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ON CANONICAL REALIZATIONS OF BOUNDED SYMMETRIC DOMAINS AS MATRIX-SPACES[1)]

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Introduction

It is the purpose of the present paper to give a natural method of realizing bounded symmetric domains as matrix-spaces. Our method yields, as special cases, the well-known bounded models of irreducible bounded symmetric domains of classical type (I)-(IV), as were already described in the original paper of E. Cartan [1] (see §3; we follow in this paper the classification table in [14], not in [1]). A direct application of this method will be to determine the *canonical* bounded models of the irreducible bounded symmetric domains of exceptional type; it will be published in another paper (see [6], [7] for the summary of the results).

In the Appendix, we indicate briefly that our version on symmetric domains can be generalized and applied to a more general class of symmetric spaces, the so-called symmetric R-spaces of non-compact type in the sense of J. Tits; this was partly stated in Nagano [13] and Takeuchi [16].

We would like to express here our deep gratitude to M. Takeuchi who read the manuscript and suggested many improvements.

NOTATION: 1) $\boldsymbol{M}_{p,q}$ denotes the complex vector space of all complex matrices of type (p, q); in particular, we write as $\boldsymbol{M}_{p,p} = \boldsymbol{M}_p$ for brevity. Similarly $\boldsymbol{M}_{p,q}(\boldsymbol{R})$ is the real vector space of all real matrices of type (p, q).

2) $\boldsymbol{C}^n$ is the complex cartesian space of n-dimensions, and in many cases, $\boldsymbol{C}^n$ is identified with $\boldsymbol{M}_{n,1}$, or with $\boldsymbol{M}_{1,n}$.

3) For hermitian matrices A, B $(\in \boldsymbol{M}_r)$, $A < B$ means that all eigen-values of $A - B$ are negative. I_r denotes the unit matrix of degree r.

4) For complex vector spaces V, W, we denote by $\mathfrak{L}(V, W)$ the complex vector space of all complex linear mappings of V into W.

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1) Part of the present work was done in 1964, when the author was staying at the Institute for Advanced Study, Princeton, under the sponsorship of the National Science Foundation.

5) For a real vector space $\mathfrak{g}$, we denote the complexification of $\mathfrak{g}$ by $\mathfrak{g}_c$.

6) $\oplus$ denotes the direct sum (not the tensor sum) of vector spaces.

7) As for terminology and notation concerning symmetric spaces we refer the reader mainly to [3]; especially we denote Lie groups by large Roman letters and Lie algebras by German letters.

§ 1. Harish-Chandra-Langlands realization.

1.1. Let $X = G/K$ denote a hermitian symmetric space of non-compact type, and $X_u = G_u/K$ the hermitian symmetric space of compact type which is *dual* to X; where G, G_u and $K = G \cap G_u$ should be all real connected closed subgroups of a simply-connected complex semi-simple Lie group G_c, and both of G and G_u are real forms of G_c (see, for detail, [3]). We know that X_u is, as a complex manifold, of the form G_c/B for a connected, complex closed subgroup B of G_c. Small German letters corresponding to the respective large Roman letters will mean the Lie algebras. Then we have the so-called *symmetric pair* (see [3]):

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}, \quad \mathfrak{g}_u = \mathfrak{k} \oplus \sqrt{-1}\,\mathfrak{m}, \tag{1}$$

$$\mathfrak{g}_c = \mathfrak{k}_c \oplus \mathfrak{m}_c. \tag{2}$$

Moreover, taking a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{k}_c$ (and also of $\mathfrak{g}_c$), we get the Cartan decompositions: $\mathfrak{g}_c = \mathfrak{h} \oplus \sum_\alpha Ce_\alpha$, $\mathfrak{k}_c = \mathfrak{h} \oplus \sum'_\beta Ce_\beta$. In the above decompositions, we can further decompose $\mathfrak{m}_c$ as

$$\mathfrak{m}_c = \mathfrak{n}^+ \oplus \mathfrak{n}^-; \; \mathfrak{n}^+ = \textstyle\sum''_{\alpha>0} Ce_\alpha, \quad \mathfrak{n}^- = \sum''_{\alpha>0} Ce_{-\alpha}: \tag{3}$$

($\sum''$ designates the summation complementary to $\sum'$ in $\sum$)

$$[\mathfrak{k}_c, \mathfrak{n}^\pm] \subset \mathfrak{n}^\pm, \quad [\mathfrak{n}^+, \mathfrak{n}^-] \subset \mathfrak{k}_c, \quad [\mathfrak{n}^\pm, \mathfrak{n}^\pm] = \{0\}. \tag{4}$$

In what follows, we will call the decompositions (2) and (3), with (4), *a complex symmetric pair* corresponding to the hermitian symmetric spaces X and X_u. Then we can here regard as $\mathfrak{b} = \mathfrak{k} \oplus \mathfrak{n}^-$ and that $\mathfrak{g}_u$ has Weyl's canonical base. Following to Harish-Chandra, we consider then the inclusion relations:

$$GB \subset N^+B \subset G_c.$$

We take the quotients of these sets by B from the right, then, using $G \cap B = K$, $N^+ \cap B = \{1\}$, it yields the new inclusions:

$$X \subset N^+ \subset X_u.$$

We denote these inclusion maps by j_1: $X \to N^+$ and by j_2: $N^+ \to X_u$ and then put $j = j_2 \circ j_1$, while N^+ is a complex vector group and so mapped isomorphically onto $\mathfrak{n}^+$ by the inverse of the exponential mapping, $\exp^{-1}$, through which we will hereafter identify N^+ with $\mathfrak{n}^+$. Thus we have an injective holomorphic mapping $\exp^{-1} \circ j_1$ of X into $\mathfrak{n}^+$, which we also denote for brevity by j_1. Hence, the above inclusions now becomes

$$X \xrightarrow{j_1} \mathfrak{n}^+ \xrightarrow{j_2} X_u. \tag{5}$$

This relation plays the fundamental role throughout the present paper; so we want to call it the *fundamental inclusion relation* for X. We note here that $j = j_2 \circ j_1$ is equivariant under the action of G. Furthermore we often identify $\mathfrak{n}^+$ with the complex cartesian space C^N ($N = \dim_C \mathfrak{n}^+$) through a suitable base of $\mathfrak{n}^+$. Then $j_1(X) = D$ is an open set of $\mathfrak{n}^+ = C^N$, and a distinguished result of Harish-Chandra says that D is relatively compact, namely D is a bounded symmetric domain in C^N.

1.2. In the original proof of Harish-Chandra for the above result, the explicit form of D is still ambiguous; it is later clarified by several authors: R.Hermann, R.Langlands and C.C.Moore (see [4], [10], [12]). Their results, which are essential in our later arguments, will be reproduced below after Langlands (see Lemma 2 in [10]).

Let τ denote the complex conjugation of $\mathfrak{g}_c$ relative to the compact real form $\mathfrak{g}_u$; we can then define, as usual, the positive definite hermitian inner product (u, v) in $\mathfrak{g}_c$ by putting

$$(u, v) = -\Phi(u, \tau v), \quad (u, v \in \mathfrak{g}_c)$$

where Φ denotes the Killing form of $\mathfrak{g}_c$. Now, for every element z of $\mathfrak{g}_c$, $\theta(z)$ will denote the adjoint operator $ad(z)$ in $\mathfrak{g}_c$ and we put $z^* = -\tau(z)$. Then we have

LEMMA 1. 1) *If* $z \in \mathfrak{n}^\pm$, *then* $z^* \in \mathfrak{n}^\mp$. 2) $\theta^*(z) = \theta(z^*)$, *where* $\theta^*(z)$ *denotes the adjoint operator of* $\theta(z)$ *with respect to the inner product introduced above.* 3) *Two hermitian operators* $\theta^*(z)\theta(z)$ *and* $\theta(z)\theta^*(z)$ *have the same norms, and for* $z \in \mathfrak{n}^+$, *we have*

$$\theta^*(z)\theta(z) = \theta([z^*, z]), \quad ([z^*, z] \in \mathfrak{k}_c).$$

on the space $\mathfrak{n}^-$

Proof. 1) is obvious from the fact that $\mathfrak{g}_u$ has the *canonical base.* 2) is verified as follows: $(\theta^*(z)u, v) = (u, \theta(z)v) = -\Phi(u, \tau[z, v]) = -\Phi(u, [\tau z, \tau v]) = \Phi([\tau z, u,], \tau v) = (\theta(z^*)u, v)$. 3) is followed from the fact that $\mathfrak{n}^-$ is an abelian subalgebra of $\mathfrak{g}_c$.

In the following, the hermitian operator $\theta^*(z)\theta(z)$, or $\theta([z^*, z])$ will be considered as that on $\mathfrak{n}^-$,[2] unless otherwise specified.

THEOREM (LANGLANDS). *The bounded domain D is explicitly given by*

$$D = \{z \in \mathfrak{n}^+ = C^N;\ \theta([z^*, z]) < 2I_N\}.$$

(cf. Notation 2) in the Introduction)

§ 2. Realization as matrix-space.

2.1. We shall now consider the irreducible hermitian symmetric space of type $(I_{p,q})$; in this case, X_u is the complex Grassmannian manifold $V_{p,q} = U(p+q)/U(p) \times U(q)$, $\mathfrak{n}^+$ can be canonically identified with $M_{p,q}$ (see Notation) and X is holomorphically isomorphic to the bounded domain $D_{p,q}$ with the ambient space $M_{p,q}$: $D_{p,q} = \{Z \in M_{p,q};\ {}^t\bar{Z}Z < I_q\}$. The fundamental inclusion relation in this case is the following one:

$$X \xrightarrow{j_1} M_{p,q} \xrightarrow{j_2} V_{p,q}, \tag{6}$$

where $j_1(X) = D_{p,q}$. All these statements shall be showed explicitly in § 3. The mapping j_2 in the above (6) is given by the following rule: For every $Z \in M_{p,q}$,

$$j_2(Z) = \left\{\begin{pmatrix} Zu \\ u \end{pmatrix} \in C^n;\ u \in C^q\right\},\quad (n = p + q),$$

where the right-hand side is a q-dimensional linear sub-space of C^n, and $V_{p,q}$ is here regarded as the totality of such sub-spaces of C^n.

In this section, we present the following commutative diagram:

$$\begin{array}{ccccc} X & \xrightarrow{j_1} & \mathfrak{n}^+ & \xrightarrow{j_2} & X_u \\ & & \downarrow \rho & & \downarrow \rho \\ D_{p,q} & \subset & M_{p,q} & \longrightarrow & V_{p,q}, \end{array} \tag{7}$$

---

[2] The hermitian operator $\theta^*(z)\theta(z)$ on $\mathfrak{g}_c$ maps $\mathfrak{n}^+$ to $\{0\}$, both $\mathfrak{k}_c$ and $\mathfrak{n}^-$ into themselves respectively. Further we can show easily that the norm of $\theta^*(z)\theta(z)$ coincides with those of $\theta^*(z)\theta(z)$ considered as the operators on $\mathfrak{k}_c$, or on $\mathfrak{n}^-$ respectively.

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namely we will introduce the mappings ρ; the left-hand ρ is a complex linear mapping and the right-hand ρ a holomorphic one.

2.2. To begin with, we take up a non-trivial irreducible holomorphic representation $\tilde{\rho}$ of G_c into $GL(n, \boldsymbol{C})$ $(n > 1)$, and denote by ρ_K the restriction of $\tilde{\rho}$ to K_c. Then, ρ_K is completely reducible; we decompose (ρ_K, V) into the direct sum of several number of representations (ρ_i, V_i) $(1 \leqslant i \leqslant s)$ after Matsushima and Murakami [11] (cf. Part II, 5):

$$(8) \qquad \begin{aligned} &\rho_K \sim \rho_1 + \rho_2 + \cdots + \rho_s, \\ &V = V_1 \oplus V_2 \oplus \cdots \oplus V_s. \end{aligned}$$

The definition of (ρ_i, V_i) is as follows: Put $V_1 = \{u \in V;\ \tilde{\rho}(x)u = 0,$ for all $x \in \mathfrak{n}^+\}$ and $V_i = \tilde{\rho}(\mathfrak{n}^-)V_{i-1}$ $(i \geqslant 2)$ inductively; namely V_i is the linear span of all $\tilde{\rho}(x)u$, for $x \in \mathfrak{n}^-$ and $u \in V_{i-1}$. Then, these V_i constitute the direct sum decomposition of V as in (8).

LEMMA 2 (MATSUSHIMA AND MURAKAMI).[3] *For the above decomposition* (8), *it holds that*

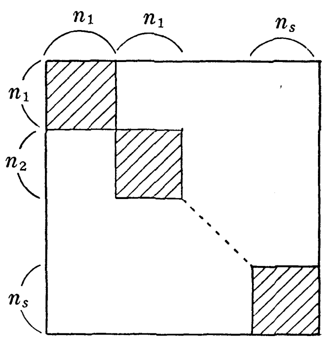
i) $\tilde{\rho}(\mathfrak{k}_c)V_i \subset V_i$ $(1 \leqslant i \leqslant s)$; $\tilde{\rho}(\mathfrak{n}^+)V_i \subset V_{i-1}$, $\tilde{\rho}(\mathfrak{n}^-)V_{i-1} \subset V_i$ $(2 \leqslant i \leqslant s)$,

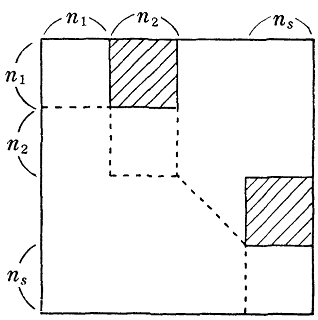
ii) (ρ_1, V_1) *is irreducible, and the highest weight of* ρ_1 *coincides with that of* $\tilde{\rho}$ *with respect to a common Cartan subalgebra* $\mathfrak{h}$ *of both* $\mathfrak{k}_c$ *and* $\mathfrak{g}_c$.

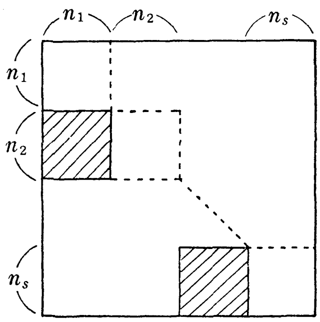
In the decomposition (8), we put $\dim V_i = n_i$ $(1 \leqslant i \leqslant s)$, and in particular $n_1 = p$, $n_2 = r$ and $n - p = q$ $(= \sum_{i=2}^{s} n_i)$. Furthermore, we take and fix, once for all, an orthonormal base of V with respect to a $\tilde{\rho}(G_u)$-invariant hermitian inner product as the totality of those of respective V_i. By use of these fixed bases of V_i and V, we shall identify every linear transformation or linear mapping with respect to V, V_i with the corresponding matrix respectively; in particular, we identify thus $GL(V)$ with $GL(n, \boldsymbol{C})$. Then, from Lemma 2 we see

[3] A somewhat different version on this lemma is found in Murakami's lecture note at Chicago University, "Cohomology groups of vector valued forms on symmetric spaces" (1966).

(9)

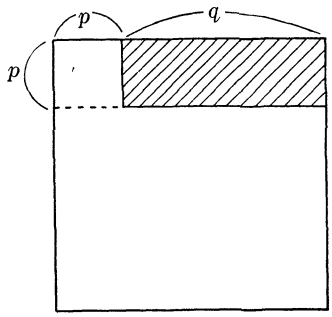
$\tilde{\rho}(\mathfrak{k}_c) \subset$ 

$\tilde{\rho}(\mathfrak{n}^+) \subset$ 

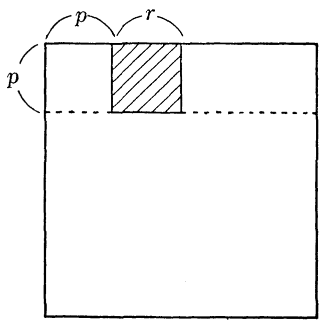
, $\tilde{\rho}(\mathfrak{n}^-) \subset$ 

.

Next we shall identify:

$\boldsymbol{M}_{p,q} = \mathfrak{L}(V_2 \oplus \cdots \oplus V_s, V_1) =$ 

,

$\boldsymbol{M}_{p,r} = \mathfrak{L}(V_2, V_1) =$ 

.

Furthermore, if we put

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$$GL(n\colon n_1, \cdots, n_s,\ \boldsymbol{C}) = \{A \in GL(V);\ A(V_i) \subset V_i \oplus \cdot \cdot \oplus V_s\ (1 \leqslant i \leqslant s)\},$$
$$GL(n\colon p, q,\ \boldsymbol{C}) = \{A \in GL(V);\ A(V_2 \oplus \cdots \oplus V_s) \subset V_2 \oplus, \cdots \oplus V_s\},$$

then, $GL(n\colon n_1, \cdots, n_s, \boldsymbol{C}) \subset GL(n\colon p, q, \boldsymbol{C})$, $\boldsymbol{V}_{p,q}$ is identified with $GL(n, \boldsymbol{C})/GL(n\colon p, q, \boldsymbol{C})$ and $\tilde{\rho}(B) \subset GL(n\colon n_1, \cdots, n_s, \boldsymbol{C})$, since $\mathfrak{b} = \mathfrak{k}_c \oplus \mathfrak{n}^-$ and $B = K_c N^-$. Hence, $\tilde{\rho}$ naturally induces the holomorphic mapping ρ:

$$\rho\colon X_u = G_c/B \longrightarrow \boldsymbol{V}_{p,q} = GL(n, \boldsymbol{C})/GL(n\colon p, q, \boldsymbol{C}).$$

We can then prove that ρ is injective, provided that G_c is simple, or more generally the restriction of $\tilde{\rho}$ to any simple component of G_c is not trivial (see [5], p. 231). From this it follows that, for any irreducible X, ρ is always an *injective* holomorphic mapping. Next, the linear mapping ρ of $\mathfrak{n}^+$ into $\boldsymbol{M}_{p,q}$ will be defined in the following way: For $Z \in \mathfrak{n}^+$ we may write $\tilde{\rho}(Z)$, as in (9),

$$\tilde{\rho}(Z) = \begin{pmatrix} 0 & Z_1 & & & \\ & & Z_2 & & \\ & & & \ddots & \\ & & & & Z_{s-1} \\ & & & & 0 \end{pmatrix};\ Z_i \in \boldsymbol{M}_{n_i, n_{i+1}}.$$

For this, we denote by $\rho(Z)$ the matrix $Z_1 \in \boldsymbol{M}_{p,r}$ which is likewise the $\boldsymbol{M}_{p,q}$-component of $\tilde{\rho}(Z)$. Then we see that ρ can be regarded as the differential at the basic point of the former mapping $\rho\colon X_u \longrightarrow \boldsymbol{V}_{p,q}$, and from this follows that the linear mapping ρ is injective and the diagram (7) is commutative.

2.3. *Remark.* In the decomposition (8), it always holds that $s \geqslant 2$; in case $s = 2$, we note that $\boldsymbol{M}_{p,q} = \boldsymbol{M}_{p,r}$. Indeed, if we take, as $\tilde{\rho}$, the irreducible representation of the lowest degree for each irreducible type of X, then it holds:

$s = 2$, for the type (I), (II), (III).

$s = 3$, for the type (IV), (V).

$s = 4$, for the type (VI).

These facts will be showed for the classical type (I)-(IV) in § 3, and for the exceptional type (V), (VI) in [6], [7].

2.4. From the arguments in § 2.2, we have a somewhat sharpened form of the diagram (7):

$$
(7') \qquad \begin{array}{ccccc} X & \xrightarrow{j_1} & \mathfrak{n}^+ & \xrightarrow{j_2} & X_u \\ & & \downarrow \rho & & \searrow \rho \\ & & D_{p,r} \subset \boldsymbol{M}_{p,r} \subset \boldsymbol{M}_{p,q} \subset \boldsymbol{V}_{p,q} & & \end{array}
$$

In what follows, we call the above (7′) the *fundamental diagram* for X and $\tilde{\rho}$; thus we get the embedding of X into $\boldsymbol{M}_{p,r}$, the mapping $\rho \circ j_1$ (we write simply $\rho \circ j_1 = \rho$ in the sequel). Through this embedding ρ we shall derive a concrete form of Langlands' theorem: For this sake, we identify $\tilde{\rho}(A) = A$ ($A \in \mathfrak{g}_c$) for brevity, and take $Z \in \tilde{\rho}(\mathfrak{n}^+)$, $X \in \tilde{\rho}(\mathfrak{n}^-)$. Then, $Z^* \in \tilde{\rho}(\mathfrak{n}^-)$ and we can write

$$
Z = \begin{pmatrix} 0 & Z_1 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & Z_{s-1} \\ & & & & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & & & & \\ X_1 & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & X_{s-1} & 0 \end{pmatrix}, \quad Z^* = \begin{pmatrix} 0 & & & & \\ Z_1^* & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & Z_{s-1}^* & 0 \end{pmatrix},
$$

where X_i, $Z_i^* \in \boldsymbol{M}_{n_{i+1}, n_i}$ $(2 \leqslant i \leqslant s)$, so that we have

$$
[Z^*, Z] = \begin{pmatrix} -Z_1 Z_1^*, & & & & \\ & Z_1^* Z_1 - Z_2 Z_2^*, & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & Z_{s-1}^* Z_{s-1} \end{pmatrix} \in \tilde{\rho}(\mathfrak{k}_c), \quad \text{(see (9))}.
$$

From this we infer that the X_1-component of $\theta([Z^*, Z])X \in \tilde{\rho}(\mathfrak{n}^-)$ is given by

$$
(10) \qquad \begin{array}{l} (Z_1^* Z_1) X_1 + X_1 (Z_1 Z_1^*), \ \text{if } s = 2, \\ (Z_1^* Z_1 - Z_2 Z_2^*) X_1 + X_1 (Z_1 Z_1^*), \ \text{if } s \geqq 3. \end{array}
$$

On the other hand, the linear mapping $X \longrightarrow X_1$, of $\tilde{\rho}(\mathfrak{n}^-)$ into $\boldsymbol{M}_{r,p}$ is injective, since the embedding ρ is injective; so we may regard $\theta([Z^*, Z])$ as the linear transformation on the space $\boldsymbol{M}_{r,p}(\mathfrak{n}^-) = \{X_1 \in \boldsymbol{M}_{r,p};\ X \in \tilde{\rho}(\mathfrak{n}^-)\}$. As is shown later in § 3 and in [6], [7], when X_u is one of the irreducible type (IV)—(VI) and $\tilde{\rho}$ is the irreducible representation of G_c of the lowest degree (hence $s \geqslant 3$), the following holds:

$$
n_1 = p = 1, \ \boldsymbol{M}_{r,p} = \boldsymbol{C}^r \ \text{and} \ \boldsymbol{M}_{r,p}(\mathfrak{n}^-) = \boldsymbol{M}_{r,p} = \boldsymbol{C}^r.
$$

Therefore, in these cases, our transformation $\theta([Z^*, Z])$ takes of the form:

$$\theta([Z^*, Z]) : X_1 \longrightarrow (Z_1Z_1^* + Z_1^*Z_1 - Z_2Z_2^*)X_1;$$

hence, to $\theta([Z^*, Z])$ corresponds the hermitian matrix

$$Z_1Z_1^* + Z_1^*Z_1 - Z_2Z_2^* \in \boldsymbol{M}_r$$

$$(Z_1Z_1^* \text{ is a scaler matrix in } \boldsymbol{M}_r).$$

We write here as $Z_1 = z \in \boldsymbol{C}^r = \boldsymbol{M}_{1,r}$; then we can state the following result:

THEOREM 1. (i) *For the irreducible bounded symmetric domains of type* (I)-(III), *the simplest bounded models in our sense are presented by*

$$D = \{Z \in \tilde{\rho}(\mathfrak{n}^+);\ Z_1^*Z_1 < I\},$$

where $\tilde{\rho}$ *is the irreducible represntation of the lowest degree.*

(ii) *For the domains of type* (IV) − (VI), *the simplest ones in the same sense as above are presented by*

$$D = \{z \in \boldsymbol{C}^r;\ Z_1Z_1^* + Z_1^*Z_1 - Z_2Z_2^* < 2I_r\}.$$

For the statement (i) in the theorem, we shall verify it case by case in the next section 3.

DEFINITION. The simplest bounded model D obtained in Theorem 1 for each type of irreducible bounded symmetric domain X will be called the *canonical bounded model* of X.

2.5. As for our realization $D = \rho(X)$, for any $\tilde{\rho}$, of a bounded symmetric domain in $\boldsymbol{M}_{p,r}$, we state here an important property as to holomorphic automorphisms: For every $g \in G$ and $x \in X$, we write as (with respect to a base of V as chosen in § 2, 2)

$$\tilde{\rho}(g) = \begin{pmatrix} A & B \\ C & D \end{pmatrix};\ A \in \boldsymbol{M}_{p,p},\ B \in \boldsymbol{M}_{p,q},\ C \in \boldsymbol{M}_{q,p},\ D \in \boldsymbol{M}_{q,q},$$

$$\rho(x) = Z,\ (Z \in \boldsymbol{M}_{p,r} \subset \boldsymbol{M}_{p,q}),$$

then we know that $\rho(g \cdot x) = \tilde{\rho}(g) \cdot \rho(x)$ (see § 2, 2).

THEOREM 2. $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z = (AZ + B)(CZ + D)^{-1}$; *namely every holomorphic automorphism* $\tilde{\rho}(g)$[4] *of* D *acts as a linear fractional transformation of the vector space* $\boldsymbol{M}_{p,q}$.

[4] It is to be noted that g belongs to the *connected* Lie group G.

Proof. To begin with, we recall that $\boldsymbol{V}_{p,q}$ is identified with the set of all q-dimensional linear subspaces of $\boldsymbol{C}^n$. This identification will be done in the following way: Let $\mathfrak{L}$ denote the set of all linear isomorphisms L of $\boldsymbol{C}^q$ into $\boldsymbol{C}^n$; namely we put

$$\mathfrak{L} = \{L \in \boldsymbol{M}_{n,q};\ \text{rank } L = q\},$$

then $GL(q, \boldsymbol{C})$ acts on $\mathfrak{L}$ from the right as linear transformations, and the quotient $\mathfrak{L}/GL(q, \boldsymbol{C})$ can be considered as the set of all q-dimensional subspaces of $\boldsymbol{C}^n$. Thus we put here $\boldsymbol{V}_{p,q} = \mathfrak{L}/GL(q, \boldsymbol{C})$ and denote by π the canonical projection of $\mathfrak{L}$ onto $\boldsymbol{V}_{p,q}$. We define further a subset $\mathfrak{L}'$ of $\mathfrak{L}$ by

$$\mathfrak{L}' = \left\{L \in \boldsymbol{M}_{n,q};\ L = \binom{u}{v},\ u \in \boldsymbol{M}_{p,q},\ v \in \boldsymbol{M}_{q,q},\ \det(v) \neq 0\right\},$$

then $\mathfrak{L}'$ is left invariant under the action of $GL(q, \boldsymbol{C})$ and the quotient $\mathfrak{L}'/GL(q, \boldsymbol{C})$ is naturally identified with $\boldsymbol{M}_{p,q}$; namely, for $L = \binom{u}{v} \in \mathfrak{L}'$, we put $\pi(L) = uv^{-1}$. On the other hand, the inclusion $\mathfrak{L}' \subset \mathfrak{L}$ induces the inclusion mapping: $\boldsymbol{M}_{p,q} \longrightarrow \boldsymbol{V}_{p,q}$, which is no other than j_2 in (7), as is easily seen. Hence we have the commutative diagram:

$$\begin{array}{ccc} \mathfrak{L}' & \subset & \mathfrak{L} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \boldsymbol{M}_{p,q} & \xrightarrow{j_2} & \boldsymbol{V}_{p,q} \end{array}$$

Now we let $\tilde{\rho}(g)$ act on $\mathfrak{L}$ from the left as a linear transformation; $\tilde{\rho}(g)\mathfrak{L}'$ is then not always contained in $\mathfrak{L}'$, but we infer that $\tilde{\rho}(g)\pi^{-1}(D) \subset \pi^{-1}(D)$ $(D \subset \boldsymbol{M}_{p,q})$ and that $\pi(\tilde{\rho}(g)L) = \tilde{\rho}(g) \cdot \pi(L)$ for $L = \binom{u}{v} \in \pi^{-1}(D)$. So, denoting $\pi(L) = uv^{-1} = Z$ $(= \rho(x)$ for some $x \in X)$, we have

$$\tilde{\rho}(g) \cdot Z = \pi(\tilde{\rho}(g)) \cdot \binom{u}{v} = \pi\binom{Au + Bv}{Cu + Dv}$$
$$= (Au + Bv)(Cu + Dv)^{-1} = (AZ + B)(CZ + D)^{-1}$$

Our theorem is thus proved.

Remark. Theorem 2 is described in [14] in the case where X is one of the classical type (I) – (III) and $\tilde{\rho}$ is the natural representation of the classical groups. We refer also to H. Klingen [8], [9] as for these facts. The proof presented above is just a rearrangement of H. Cartan [2] for the case of type (III). We note here that T. Nagano communicated to me that Theorem 2 had been obtained by T. Yokonuma independently.

2.6. We know since A. Korànyi and J.A. Wolf (Ann. of Math., 81 (1965), 265–288) that every bounded symmetric domain X has the *unbounded model;* namely it is realized as a Siegel domain of the second kind in $\mathfrak{n}^+$, which is a generalization of the so-called Siegel's generalized upper half-plane $\{Z \in \boldsymbol{M}_n;\ {}^tZ = Z,\ Im(Z) > 0\}$. Such an unbounded domain D^c is obtained from Harish-Chandra's domain D (see §1) through a transformation $c \in G_u$ which is called the *Cayley transform* of D: $D^c = j_2^{-1} c j_2(D)$. Therefore the conjugate group $cGc^{-1} = G^c$ acts on D^c as the automorphism group. Then, taking the maximal compact subgroup cKc^{-1} of G_c instead of K, we have an analogous decomposition of V as in (8): $V = V_1' \oplus \cdots \oplus V_s'$, $V_i' = \tilde{\rho}(c) \cdot V_i$ $(1 \leqslant i \leqslant s)$. Thus, writing $\tilde{\rho}(cgc^{-1}) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} (g \in G)$ with reference to this decomposition as in Theorem 2, we see immediately that the proof of Theorem 2 is valid also for this case, and that $\tilde{\rho}(cgc^{-1})$ acts on D^c as a linear fractional transformation. This fact is well-known for Siegel's generalized upper half-plane (see [14]).

§3. The canonical models of irreducible bounded symmetric domains of calssical type.

In this section we shall determine the canonical models of the domains of classical type. As we have clarified in the preceding section, the irreducible domains of type (I) – (III) and that of type (IV) are somewhat different to handle (see Theorem 1); so we shall devide the following arguments into two cases:

(1°) The domains of type (I) – (III). The Lie algebra $\mathfrak{g}_c$ is of classical type and we choose, as $\tilde{\rho}$, the *identity* representation that is of the lowest degree; so we identify $\tilde{\rho}(\mathfrak{g}_c)$ with $\mathfrak{g}_c$ itself, etc. Then we see $s=2$, $V = V_1 \oplus V_2$ as for the notation in §2; in fact, we have:

$$\mathfrak{g}_c = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \boldsymbol{M}_{p+q};\ A \in \boldsymbol{M}_p,\ B \in \boldsymbol{M}_{p,q},\ C \in \boldsymbol{M}_{q,p},\ D \in \boldsymbol{M}_q \text{ which satisfy the condition (11) given below} \right\},$$

$$\mathfrak{k}_c = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in \mathfrak{g}_c \right\},\quad \mathfrak{m}_c = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \in \mathfrak{g}_c \right\},$$

$$\mathfrak{n}^+ = \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \in \mathfrak{m}_c \right\},\quad \mathfrak{n}^- = \left\{ \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \in \mathfrak{m}_c \right\},$$

where $\mathfrak{n}^\pm$ are to be identified with the totality of B, or C, respectively.

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While the compact form $\mathfrak{g}_u$ is here presented by $\mathfrak{g}_u = \mathfrak{g}_c \cap \mathfrak{u}(p+q)$, so the complex conjugation τ with respect to $\mathfrak{g}_u$ is given by

$$\tau: \begin{pmatrix} A & B \\ C & D \end{pmatrix} \longrightarrow \begin{pmatrix} -{}^t\bar{A}, & -{}^t\bar{C} \\ -{}^t\bar{B}, & -{}^t\bar{D} \end{pmatrix}$$

The condition in the definition of $\mathfrak{g}_c$ are given as follows:

$$(11) \quad \begin{cases} \text{For the type } (\mathrm{I}_{p,q}), \ \mathrm{Trace}(A+D) = 0. \\ \text{For the type } (\mathrm{II}_n), \ p = q = n, \ D = -{}^tA, \ {}^tB = -B, \ {}^tC = -C. \\ \text{For the type } (\mathrm{III}_n), \ p = q = n, \ D = -{}^tA, \ {}^tB = B, \ {}^tC = C. \end{cases}$$

Now, for $Z = \begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix} \in \mathfrak{n}^+$, the hermitian operator $\theta^*(Z)\theta(Z)$ is given by

$$\theta^*(Z)\theta(Z): X \longrightarrow (Z^*Z)X + X(ZZ^*),$$

where $X = \begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix} \in \mathfrak{n}^-$ (see § 2) and $Z^* = -\tau(Z) = {}^t\bar{Z}$.

Now, for the type $(\mathrm{I}_{p,q})$, the transformations $X \longrightarrow (Z^*Z)X$ and $X \longrightarrow X(ZZ^*)$ commute with each other, and the eigen-values of the former one are p-copies of those of the hermitian matrix Z^*Z, and, in like manner, the eigen-values of the latter are q-copies of these of ZZ^*. On the other hand, both Z^*Z and ZZ^* have non-negative common eigen-values with their multiplicity. Hence we have the canonical model:

$$D_{p,q} = \rho(X) = \{Z \in \boldsymbol{M}_{p,q};\ Z^*Z < I_q \ (\text{or, } ZZ^* < I_p)\}.$$

For the type (II_n), $\tilde{\rho}(\mathfrak{n}^+) = \mathfrak{n}^+$ is identified with $\{Z \in \boldsymbol{M}_n;\ {}^tZ = -Z\}$ and for the type (III_n), $\tilde{\rho}(\mathfrak{n}^+)$ with $\{Z \in \boldsymbol{M}_n;\ {}^tZ = Z\}$; so in each case, the operator $\theta^*(Z)\theta(Z)$ is the natural prolongation of the hermitian operator ZZ^* in $\boldsymbol{C}^n$ to the respective matrix-space (= the tensor space of type $(1, 1)$ consisting of skew-symmetric ones with respect to the canonical non-degenerate inner-product, for the type (II_n); that of symmetric ones, for the type (III_n)). From this we infer that, for the type (II_n), the eigen-values of $\theta^*(Z)\theta(Z)$ consists of $\lambda_i + \lambda_j$ $(1 \leqslant i < j \leqslant n)$, and for the type (III_n), those consist of $\lambda_i + \lambda_j$ $(1 \leqslant i,\ j \leqslant n)$, where λ_i $(1 \leqslant i \leqslant n)$ denote the eigen-values of Z^*Z (or, of ZZ^*). While, in the former case (II_n), we see the following fact:

LEMMA 3. *For any skew-symmetric matrix Z of degree $n \geqslant 2$, every positive eigen-value of Z^*Z has the multiplicity not less than two.*

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Hence, for the both types (II_n) and (III_n), the canonical models of our domains have to be written as

$$D = \rho(X) = \{Z \in \tilde{\rho}(\mathfrak{n}^+);\ Z^*Z < I_n \ \text{(or, } ZZ^* < I_n)\},$$

under the identification of $\tilde{\rho}(\mathfrak{n}^+)$ indicated above.

(2°) The domain of type (IV). The Lie algebra $\mathfrak{g}_c$ is isomorphic to $\mathfrak{o}(n+2, \boldsymbol{C})$, and $\mathfrak{k}_c$ to $\mathfrak{o}(n, \boldsymbol{C}) \oplus \mathfrak{o}(2, \boldsymbol{C})$. We shall be confined here to the case $n = 2m$ $(m > 1)$ for brevity (the case $n = 2m+1$ will be treated in parallel); namely

$$\mathfrak{g}_c = \mathfrak{o}(2m+2, \boldsymbol{C}) = \left\{\begin{pmatrix} X, & Y \\ Z, & -{}^tX \end{pmatrix};\ X, Y, Z \in M_{m+1},\ {}^tY = -Y,\ {}^tZ = -Z\right\}.$$

Now we take the representation $\tilde{\rho}$ of $\mathfrak{g}_c$ as

$$\tilde{\rho}(A) = T^{-1}AT;\ T = \begin{pmatrix} I_{m+1}, & 0 \\ 0, & I'_{m+1} \end{pmatrix},\ I'_{m+1} = \begin{pmatrix} 0, & 1 \\ I_m, & 0 \end{pmatrix}.$$

Then, we can put

$$\tilde{\rho}(\mathfrak{k}_c) = \left\{\begin{pmatrix} a & & \\ & \begin{matrix} X & Y \\ Z & -{}^tX \end{matrix} & \\ & & -a \end{pmatrix};\ \begin{pmatrix} X & Y \\ Z & -{}^tX \end{pmatrix} \in \mathfrak{o}(n, \boldsymbol{C}),\ a \in \boldsymbol{C}\right\},$$

$$\tilde{\rho}(\mathfrak{n}^+) = \left\{\begin{pmatrix} 0 & \boldsymbol{z}', & \boldsymbol{z}'', & 0 \\ & & & -{}^t\boldsymbol{z}'' \\ & & & -{}^t\boldsymbol{z}' \\ & & & 0 \end{pmatrix};\ \boldsymbol{z}' = (z_1 \cdots, z_m),\quad \boldsymbol{z}'' = (z_{m+1}, \cdots, z_n) \in \boldsymbol{C}^m\right\},$$

$$\tilde{\rho}(\mathfrak{n}^-) = \left\{\begin{pmatrix} 0 & & & \\ \boldsymbol{y}' & & & \\ \boldsymbol{y}'' & & & \\ 0, & -{}^t\boldsymbol{y}'', & -{}^t\boldsymbol{y}', & 0 \end{pmatrix};\ \boldsymbol{y}' = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix},\ \boldsymbol{y}'' = \begin{pmatrix} y_{m+1} \\ \vdots \\ y_n \end{pmatrix} \in \boldsymbol{C}^m\right\},$$

where the compact form $\tilde{\rho}(\mathfrak{g}_u)$ is of the form $\tilde{\rho}(\mathfrak{g}_c) \cap \mathfrak{u}(n+2)$. Hence, for the corresponding complex conjugation τ, we see that, for

$$Z = \begin{pmatrix} 0, & \boldsymbol{z}', & \boldsymbol{z}'', & 0 \\ & & & -{}^t\boldsymbol{z}'' \\ & & & -{}^t\boldsymbol{z}' \\ & & & 0 \end{pmatrix} \in \tilde{\rho}(\mathfrak{n}^+);\ \boldsymbol{z}' = (z_1, \cdots, z_m),\ \boldsymbol{z}'' = (z_{m+1}, \cdots, z_n),$$

$$Z^* = -\tau(Z) = \begin{pmatrix} 0 & & & \\ {}^t\bar{\boldsymbol{z}}' & & & \\ {}^t\bar{\boldsymbol{z}}'' & & & \\ 0, & -\bar{\boldsymbol{z}}'', & -\bar{\boldsymbol{z}}', & 0 \end{pmatrix} \in \tilde{\rho}(\mathfrak{n}^-).$$

Namely, using the notation in § 2, we get

$$Z_1 = (\boldsymbol{z}', \boldsymbol{z}'') = \boldsymbol{z},\quad Z_1^* = \begin{pmatrix} {}^t\bar{\boldsymbol{z}}' \\ {}^t\bar{\boldsymbol{z}}'' \end{pmatrix} = {}^t\bar{\boldsymbol{z}},$$

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$$Z_2 = \begin{pmatrix} -{}^t z'' \\ -{}^t z' \end{pmatrix} = -J\,{}^t z, \quad Z_2^* = (-\bar{z}'', -\bar{z}') = -\bar{z}J \quad \text{for } J = \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix}.$$

It follows then that

$$Z_1 Z_1^* = \left(\textstyle\sum_{i=1}^n |z_i|^2\right) I_n = \|z\|^2 \cdot I_n,$$
$$Z_1^* Z_1 = {}^t\bar{z}z = (\bar{z}_i z_j) \in M_n,$$
$$Z_2 Z_2^* = J\,{}^t z\bar{z}J = J(z_i\bar{z}_j)J \in M_n.$$

Thus, the hermitian matrix to be considered is of the form:

$$Z_1 Z_1^* + Z_1^* Z_1 - Z_2 Z_2^* = \|z\|^2 \cdot I_n + H_z,$$

where $H_z = Z' - J\bar{Z}'J$, $Z' = Z_1^* Z_1$. We note that $Z' = (\bar{z}_i z_j)$ may have only one non-zero eigen-value $\alpha = \|z\|^2$ $(\geqslant 0)$; hence we infer that

$$\operatorname{rank} H_z \leqslant \operatorname{rank} Z' + \operatorname{rank} J\bar{Z}'J \leqslant 2.$$

Let λ be a non-zero eigen-value of H_z, then $H_z u = \lambda u$ for some vector $u \in C^n (u \neq 0)$. So, using $\bar{H}_z \bar{u} = \lambda \bar{u}$ and $\bar{H}_z = -JH_zJ$, we get

$$H_z J\bar{u} = (-\lambda)J\bar{u};\ J\bar{u} \neq 0.$$

This shows that $-\lambda$ is also an eigen-value of H_z and that the eigen-values of H_z consist of $\{\lambda\ (\geqslant 0), -\lambda \text{ and } 0, \cdots, 0\}$. Now we can compute the (possible) non-zero eigen-value λ of H_z. The eigen-values of H_z^2 consist of $\{\lambda^2, \lambda^2, 0 \cdots, 0\}$, so we have $\lambda^2 = \frac{1}{2} \operatorname{Trace} H_z^2$. On the other hand, from $H_z^2 = Z'^2 + J\bar{Z}'^2 J - Z'J\bar{Z}'J - J\bar{Z}'JZ'$ follows that $\operatorname{Trace} H_z^2 = 2\{\operatorname{Trace} Z'^2 - \operatorname{Trace} (Z'J\bar{Z}'J)\} = 2\lambda^2$. Thus we have to compute $\lambda^2 = \operatorname{Trace} Z'^2 - \operatorname{Trace} (Z'J)(\overline{Z'J})$; in fact, we have

$\operatorname{Trace} Z'^2 = \sum_{i,j=1}^n |z_i z_j|^2$, $\operatorname{Trace} (Z'J)(\overline{Z'J}) = 4\sum_{i,j=1}^m (\bar{z}_i z_j \bar{z}_{i+m} z_{j+m})$. Namely, the canonical model of our domain is the set of all $z \in C^n$ satisfying the inequality $\alpha + \lambda < 2$. However, we transform the coordinates $(z_1, \cdots, z_m, \cdots, z_n)$ of z by

$$(z_1, \cdots, z_n) = (z_1', \cdots, z_n') \begin{pmatrix} I_m, & I_m \\ \sqrt{-1}\, I_m, & -\sqrt{-1}\, I_m \end{pmatrix}.$$

As for the new coordinates $(z_1', \cdots, z_n')\ (= z')$, we see

$$\|z\|^2 = 2\|z'\|^2, \quad \textstyle\sum_{i,j=1}^n |z_i z_j|^2 = 4\sum_{i,j=1}^n |z_i' z_j'|^2$$
$$\textstyle\sum_{i,j=1}^m (\bar{z}_i z_j \bar{z}_{i+m} z_{j+m}) = \sum_{i,j=1}^n (\bar{z}_i' z_j')^2.$$

Hence we get $\lambda^2 = 4\sum_{i,j=1}^{n}\{|z_i'z_j'|^2 - (\bar{z}_i'z_j')^2\} = -4\sum_{i<j}\{\bar{z}_i^{2\prime}z_j^{2\prime} + z_i^{2\prime}\bar{z}_j^{2\prime} - 2z_i'\bar{z}_i'z_j'\bar{z}_j'\} = -4\sum_{i<j}(z_i'\bar{z}_j' - \bar{z}_i'z_j')^2 = 16\sum_{i<j}\{\mathrm{Im}\,(z_i'\bar{z}_j')\}^2$; namely $\lambda = 4[\sum_{i<j}\{\mathrm{Im}\,(z_i'\bar{z}_j')\}^2]^{\frac{1}{2}}$. Thus the inequality in Theorem 1, ii) is

$$\alpha + \lambda = 2\|\boldsymbol{z}'\|^2 + 4[\textstyle\sum_{i<j}\{\mathrm{Im}\,(z_i'\bar{z}_j')\}^2]^{\frac{1}{2}} < 2.$$

Thus the canonical model of our domain is the set of all $z \in \boldsymbol{C}^n$ satisfying this inequality; therefore it is equivalent to

$$D = \{z \in \boldsymbol{C}^n;\ \|\boldsymbol{z}\|^2 + 2[\textstyle\sum_{i<j}\{\mathrm{Im}\,(z_i\bar{z}_j)\}^2]^{\frac{1}{2}} < 1\}.$$

This realization of the domain of type (IV_n) coincides with the usual one which has been known since E. Cartan [1], because of the following easily-checked lemma:

LEMMA 4. *For* $z \in \boldsymbol{M}_{n,1} = \boldsymbol{C}^n$, *the condition*

$${}^t\bar{\boldsymbol{z}}\boldsymbol{z} < \frac{1}{2}(1 + |{}^t\boldsymbol{z}\boldsymbol{z}|^2) < 1$$

is equivalent to the following single inequality:

$${}^t\bar{\boldsymbol{z}}\boldsymbol{z} + 2[\textstyle\sum_{i<j}\{\mathrm{Im}\,(z_i\bar{z}_j)\}^2]^{\frac{1}{2}} < 1.$$

Proof. This lemma is immediately derived from the relation

$$|{}^t\boldsymbol{z}\boldsymbol{z}|^2 = ({}^t\bar{\boldsymbol{z}}\boldsymbol{z})^2 + \textstyle\sum_{i<j}(z_i\bar{z}_j - \bar{z}_iz_j)^2;$$

hence we leave it to the reader.

Appendix

1. In this Appendix, we shall sketch a generalization of our arguments in §§1–3 to a class of *real* symmetric spaces——the so-called *symmetric R-spaces* (see [16]). Materials are mostly provided in [16], so we will recall here some notions stated in [16]; Chap. III, §1 (see also [13]). We denote by $X = G/K$ and by $X_u = G_u/K$, respectively, the non-compact form and the compact form of such a space. Typical examples are the irreducible symmetric spaces of type $(BDI)_{p,q}$ in the classification table of E. Cartan (see [3]); namely $X = SO_0(p, q, \boldsymbol{R})/SO(q, \boldsymbol{R}) \times SO(p, \boldsymbol{R})$ and $X_u = O(p+q, \boldsymbol{R})/O(p, \boldsymbol{R}) \times O(q, \boldsymbol{R})$. T. Nagano, H. Matsumoto and M. Takeuchi have proved, analogously to the case of hermitian symmetric spaces, that there exist likewise the canonical embedding relations for any symmetric R-spaces X and X_u;

$$X \xrightarrow{j_1} \mathfrak{n}^+ \xrightarrow{j_2} X_u. \tag{12}$$

To be more pricise, X_u can be written as $X_u = G'/U'$ for a real semi-simple (or, reductive) Lie group G' and its parabolic subgroup U', and furthermore if we take a maximal compact subgroup G_u of G', then $G' = G_u U'$ and $X_u = G_u/K$, $K = G_u \cap U'$. While, we have a subgroup G of G' which is isomorphic to a real form of the complexification of G_u and contains K as a maximal compact subgroup, for which we get the non-compact symmetric space $X = G/K$ that is dual to $X_u = G_u/K$. Under these situations, we can show the following relations: First, the Lie algebra $\mathfrak{g}'$ is decomposed into the eigen-spaces of $\operatorname{ad} Z$ (where Z denotes some element in a Cartan subalgebra; see [16]); namely

$$\mathfrak{g}' = \mathfrak{n}^+ \oplus \mathfrak{k}' \oplus \mathfrak{n}^-,$$

where $\mathfrak{n}^\pm$ denote the sum of eigen-spaces corresponding to positive (resp. negative) eigen-values, and $\mathfrak{k}'$ that corresponding to zero eigen-value. Then $\mathfrak{k}' \oplus \mathfrak{n}^- = \mathfrak{u}'$ may be considered as the Lie algebra of U', while $\mathfrak{n}^\pm$ generate vector groups $N^\pm$ and $\mathfrak{k}'$ the reductive subgroup K_0' of G'. Further, there exist (not nec. connected) subgroup K' of G' such that $U' = K'N^-$ (semi-direct product) and its connected component of the identify is K_0'. For these Lie subgroups of G', the following relations hold:

$$U' \cap G = K' \cap G = G \cap G_u = K, \quad G \subset N^+U' \ (N^+ \cap U' = \{1\}).$$

From the last inclusion relation we have $GU' \subset N^+U' \subset G'$; thus it yields the following:

$$G/K \subset N^+ \subset G'/U',$$

where we can identify N^+ with $\mathfrak{n}^+$ through the exponential map; thus getting the relation (12) analogously to (5).

2. We will now illustrate the subgroups and subalgebras introduced above in the case of the spaces of type $(BDI)_{p,q}$: In this case we put $n = p + q \ (p \geqslant q \geqslant 1, \ p + q > 4)$, then

$$G' = GL(n, \boldsymbol{R}), \quad G_u = O(n, \boldsymbol{R}),$$

$$G = \{g \in G';\ {}^t g I_{p,q} g = I_{p,q}\};\ I_{p,q} = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix},$$

$$U' = \left\{\begin{pmatrix} A & 0 \\ C & D\end{pmatrix};\ A\in GL(p,\boldsymbol{R}),\ \ C\in \boldsymbol{M}_{q,p}(\boldsymbol{R}),\ \ D\in GL(p,\boldsymbol{R})\right\},$$

$$K' = \left\{\begin{pmatrix} A & 0 \\ 0 & D\end{pmatrix}\in U'\right\} \cong GL(p,\boldsymbol{R}) \times GL(q,\boldsymbol{R}),$$

$$K = \left\{\begin{pmatrix} A & 0 \\ 0 & D\end{pmatrix}\in K';\ A\in O(p,\boldsymbol{R}),\ \ D\in O(q,\boldsymbol{R})\right\} \cong O(p,\boldsymbol{R})\times O(q,\boldsymbol{R}),$$

$$N^+ = \left\{\begin{pmatrix} I_p & B \\ 0 & I_q\end{pmatrix};\ B\in \boldsymbol{M}_{p,q}(\boldsymbol{R})\right\},\ \ N^- = \left\{\begin{pmatrix} I_p & 0 \\ C & I_q\end{pmatrix};\ C\in \boldsymbol{M}_{q,p}(\boldsymbol{R})\right\}.$$

Therefore, as for the corresponding Lie algebras we have, for instance, as below:

$$\mathfrak{g} = \left\{\begin{pmatrix} A & B \\ {}^tB & D\end{pmatrix};\ A\in\mathfrak{o}(p,\boldsymbol{R}),\ \ D\in\mathfrak{o}(q,\boldsymbol{R}),\ \ B\in \boldsymbol{M}_{p,q}(\boldsymbol{R})\right\},$$

$$\mathfrak{k} = \left\{\begin{pmatrix} A & 0 \\ 0 & D\end{pmatrix}\in\mathfrak{g}\right\}\in\mathfrak{o}(p,\boldsymbol{R})\oplus\mathfrak{o}(q,\boldsymbol{R}),$$

$$\mathfrak{n}^+ = \left\{\begin{pmatrix} 0 & B \\ 0 & 0\end{pmatrix};\ B\in \boldsymbol{M}_{p,q}(\boldsymbol{R})\right\},\ \ \mathfrak{n}^- = \left\{\begin{pmatrix} 0 & 0 \\ C & 0\end{pmatrix};\ C\in \boldsymbol{M}_{q,p}(\boldsymbol{R})\right\}.$$

Further, put $\mathfrak{m} = \left\{\begin{pmatrix} 0 & B \\ {}^tB & 0\end{pmatrix}\in\mathfrak{g}\right\} \cong \boldsymbol{M}_{p,q}(\boldsymbol{R})$, then $(\mathfrak{k},\mathfrak{m})$ provides the symmetric pair correspondiig to the symmetric space $X = G/K$.

From these materials, the inclusion relation (12) now yields the following special one:

$$X \xrightarrow{\ j_1\ } \boldsymbol{M}_{p,q}(\boldsymbol{R}) \xrightarrow{\ j_2\ } X_u = \boldsymbol{V}_{p,q}(\boldsymbol{R}), \tag{13}$$

where we have identified $\mathfrak{n}^+$ with $\boldsymbol{M}_{p,q}(\boldsymbol{R})$ as in § 2 and denoted by $\boldsymbol{V}_{p,q}(\boldsymbol{R})$ the real Grassmann manifold. Then we can show, as in § 3, that $j_1(X) = D$ is realizable as a real bounded domain:

$$D_{p,q} = \{Z\in \boldsymbol{M}_{p,q}(\boldsymbol{R});\ {}^tZZ < I_q\};$$

in case $q = 1$, $X \cong D_{p,1}$ is the real hyperbolic space and $X_u = \boldsymbol{V}_{p,1}(\boldsymbol{R})$ the real projective space, of p-dimensions, respectively.

3. Now let ρ be an irreducible representation of G' into $GL(n,\boldsymbol{R})$ $(n = p+q)$ such that ρ sends U' into $GL(n\colon p,q,\boldsymbol{R})$. It then induces a real analytic mapping ρ of $X_u = G'/U'$ into $\boldsymbol{V}_{p,q}(\boldsymbol{R}) = GL(n,\boldsymbol{R})/GL(n\colon p,q,\boldsymbol{R})$, which gives rise to the following commutative diagram in the quite same manner as in (7):

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$$
(14)\qquad \begin{array}{ccccc} X & \longrightarrow & \mathfrak{n}^+ & \longrightarrow & X_u \\ & & \downarrow \rho & & \downarrow \rho \\ D_{p,q} \subset & & \boldsymbol{M}_{p,q}(\boldsymbol{R}) & \longrightarrow & \boldsymbol{V}_{p,q}(\boldsymbol{R}). \end{array}
$$

All these procedure are carried out by using the complexification of (14), which is no other than a diagram of type (7), as is readily seen from [16], p. 181 (we leave the detail to the reader). In particular, by taking the complexified representation of ρ, we can show that our mapping ρ's are injective; X is therefore mapped injectively into $\boldsymbol{M}_{p,q}(\boldsymbol{R})$. Here we note that the *real analogue* of Lemma 2 in § 2 will be also valid (cf. Foot note 3)), and so we infer that the image $\rho(X) = D$ is a real bounded domain in $\boldsymbol{M}_{p,r}(\boldsymbol{R})$ as is known from [16], Theorem 5, p. 182, or by using the complexification and the arguments in § 2. In the bounded model D of X thus obtained, every element of $\rho(G) \subset GL(p+q, \boldsymbol{R})$ also acts on D as a linear fractional transformation:

$$Z \longrightarrow (AZ+B)(CZ+D)^{-1}.$$

The proof of this fact is done in the same way as that of Theorem 2, or by using the complexification of (14) and Theorem 2. A simple example of this result is exhibited in Takahashi [15], p. 372, where X is of type $(BDI)_{p,q}$ with $p=4$, $q=1$, by taking as the ambient space $\boldsymbol{M}_{4,1}(\boldsymbol{R})$ the real quoternion algebra $\boldsymbol{Q}$; indeed D is there given by $D = \{u \in \boldsymbol{Q};\ \|u\| < 1\}$, $\|u\|$ denoting the norm in the sense of quoternions.

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