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SINGULAR ISOMETRIES IN ORTHOGONAL GROUPS

BY
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In this paper, we study the behaviour of singular isometries in orthogonal groups. These are isometries whose path is a singular subspace. We shall prove that the path of such a singular isometry is always even-dimensional. We shall use this result to show that the subgroup of the orthogonal group $O_n(K, Q)$ which is generated by the singular isometries is the commutator subgroup $\Omega_n(K, Q)$. In particular, in the case that $n=4$ and index $Q=2$, we shall obtain a nice geometric interpretation for the well-known result that $P\Omega_n(K, Q)$ is isomorphic to $PSL_2(K)\times PSL_2(K)$. In addition, we shall discuss some subgroups of the commutator group.

1. **Introduction.** Let (V, Q) be an n-dimensional metric vector space over a field K, with a quadratic form Q. Let f be the bilinear form associated with Q. The form f induces an orthogonality on V. We say that a subspace H is *regular* if $\operatorname{Rad} H=H\cap H^\perp=0$ and H is *isotropic* if $\operatorname{Rad} H=H$. A subspace H is *singular* if $Q(h)=0$ for all $h\in H$. The *index* of Q is the dimension of the maximal singular subspaces of V.

In this paper, we shall assume that V is regular, that $K\neq GF(2)$, and that the index of Q is at least 2. We use the notation of Ellers ([5], [6] and [7]) throughout this text.

A transformation π of V is called an *isometry* if it preserves Q (and hence preserves f). The set of all isometries is the orthogonal group $O_n(K, Q)$. With every isometry we can associate two subspaces:

$$\text{the } \textit{path} \text{ of } \pi:=B(\pi)=\{\pi(x)-x \mid x\in V\}$$

and

$$\text{the } \textit{fix} \text{ of } \pi:=F(\pi)=\{x\in V \mid \pi(x)=x\}.$$

Clearly, $\dim B(\pi)+\dim F(\pi)=n$, and $B(\pi)^\perp=F(\pi)$. An isometry σ is *simple* if $\dim B(\sigma)=1$; these simple isometries generate the group $O_n(K, Q)$. We say that an isometry π is *singular* if $B(\pi)$ is a singular subspace; otherwise, we call π non-singular.

We let $l(\pi)$ (the *length* of π) be the minimum number of simple isometries whose product yields π. If $l(\pi)=2$, we call π a *rotation*. We shall use projective

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language, and say "point", respectively "line" for a one, respectively two-dimensional subspace. A line can then contain 0, 1, or 2 singular points, or consist entirely of singular points. We call such lines respectively *elliptic*, *parabolic*, *hyperbolic*, or singular, and say that a rotation is *elliptic*, *parabolic*, *hyperbolic*, or singular as its path is a line of the corresponding type. In the following theorem, we gather some well-known results.

THEOREM 1.1

(a) *Let π be an isometry, and σ be a simple isometry. Then $B(\pi\sigma)=B(\pi)+B(\sigma)$ if and only if $B(\sigma)\not\subset B(\pi)$, and* $\dim B(\pi\sigma)=\dim B(\pi)-1$ *if and only if $B(\sigma)\subset B(\pi)$.*

(b) *If π is non-singular, then $l(\pi)=\dim B(\pi)$.*

(c) *If π is singular, then $l(\pi)=\dim B(\pi)+2$.*

(d) *If σ is simple and $B(\sigma)=Kp$, then*

$$\sigma(x)=x-\frac{f(x,p)}{Q(p)}p.$$

(e) *If ρ is a parabolic rotation with $B(\rho)=\langle s,z\rangle$ with $Q(s)=f(s,z)=0$ and $Q(z)=\lambda\neq 0$, then*

$$\rho(x)=x+\frac{\delta}{\lambda}[f(x,s)z-f(x,z+\delta s)s]\quad \text{for some } \delta\in K.$$

(f) *If ρ is a hyperbolic rotation, $B(\rho)=\langle s,t\rangle$ with $Q(s)=Q(t)=0$ and $f(s,t)=1$, then $\rho(s)=\alpha s$, $\rho(t)=\alpha^{-1}t$, for some $\alpha\in K$, and ρ is uniquely determined by its effect on s.*

The proof of these results will be omitted. See [4] for details; also [8] for a proof of (b) and (c).

If H is any subspace, we let $O(H)=\{\pi\in O_n(K,Q)\mid B(\pi)\subset H\}$, and $O^+(H)=\{\pi\in O(H)\mid l(\pi)$ is even$\}$. In particular, if H is a line, $O^+(H)$ is the group of all rotations whose path is the line H.

As in Ellers [5], we also define, for any isometry π, the set $P(\pi)$ by

$$P(\pi):=\{x\in V\mid f(\pi(x)-x,x)=0\}.$$

The condition "$x\in P(\pi)$" is clearly equivalent to "$Q(\pi(x)-x)=0$", so that the size of $P(\pi)$ gives a measure of the occurrence of singular vectors in $B(\pi)$. Clearly, the isometry π is singular if and only if $P(\pi)=V$.

2. **Representation of singular isometries.** In 1.1d, we see that a simple isometry can never be singular. This raises the question: for what singular subspaces H does there exist a singular isometry whose path is H? Before answering this, we prove

LEMMA 2.1. *Let A be a k-dimensional non-singular subspace containing a* *singular k-1-dimensional subspace B. The one of two cases occurs:*

(a) $B \subset \mathrm{rad}\, A$, *and all singular points of A lie in B.*

(b) $B \not\subset \mathrm{rad}\, A$, *and A contains a second singular k-1-dimensional subspace $C \neq B$.*

Every singular point of A lies in either A or B. If σ is any simple isometry such that $B(\sigma) \subset A$, then $\sigma(B) = C$.

Proof

(a) This is immediate, as otherwise A would be singular.

(b) Now $B \not\subset \mathrm{rad}\, A$. Let a be a non-singular vector of A, and b a vector of B such that $f(a, b) \neq 0$. Then $\langle a, b\rangle$ is a hyperbolic line, and hence contains a second singular vector $c \notin B$. Then $Kc \oplus (c^{\perp} \cap B) = C$ is a second singular k-1-dimensional subspace of A. Now suppose that $x = b + \alpha c$ with $b \in B$ is any vector of A. Then $Q(b + \alpha c) = \alpha f(b, c) = 0$ if and only if $\alpha = 0$ and $x \in B$, or $f(b, c) = 0$ and $x \in C$. Finally, let $B(\sigma) = Ka \subset A$. Then there exists a hyperbolic line $\langle b', c'\rangle$ with $b' \in B$ and $c' \in C$ containing a such that $\sigma(\langle b'\rangle) = \langle c'\rangle$. Hence $\sigma(B) = C$.

This enables us to prove

LEMMA 2.2. *Let π be singular and σ be simple. Then $\pi\sigma$ is non-singular.*

Proof. Since π is singular, we know that $P(\pi) = V$, and hence $f(\pi\sigma(x), \sigma(x)) = f(x, x)$ for all $x \in V$. If $\pi\sigma$ were also singular, we would also have $f(\pi\sigma(x), x) = f(x, x)$ for all $x \in V$, implying that $f(\pi\sigma(x), B(\sigma)) = 0$ for all $x \in V$. But then $B(\sigma) \subset \mathrm{rad}\, V$, contrary to the assumption that V is regular.

We are now able to answer the question posed at the top in

THEOREM 2.3. *Let π be a singular isometry. Then* $\dim B(\pi)$ *is even.*

Proof. Suppose $l(\pi) = k + 2$. Let Kp be a non-singular point such that $p \notin B(\pi)^{\perp}$, and let $B(\sigma) = Kp$. By 2.2, we can write $\pi = \rho\sigma$, where ρ is a non-singular isometry of length $k + 1$. Clearly, $B(\pi)$, $B(\sigma) \subset B(\rho) = A$, and by our choice of p, we ensure that $B(\pi) \not\subset \mathrm{rad}\, A$. Thus Theorem 2.1 pertains. Now $\pi(B(\pi)) = B(\pi)$. But if $l(\pi)$ were odd, then $\pi(B(\pi))$ would be the image of $B(\pi)$ under an odd number of simple isometries, which is distinct from $B(\pi)$ by 2.1. Hence $l(\pi) = k + 2$ is even, and thus $\dim B(\pi) = k$ is even.

We observe that if π is a singular isometry, then $B(\pi)$ is singular, and hence isotropic, implying that $B(\pi) \subset F(\pi)$. If char $K = p$, then let π be any singular isometry. Then

$$\begin{aligned}\pi^p(x) - x &= \pi^p(x) - \pi^{p-1}(x) + \pi^{p-1}(x) - \cdots + \pi(x) - x \\ &= \sum_{i=1}^{p} \pi^{i-1}(\pi(x) - x) = \sum_{i=1}^{p} (\pi(x) - x) = 0.\end{aligned}$$

Thus we have proved

LEMMA 2.4. *If* Char $K=p$, *and* π *is singular, then* $\pi^p=1$.

We now state the representation theorem for singular isometries.

THEOREM 2.5. *Let* π *be an isometry. Then* π *is singular if and only if*

$$\pi(x)=x+\sum_{i,j=1}^{k}\alpha_{ij}f(x,s_j)s_i$$

where the matrix (α_{ij}) *is skew-symmetric with zeros down the main diagonal and* $S=\langle s_1,\ldots,s_k\rangle$ *is a k-dimensional singular subspace.*

The proof of this theorem involves straightforward calculation, and thus is omitted. Now, if H is any singular subspace, we let $O(H)$ be the group generated by all singular isometries whose path lies in H. As corollaries of 2.5 we have

COROLLARY 2.6. *If* H *is a singular subspace of even dimension, then there exists a singular isometry whose path is* H.

COROLLARY 2.7. *If* H *is a singular line, then* $O(H)$ *is isomorphic to* $(K,+)$. *If* $\pi\in O(H)$, *then* $\pi(x)=x+\alpha[f(x,s)t-f(x,t)s]$, *where* $\langle s,t\rangle=H$.

The singular rotations are important because they generate the singular isometries. This is the content of the next

THEOREM 2.8. *Let* π *be a singular isometry. Let* $\dim B(\pi)=2k$. *Then* π *is the product of k singular rotations.*

Proof. Choose $s\in B(\pi)$. Then there exists some $a\in V$ such that $\pi(a)=a+s$. But then $Q(\pi(a))=Q(a)=Q(a+s)=Q(a)+f(a,s)$, implying that $f(a,s)=0$. Clearly $B(\pi)\not\subset a^{\perp}$, and so we can find some $t\in B(\pi)$ for which $f(a,t)=1$. Let τ be the singular rotation given by $\tau(x)=x+f(x,s)t-f(x,t)s$. Then $a\in F(\tau\pi)$. Since $F(\pi)\subset F(\tau\pi)$, we thus see that $F(\pi)$ is properly contained in $F(\tau\pi)$. Since $\tau\pi$ is singular, we thus conclude that $\dim B(\tau\pi)=\dim B(\pi)-2$. The result now follows by induction.

In the next lemma, we study products of two singular or two parabolic rotations.

LEMMA 2.9

(a) *Let* τ_1, τ_2 *be two singular rotations such that* $B(\tau_1)=\langle s,t\rangle$ *and* $B(\tau_2)=\langle s,r\rangle$. *If* $f(t,r)=0$, *then the product* $\tau_1\tau_2$ *is a singular rotation. If* $f(t,r)\neq 0$, *the product is a parabolic rotation. In either case, we have* $\tau_1\tau_2=\tau_2\tau_1$.

(b) Let ρ *be a parabolic rotation with* $B(\rho)=\langle s,z\rangle$, *such that* $Q(s)=0=f(s,z)$, *and* $Q(z)\neq 0$. *Then* ρ *is the product of two singular rotations* τ_1, τ_2 *such that* $B(\tau_1)\cap B(\tau_2)=Ks$.

(c) *Let* ρ_1, ρ_2 *be two parabolic rotations with* $B(\rho_1)\cap B(\rho_2)=Ks$, *and* $Q(s)=0$. *Then* $\rho_1\rho_2=\rho_2\rho_1$, *and the product is either singular or parabolic.*

(d) *Let τ be a singular rotation. Then $\tau=\rho_1\rho_2$, where the ρ_i are parabolic rotations with $B(\rho_1)\cap B(\rho_2)=Ks\subset B(\tau)$.*

Proof

(a) If $\tau_1\neq\tau_2$, then by 1.1b, c, we have $1(\tau_1\tau_2)=2$. If $f(t,r)=0$, then $B(\tau_1\tau_2)\subset\langle s,t,r\rangle$, which is a singular subspace, implying the result. We can assume that $B(\tau_1)\neq B(\tau_2)$, as otherwise 2.7 yields the result. But then $F(\tau_1\tau_2)=\{x\in V\mid\tau_1\tau_2(x)=x\}\subset\{x\mid\tau_2(x)-x\in Ks\}$. But $\tau_2(x)=x+\alpha s$ implies that $f(x,s)=0$. Hence $F(\tau_1\tau_2)\subset s^\perp$, or $s\in B(\tau_1\tau_2)$. Hence $\tau_1\tau_2$ is a rotation fixing s. If $\tau_1\tau_2\neq$ id., then in view of 1.1f, we can conclude that $\tau_1\tau_2$ is parabolic. A simple computation shows that $\tau_1\tau_2=\tau_2\tau_1$.

(b) In view of 1.1e, we can write $\rho(x)=x+\delta/\lambda[f(x,s)z-f(x,z+\delta s)s]$, where $\lambda=Q(z)$. Choose a singular vector $t\in s^\perp$ such that $f(t,z)=1$, and let τ be the singular isometry $\tau(x)=x+\delta[f(x,t)s-f(x,s)t]$. Then a simple computation shows that the product $\tau\rho$ is again a singular rotation τ' with path $B(\tau')=\langle s,\delta z-\delta\lambda t\rangle$.

(c) As in (a), we see that the product $\rho_1\rho_2$ is again a rotation whose path contains Ks, and which fixes the vector s. Hence, either $\rho_1\rho_2=$ id, or, by 1.1f, $\rho_1\rho_2$ is not hyperbolic. Again, a simple computation shows that $\rho_1\rho_2=\rho_2\rho_1$.

(d) As in (b), this again follows constructively.

3. **The group of a singular point.** We can now use Lemma 2.9 to define a certain family of subgroups of the group $O_n(K,Q)$. We let Ks be any singular point, and we define the set

$I(s)=\bigcup O^+(H)$, *where this union is taken over all lines H that contain the point* Ks. We also define a subset of $I(s)$ by $R(s)=\bigcup O^+(H)$, where the line H contains the point Ks and is itself contained in the hyperplane $s^\perp$. We prove the following

THEOREM 3.1. *The set $I(s)$ is a group. $R(s)$ is a normal abelian subgroup of $I(s)$, and the factor group $I(s)/R(s)$ is isomorphic to the multiplicative group of the field K.*

Proof. Let $\tau\neq\text{id}\neq\rho$ be elements of $I(s)$. Since $s\in B(\tau)\cap B(\rho)$, we see that $\dim B(\tau\rho)\leq 3$. Also, $\dim B(\tau\rho)$ is even, and hence the product $\tau\rho$ is a rotation or $\tau\rho=$ id. If $B(\tau)=B(\rho)$, there is nothing to prove. So assume that $B(\tau)\cap B(\rho)=Ks$. Then $F(\tau\rho)=\{x\in V\mid\tau\rho(x)=x\}\subset\{x\in V\mid\rho(x)-x\in Ks\}$. But $\rho(x)=x+\alpha s$ implies that $f(x,s)=0$, and thus $F(\tau\rho)\subset s^\perp$, or equivalently, $s\in B(\tau\rho)$. Thus $I(s)$ is a subgroup. The same reasoning shows that $R(s)$ is a subgroup which is clearly normal in $I(s)$. The elements of $R(s)$ are either singular or parabolic rotations, and so we see from 2.9 that the elements of $R(s)$ all commute. We now fix a hyperbolic line through s, and let τ be any hyperbolic rotation whose path is a second hyperbolic line. Then $\tau(s)=\alpha(s)$ for some

$\alpha \in K$. We let τ' be the hyperbolic rotation whose path is the given hyperbolic line H such that $\tau'(s) = \alpha^{-1}s$. Then $\tau\tau'(s) = s$ and hence $\tau\tau'$ is a parabolic or singular rotation. Also, the rotation τ uniquely determines the rotation τ'. We have thus shown that every rotation τ can be expressed as a product of a rotation in $O^+(H)$ and a rotation in $R(s)$, and that this decomposition is unique. We define a map $\psi: I(s) \to O^+(H)$, where $\psi(\tau)$ is the uniquely determined element of $O^+(H)$ in the decomposition described above. Clearly, $\psi(\mathrm{id}) = \mathrm{id}$, and $\tau\tau' = \psi(\tau)\rho \cdot \psi(\tau')\rho'$, where $\rho, \rho' \in R(s) = \psi(\tau)\psi(\tau')[\psi(\tau')^{-1}\rho\psi(\tau')\rho']$, so that $\psi(\tau\tau') = \psi(\tau)\psi(\tau')$, since $\psi(\tau')^{-1}\rho\psi(\tau')\rho' \in R(s)$. Hence the map ψ is a homomorphism whose kernel is clearly $R(s)$. Hence we have $I(s)/R(s)$ is isomorphic to $O^+(H)$, where H is a hyperbolic line, and from Dieudonné [4] we know that $O^+(H)$ is isomorphic to the multiplicative group of the field.

If $\dim V = 3$ and index $Q = 1$, we have a nice geometric interpretation of the group $I(s)$, which is due to Dr. H. Mäurer at the Technische Hochschule Darmstadt. For now the quadric is a Möbius plane, if we define the cycles as the plane sections of V with the quadric. Then the derived plane in the point Ks is an affine plane. If H is a hyperbolic line through Ks which meets the quadric in a second point Kt, then all the hyperbolic rotations with path H induce central dilatations with center Kt in the derived affine plane. If H is a parabolic line, on the other hand, then the corresponding parabolic rotations induce all the translations in a given direction. Thus $R(s)$ is isomorphic to the translation group of this affine plane, and this group is, as we know, an abelian group.

4. **A class of subgroups of O_n.**

In this section we describe another class of subgroups of O_n. Before doing this, we require a general

LEMMA 4.1. *Let A be an n-2-dimensional subspace of an n-dimensional vector space. Define $T(A) := \langle\{\tau \in SL_n(K) \mid \tau$ is a transvection and $A \subset F(\tau)\}\rangle$, and $BT(A) := \langle\{\pi \in T(A) \mid B(\pi) \subset A\}\rangle$. Then $T(A)$, $BT(A)$ are groups; $BT(A)$ is a normal subgroup of $T(A)$, and the factor group $T(A)/BT(A)$ is isomorphic to the group $SL_2(K)$.*

Proof. Clearly, $T(A)$ and $BT(A)$ are subgroups, and $BT(A)$ is normal in $T(A)$. We now define a map δ as follows

$$\delta: \begin{cases} T(A) \to GL_2(K) \\ \quad \pi \to \delta(\pi) \end{cases}$$

where

$$\delta(\pi): \begin{cases} V/A \to V/A \\ x + A \to \pi(x) + A \end{cases}$$

This is a well-defined homomorphism, since $A \subset F(\pi)$ for all $\pi \in T(A)$. Now

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$\ker\delta=\{\pi \mid \pi(x)+A=x+A$ for all $x\in V\}=\{\pi \mid B(\pi)\subset A\}$. Hence $\ker\delta = BT(A)$. Now choose any transvection $\tau\in T(A)$. We know that $\tau(x)=x+\psi(x)b$, where $A\subset\ker\psi$. Then $\delta(\tau)(x+A)=x+\psi(x)b+A=x+A+\psi(x+A)(b+A)$, and $\delta(\tau)(b+A)=b+A$. Thus we see that $\delta(\tau)$ is a transvection of V/A, and hence δ is a homomorphism into $SL_2(K)$. Now let T be any transvection of V/A, say $T(x+A)=(x+A)+\phi(x+A)(c+A)$ with $c\notin A$. Then $A+Kc$ is a hyperplane of V, and thus the map $\tau'(x)=x+\phi'(x)c$ is a transvection in $T(A)$, where $H=\ker\phi'$ and $\phi'(y)=1=\phi(y+A)$ for a suitable $y\notin A+Kc$. Then we see at once that $\delta(\tau')=T$. Hence δ is surjective. Thus the theorem is proved.

We now return to the study of the orthogonal group, and use Theorem 4.1 to prove

THEOREM 4.2. *Let H be a fixed singular line. Let $L(H):=\{H' \mid \dim H'=2$ and $H\cap H'\neq 0$, and H' singular$\}$. Define $G(H):=\langle\{\pi\in O_n \mid B(\pi)\in L(H)\}\rangle$. Then $O(H)$ is a normal subgroup of $G(H)$, and the factor group $G(H)/O(H)$ is isomorphic to the group $T(H^\perp/H)$, which is defined as in the statement of Theorem* 4.1.

Proof. We first observe that if τ is any singular rotation whose path is a line of $L(H)$, then τ fixes H. This is a direct consequence of 2.1b. As in the proof of Theorem 4.1, we define a map ρ as follows

$$\rho:\begin{cases} G(H)\to GL_{n-2}(V/H)\\ \pi\to\rho(\pi)\end{cases}$$

where $\rho(\pi)$ is defined by

$$\rho(\pi):\begin{cases} V/H\to V/H\\ x+H\to \pi(x)+H\end{cases}$$

Now we argue as in 4.1. We see that ρ is a homomorphism with kernel $O(H)$. If τ is any singular rotation with path *a* line in $L(H)$, then as in 4.1, we see that $\rho(\tau)$ is a transvection of V/H. Indeed, if $H\cap B(\tau)=Ks$, then $\rho(\tau)$ is a transvection whose path is $K(t+H)$, where t is a second vector of $B(\tau)$, and whose fix is the hyperplane $s^\perp/H$ of V/H. But $s\in H$ implies that $H^\perp\subset s^\perp$, and thus $\rho(\tau)$ is a transvection of $T(H^\perp/H)$. Thus ρ is a homomorphism into $T(H^\perp/H)$. Now consider any $s\in H$, and any non-singular $z\in s^\perp$. Then $\langle s, z\rangle$ is a parabolic line, and let σ be a parabolic rotation with this line as path. Then by 2.9b, we know that σ can be written as the product of two singular rotations which are clearly elements of $G(H)$. Hence $\sigma\in G(H)$. But then we can check that $\rho(\tau)$ is a transvection of V/H whose path is $K(z+H)$, and whose fix is $s^\perp/H$. Thus these parabolic rotations yield all the transvections of $T(H^\perp/H)$ whose path is a non-singular point. Conversely, if T is any transvection in $T(H^\perp/H)$, then T is induced by a singular or a parabolic rotation in $G(H)$,

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depending whether $B(T)$ is a singular or non-singular point. Thus ρ is surjective, and the theorem is proved. The following corollary is of particular interest.

COROLLARY 4.3. *If* dim $V=4$, *then* $G(H)/O(H)$ *is isomorphic to* $SL_2(K)$.

Proof. In this case, $H=H^\perp$, and hence $H^\perp/H=0$. But then 4.1 allows us to deduce that $BT(H^\perp/H)=1$, and hence $T(H^\perp/H)/BT(H^\perp/H)$ is isomorphic to $T(H^\perp/H)$, which in turn is isomorphic to $SL_2(K)$.

5. **The group generated by singular isometries.** We now let S be the set of all singular isometries, and $G(S)$ be the subgroup of O_n which is generated by the elements of S. We let Ω_n denote the commutator subgroup of $O_n(K, Q)$, and let $O_n^+(K, Q)$ be the subgroup of $O_n(K, Q)$ which is generated by the rotations. As in [4], we let θ be the *spinor norm*, which is the map $\theta: O_n^+(K, Q)\to K^*/K^{*^2}$. The spinor norm is defined as follows: suppose $\pi\in O_n^+(k, Q)$, and $\pi=\sigma_1\cdots\sigma_k$, where the σ_i are simple isometries and $B(\sigma_i)=Kp_i$. Then $\theta(\pi)=Q(p_1)\cdots Q(p_k)\cdot K^{*^2}$. From [4], we know that θ is a surjective homomorphism whose kernel is $\Omega_n(K, Q)$, and hence $O_n^+(K, Q)/\Omega_n(K, Q)$ is isomorphic to K^*/K^{*2}.

Now let ρ be any parabolic rotation with path $\langle s, z\rangle$ where $Q(s)=f(s, z)=0$, and $Q(z)\neq 0$. Then any other vector in $\langle s, z\rangle$ can be expressed in the form $x=\alpha(\beta s+z)$, and so $Q(x)=\alpha^2Q(z)$. But ρ is a product of two simple isometries whose path lies in $\langle s, z\rangle$ and thus $\theta(\rho)=\alpha^2\beta^2Q(z)^2K^{*2}=K^{*2}$, implying that the parabolic rotations all lie in $\Omega_n(K, Q)$. But by 2.9d, we see that every singular rotation can be expressed as the product of two parabolic rotations. Thus we have proved

LEMMA 5.1. *The group* $G(S)$ *is a subgroup of* $\Omega_n(K, Q)$.

We can improve upon this lemma. We do this in the next two theorems.

THEOREM 5.2. *If* $n\geq 5$, *then* $G(S)=\Omega_n(K, Q)$.

Proof. If char $K=2$, we know from [4] that $\Omega_n(K, Q)$ is a simple group for $n\geq 5$. Since $G(S)$ is clearly a normal subgroup of $\Omega_n(K, S)$, we therefore conclude the result. If char $K\neq 2$, we need only check that $G(S)$ is not contained in the center Z_n of $O_n(K, Q)$. But $Z_n=\{1, -1\}$, and clearly no singular rotation lies in Z_n. Hence we may apply Theorem 5.27 of Artin [1], to conclude that $G(S)$ contains the group $\Omega_n(K, Q)$. This again proves the result.

We deal separately with the case that $n=4$ and index $Q=2$. In this case the quadric is a hyperboloid and contains two families of singular lines. We denote these two families L_1 and L_2. Then any two distinct lines of L_1, respectively L_2, span the whole space, and every line of L_1 meets every line of L_2. Also, every singular point lies on exactly one line of each family. We define $G(L_1)$, respectively $G(L_2)$, to be the group generated by all the singular rotations whose path lies in L_1, respectively in L_2. Let τ_1 be a singular rotation in $G(L_1)$, and τ_2 be a singular rotation whose path lies in L_2. In view of 2.9a, we know that τ_1 and τ_2 commute. Hence we have proved

LEMMA 5.3. *Let* $\pi_1 \in G(L_1)$, *and* $\pi_2 \in G(L_2)$. *Then* $\pi_1\pi_2 = \pi_2\pi_1$.

We now strengthen this result in

LEMMA 5.4. *Suppose that* $\pi \in G(L_1)$, *and that* π *fixes all the lines in* L_1. *Then* $\pi \in Z_4 =$ *center of the group* $O_4(K, Q)$.

Proof. By 5.3, we know that π fixes all the lines in L_2. If π also fixes all the lines of L_1 then π fixes all the point of the quadric, since every singular point lies on two fixed lines. But then π is a homothety (see [1]). Since π is also an isometry, we conclude that $\pi = 1$ or -1.

We now check that $-1 \in G(L_1)$. For this purpose we decompose V into two hyperbolic planes $V = \langle s_1, s_2\rangle \oplus \langle s_3, s_4\rangle$. Then we define the following three singular isometries: (we assume char $K \neq 2$, as otherwise the result is trivial)

$$\pi_1(x) = x + 2[f(x, s_3)s_1 - f(x, s_1)s_3]$$

$$\pi_2(x) = x + 2[f(x, s_4)s_2 - f(x, s_2)s_4]$$

$$\pi_3(x) = x + 2[f(x, s_1 + s_4)(s_2 - s_3) - f(x, s_2 - s_3)(s_1 + s_4)]$$

These are three singular rotations, and it is easy to see that they belong to $G(L_1)$. Now an easy, though laborious, calculation shows that the product $\pi_1\pi_2\pi_3 = -1$.

We now choose a particular line H in L_2. Recalling the definition of the group $G(H)$ (Theorem 4.2), we see at once from the preceding results that $G(H)$ is isomorphic to the direct product of $G(L_1)$ with $O(H)$. From this we deduce at once that the factor group $G(H)/O(H)$ is isomorphic to the group $G(L_1)$. Combining this with Corollary 4.3, we see that the group $G(L_1)$ (and hence $G(L_2)$) is isomorphic to the group $Sl_2(K)$. Thus we have proved

THEOREM 5.5. *The group* $G(S)$ *is isomorphic to* $G(L_1) \times G(L_2)$ *if* char $K = 2$. *If* char $K \neq 2$, *then* $G(S)/Z_4$ *is isomorphic to* $G(L_1)/Z_4 \times G(L_2)/Z_4$. *The groups* $G(L_i)$ *are isomorphic to* $SL_2(K)$.

Thus we see that the group $G(S)$ is always the commutator subgroup $\Omega_n(K, Q)$, (see Dieudonne [4]). Thus we see why the commutator displays the uncharacteristic behaviour of being the direct product of two groups in the case that dim $V = 4$. This is a consequence of the geometric result that a hyperboloid is a ruled quadric, admitting the two families of lines.

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