0^1 \frac{(A-B)r^k t^{k-1}}{1+Br^k t^k}\,dt = (1+Br^k)^{(A-B)/kB} .$$

The lower bound may be obtained similarly. The case $B = 0$ is trivial. To prove (ii), we note that

$$|f'(z)| = \left|\frac{f(z)}{z}\right| \, |p(z)| \quad , \quad p(z) \in P_k(A,B) .$$

Hence, applying the above results and (2.9), the assertions follow.

All the bounds are sharp for

$$f(z) = z(1+Bz^k)^{(A-B)/kB} , \quad \text{if} \quad B \neq 0 ,$$

$$f(z) = z \exp\left(\frac{Az^k}{k}\right) , \quad \text{if} \quad B = 0 .$$

The corollary of Theorem 1 of Zawadzki [16] corresponds to the special case $A = 1 - 2\alpha$, $B = -1$ .

Letting $r \to 1$ in the lower bound for $|f(z)|$ we obtain the following covering theorem for $S_k^*(A, B)$ .

COROLLARY 3.4. *The image of the unit disc under a function* $f(z) \in S_k^*(A, B)$ *contains the disc of centre* $0$ *and radius* $(1-B)^{(A-B)/kB}$ *if* $B \neq 0$, $\exp(-A/k)$ *if* $B = 0$.

## 4. Coefficient Bounds for $S_k^*(A,B)$

It is known that if $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ belongs to $P$, then $|p_n| \leqslant 2$ for $n = 1,2,3,\ldots$ For the next theorem of this section, we generalise this result to the class $P(A,B)$. The method of proof is essentially due to Clunie [2].

THEOREM 4.1. *If* $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ *belongs to* $P(A,B)$, *then* $|p_n| \leqslant A - B$ *for* $n = 1,2,3,\ldots$ *The estimates are sharp for each n.*

Proof. From the definition of $P(A,B)$, we can write that

$$p(z) - 1 = (A - Bp(z))\,w(z), \quad w(z) \in B.$$

That is,

$$\sum_{k=1}^{\infty} p_k z^k = \left(A - B \sum_{k=0}^{\infty} p_k z^k\right) w(z) .$$

This equation can be put in an equivalent form as

$$(4.1)\qquad \sum_{k=1}^{n} p_k z^k + \sum_{k=n+1}^{\infty} c_k z^k = (A - B - B\sum_{k=1}^{n-1} p_k z^k)\, w(z) ,$$

where the second series on the left-hand side is also uniformly and absolutely convergent on compact subsets of $\Delta$. Since (4.1) has the form $F(z) = G(z)w(z)$, where $|w(z)| < 1$, it follows that

$$(4.2)\qquad \frac{1}{2\pi}\int_0^{2\pi} |F(r\, e^{i\theta})|^2 d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} |G(r\, e^{i\theta})|^2 d\theta .$$

In view of Parseval's identity (see Nehari [10, p.100]), (4.2) is equivalent to

$$\sum_{k=1}^{n} |p_k|^2 r^{2k} + \sum_{k=n+1}^{\infty} |c_k|^2 r^{2k} \leq \frac{1}{2\pi}\int_0^{2\pi} |A - B - B\sum_{k=1}^{n-1} p_k r^k e^{ik\theta}|^2\, d\theta$$

$$= (A - B)^2 + B^2 \sum_{k=1}^{n-1} |p_k|^2 r^{2k} .$$

Thus,

$$\sum_{k=1}^{n} |p_k|^2 r^{2k} \leq (A - B)^2 + B^2 \sum_{k=1}^{n-1} |p_k|^2 r^{2k} .$$

Letting $r \to 1$, we obtain

$$\sum_{k=1}^{n} |p_k|^2 \leq (A - B)^2 + B^2 \sum_{k=1}^{n-1} |p_k|^2 ,$$

or equivalently,

$$|p_n|^2 \leq (A - B)^2 + (B^2 - 1) \sum_{k=1}^{n-1} |p_k|^2 .$$

Since $B < 1$, it follows that $|p_n| \leq A - B$. The function

$$p(z) = \frac{1 + Az^n}{1 + Bz^n} = 1 + (A - B)z^n + \ldots$$

in $P(A,B)$ shows that the result is sharp.

We next apply the above theorem to derive coefficient estimates for $k$-fold symmetric starlike functions of order $\alpha$, that is, for functions in the class $S_k^*(1 - 2\alpha, -1)$.

THEOREM 4.2. *If* $f(z) = z + a_{k+1}z^{k+1} + a_{2k+1}z^{2k+1} + \dots$ *belongs to* $S_k^*(1 - 2\alpha, -1)$,

$$|a_{nk+1}| \leq \frac{1}{n!} \prod_{\nu=0}^{n-1} \left[\frac{2(1-\alpha)}{k} + \nu\right], \quad n = 1,2,3,\dots$$

*The estimates are sharp for each* $n$.

Proof. If we put $\xi = z^k$ and define a function

$$g(\xi) = [f(z)]^k,$$

then $g(\xi)$ is regular in $\Delta$ and

$$\frac{\xi\, g'(\xi)}{g(\xi)} = \frac{zf'(z)}{f(z)}$$

Thus $g(\xi)$ is starlike of order $\alpha$ for $|\xi| < 1$. Expanding in a power series, we find that

$$(4.3) \quad \frac{\xi\, g'(\xi)}{g(\xi)} = 1 + k\xi \frac{a_{k+1} + 2a_{2k+1}\xi + \dots + n\, a_{nk+1}\xi^{n-1} + \dots}{1 + a_{k+1}\xi + a_{2k+1}\xi^2 + \dots + a_{nk+1}\xi^n + \dots}$$

$$= 1 + d_1\xi + d_2\xi^2 + \dots$$

In view of Theorem 4.1 with $A = 1 - 2\alpha$, $B = -1$, we obtain

$$|d_n| \leq 2(1 - \alpha), \quad n = 1,2,3,\dots$$

It then follows that

$$(4.4) \quad \frac{\xi\, g'(\xi)}{g(\xi)} \ll 1 + \frac{2(1-\alpha)\xi}{1 - \xi}$$

Here, for simplicity, we write $\sum_{n=0}^{\infty} a_n z^n \ll \sum_{n=0}^{\infty} b_n z^n$ if $b_n \geq 0$ and $|a_n| \leq b_n$ for every $n$.

From (4.3) and (4.4) we see that

$$\frac{a_{k+1} + 2a_{2k+1}\xi + \dots}{1 + a_{k+1}\xi + a_{2k+1}\xi^2 + \dots} \ll \frac{2(1-\alpha)}{k} \frac{1}{1 - \xi},$$

that is

$$(4.5) \quad \log(1 + a_{k+1}\xi + a_{2k+1}\xi^2 + \dots) \ll -\frac{2(1-\alpha)}{k} \log(1 - \xi),$$

taking a branch of log such that log $1 = 0$. It follows from (4.5) that

$$1 + a_{k+1}\xi + a_{2k+1}\xi^2 + \ldots \ll \frac{1}{(1-\xi)^{2(1-\alpha)/k}}$$

from which the result can be derived. To see that the estimates are sharp, we consider the function

$$f(z) = \frac{z}{(1-z^k)^{2(1-\alpha)/k}} = z + \sum_{n=1}^{\infty} \frac{1}{n!}\frac{2(1-\alpha)}{k}\left(\frac{2(1-\alpha)}{k}+1\right)\ldots\left(\frac{2(1-\alpha)}{k}+n-1\right) \times z^{nk+1}.$$

The method of proof used in the above theorem unfortunately does not work for the general class $S_k^*(A,B)$. However, the above coefficient bounds for $S_k^*(1-2\alpha, -1)$ do suggest the form of coefficient bounds for functions in $S_k^*(A,B)$. In fact, we have the following theorem, the proof of which is under the influence of MacGregor [9].

THEOREM 4.3. *Let* $f(z) = z + a_{k+1}z^{k+1} + a_{2k+1}z^{2k+1} + \ldots$ *be in* $S_k^*(A,B)$ *and put* $M = [\frac{A-B}{k(1+B)}]$, *the largest integer not greater than* $(A-B)/k(1+B)$.

(a) *If* $A - B > k(1+B)$, *then*

$$|a_{nk+1}| \leq \frac{1}{n!}\prod_{\nu=0}^{n-1}\left|\frac{A-B}{k} - \nu B\right|, \quad n = 1,2,\ldots, M+1, \tag{4.6}$$

$$|a_{nk+1}| \leq \frac{1}{nM!}\prod_{\nu=0}^{M}\left|\frac{A-B}{k} - \nu B\right|, \quad n \geq M+2. \tag{4.7}$$

(b) *If* $A - B \leq k(1+B)$, *then*

$$|a_{nk+1}| \leq \frac{A-B}{nk}, \quad n = 1,2,3,\ldots \tag{4.8}$$

The estimates (4.6) and (4.8) are sharp.

Proof. From the definition of $S_k^*(A,B)$, we have that

$$\frac{zf'(z)}{f(z)} = \frac{1+Aw(z)}{1+Bw(z)}, \quad w(z) \in B_k,$$

that is,

$$zf'(z) - f(z) = w(z)(Af(z) - Bzf'(z)) ,$$

or, in their series expansion,

(4.9) $$\sum_{n=1}^{\infty} nka_{nk+1}z^{nk+1} = w(z)((A - B)z + \sum_{n=1}^{\infty} (A-B(nk + 1))a_{nk+1}z^{nk+1}) .$$

This equation can be put in an equivalent form as

$$\sum_{n=1}^{N} nk\ a_{nk+1}\ z^{nk+1} + \sum_{n=N+1}^{\infty} d_{nk+1}z^{nk+1} = w(z)((A-B)z + \sum_{n=1}^{N-1} (A-B(nk+1))a_{nk+1}z^{nk+1}) ,$$

where $N = 1,2,3,\ldots$ and the second series on the left-hand side is again uniformly and absolutely convergent on compact subsets of $\Delta$.

With the same argument as in the proof of Theorem 4.1, using Parseval's identity and the fact that $|w(z)| < 1$, we arrive at the inequality

$$\sum_{n=1}^{N} n^2k^2|a_{nk+1}|^2 \leqslant (A - B)^2 + \sum_{n=1}^{N-1} (A - B(nk + 1))^2|a_{nk+1}|^2 ,$$

or equivalently,

(4.10) $$N^2k^2|a_{Nk+1}|^2 \leqslant (A - B)^2 + \sum_{n=1}^{N-1} [(A-B(nk + 1))^2 - n^2k^2]|a_{nk+1}|^2 .$$

Since $(A - B(nk + 1))^2 - n^2k^2 \geqslant 0$ if and only if $n \leqslant (A - B)/k(1 + B)$, the following four cases can arise:

(i) $n \leqslant \frac{A - B}{k(1 + B)}$ and $A - B > k(1 + B)$ ,

(ii) $n > \frac{A - B}{k(1 + B)}$ and $A - B > k(1 + B)$ ,

(iii) $n \leqslant \frac{A - B}{k(1 + B)}$ and $A - B \leqslant k(1 + B)$ ,

(iv) $n > \frac{A - B}{k(1 + B)}$ and $A - B \leqslant k(1 + B)$ .

Case (iii) holds only if $n = 1$. In view of (4.9), we have

$k\ a_{k+1} = (A - B)b_k$,

where $w(z) = b_kz^k + b_{2k}z^{2k} + \ldots$ Since $|w(z)| < 1$, it follows that

$$\sum_{n=1}^{\infty} |b_{nk}|^2 \le 1.$$

Thus $|b_k|^2 \leqslant 1$. And so,

(4.11) $$|a_{k+1}| \leqslant \frac{A - B}{k}.$$

Let us now consider each of the remaining cases.

(i) In view of (4.10), we want to establish that

(4.12) $$N^2k^2|a_{Nk+1}|^2 \leqslant \left[\frac{k}{(N - 1)!} \prod_{n=0}^{N-1} \left(\frac{A - B}{k} - nB\right)\right]^2 .$$

This inequality holds for $N = 1$ in view of (4.11). Suppose that it is true up to $N - 1$. Then for $N \leqslant M + 1$,

$$N^2k^2|a_{Nk+1}|^2 \leqslant (A - B)^2 + \sum_{n=1}^{N-1} ((A - B(nk + 1))^2 - n^2k^2)|a_{nk+1}|^2$$

(4.13) $$\le (A - B)^2 + \sum_{n=1}^{N-1} \left\{\left[\frac{1}{n!} \prod_{\nu=0}^{n-1} \left(\frac{A - B}{k} - \nu B\right)\right]^2 \left[(A - B(nk + 1))^2 - n^2k^2\right]\right\}.$$

Put the expression on the right-hand side of (4.13) equal $S(N - 1)$. If we can establish that

(4.14) $$S(N - 1) = \left[\frac{k}{(N - 1)!} \prod_{n=0}^{N-1} \left(\frac{A - B}{k} - nB\right)\right]^2,$$

then (4.12) is true for all $N \leqslant M + 1$. We again prove (4.14) by induction. For $N = 2$, we have

$$S(1) = (A - B)^2 + \left(\frac{A - B}{k}\right)^2 ((A - B(k + 1))^2 - k^2)$$

$$= \left(\frac{A - B}{k}\right)^2 (A - B(k + 1))^2$$

which is the right-hand side of (4.14). Thus (4.14) holds for $N = 2$. Suppose that it is true up to $N - 1$. Then for $N$,

$$S(N) = S(N - 1) + \left(\frac{1}{N!} \prod_{\nu=0}^{N-1} \left(\frac{A - B}{k} - \nu B\right)\right)^2 ((A - B(Nk+1))^2 - N^2k^2)$$

$$= \left(\frac{k}{(N - 1)!} \prod_{n=0}^{N-1} \left(\frac{A-B}{k} - nB\right)\right)^2 + \left(\frac{1}{N!} \prod_{\nu=0}^{N-1} \left(\frac{A-B}{k} - \nu B\right)\right)^2 ((A-B(Nk+1))^2 - N^2k^2)$$

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$$= \left[\frac{1}{(N-1)!}\prod_{n=0}^{N-1}\left(\frac{A-B}{k}-nB\right)\right]^2 \left[k^2+\frac{1}{N^2}\left(\left(A-B(Nk+1)\right)^2-N^2k^2\right)\right]$$

$$= \left(\frac{k}{N!}\prod_{n=0}^{N}\left(\frac{A-B}{k}-nB\right)\right)^2 .$$

Thus (4.14) is true for all $N$. This establishes (4.12). Note that $(A - B)/k - nB \geq 0$ is equivalent to $nk \leq (A - B)/B$ if $B > 0$ [The inequality is obvious if $B \leq 0$]. In case (i), $nk \leq (A - B)/(1+B) < (A-B)/B$ as $A - B > 0$. Thus, inequality (4.6) of the theorem follows from (4.12).

(ii) Again, from (4.10), we have that

$$N^2k^2|a_{Nk+1}|^2 \leq (A-B)^2 + \sum_{n=1}^{M}\left(\left(A-B(nk+1)\right)^2-n^2k^2\right)|a_{nk+1}|^2$$

$$+ \sum_{n=M+1}^{N-1}\left((A-B(nk+1))^2-n^2k^2\right)|a_{nk+1}|^2, \; N \geq M+2$$

$$\leq (A-B)^2 + \sum_{n=1}^{M}\left((A-B(nk+1))^2-n^2k^2\right)|a_{nk+1}|^2$$

$$\leq (A-B)^2 + \sum_{n=1}^{M}\left[\frac{1}{n!}\prod_{\nu=0}^{n-1}\left(\frac{A-B}{k}-\nu B\right)\right]\left((A-B(nk+1))^2-n^2k^2\right)$$

in view of (4.6)

$$= \left[\frac{k}{M!}\prod_{n=0}^{M}\left(\frac{A-B}{k}-nB\right)\right]^2 \qquad \text{from (4.14)}$$

Thus, $|a_{Nk+1}| \leq \frac{1}{NM!}\prod_{n=0}^{M}\left(\frac{A-B}{k}-nB\right)$ for $N \geq M+2$.

This is inequality (4.7) of the theorem.

(iv) In this case, it follows easily from (4.10) that

$N^2k^2|a_{Nk+1}|^2 \leq (A-B)^2$, $N \geq 2$.

That is,

$$|a_{Nk+1}| \leq \frac{A-B}{Nk}, \; N \geq 2.$$

This with (4.11) above yields inequality (4.8) of the theorem.

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Inequality (4.6) is sharp for the function

$$f(z) = z(1 + Bz^k)^{(A - B)/kB}, \quad \text{if} \quad B \neq 0,$$

$$f(z) = z \exp(Az^k/k) \quad , \quad \text{if} \quad B = 0,$$

while inequality (4.8) is sharp for the function

$$f(z) = z \exp\left(\frac{A - B}{nk} z^{nk}\right) .$$

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