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SMOOTHNESS PROPERTIES OF BOUNDED SOLUTIONS OF DIRICHLET'S PROBLEM FOR ELLIPTIC EQUATIONS IN REGIONS WITH CORNERS ON THE BOUNDARY

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We study here the smoothness of solutions of the Dirichlet problem for elliptic equations in a region G with a piece-wise smooth boundary. The smoothness of the solution given depends on the smoothness of the coefficients of the equation, the boundary, the boundary function and the values of the angles on the boundary and the values of the coefficients of the second derivatives at the corners.

§1. **Introduction.** Boundary value problems for the linear second order elliptic equation

$$(1.1)\qquad \sum_{i,j=1}^{2} a_{ij}(x_1, x_2)\frac{\partial^2 u}{\partial x_i\, \partial x_j} + \sum_{i=1}^{2} a_i(x_1, x_2)\frac{\partial u}{\partial x_i} + a(x_1, x_2)u = f(x_1, x_2)$$

in a domain with smooth boundary have been thoroughly investigated. In the works of Agmon, Douglis, Nirenberg [1] and Browder [2], the normal solvability of such problems for general boundary conditions, satisfying the Sapiro-Lopatinskii condition has been established. It was also proven that, if the right hand side of (1.1), the coefficients of the equation, the boundary and the boundary operators are infinitely differentiable, then the solution of the problem is also infinitely differentiable (see also [4], [7], [10]). If the boundary contains a corner, this is no longer true. The reason is that, it is not possible in this case to smooth the boundary by means of a smooth transformation. Moreover, from the simplest examples, it is apparent that when the boundary contains angular points the solution of the problem may not be infinitely differentiable for infinitely differentiable right hand side, coefficients and boundary functions. In [5, 6] Kondratev considered some special Sobolev spaces with weight functions. In these spaces, he studied general boundary value problems for equation (1.1) in a domain whose boundary is piece-wise smooth. We study here the first boundary value problem for equation (1.1) in the space $C_{m+\alpha}$.

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DEFINITIONS 1.1. The function $g(x_1, x_2)$ is said to belong to $C_{m+\alpha}(G)$ $m \geq 0$ an integer, $0<\alpha<1$, if $g(x_1, x_2)$ has m continuous derivatives in G, and the derivatives of order m satisfy in G a Hölder condition with exponent α.

1.2. The Hölder coefficient of the derivatives of order k of $g(x_1, x_2)$ in the domain G is defined as

$$H_\alpha^G(D^k g) = \text{l.u.b.} \frac{|D^k g(P) - D^k g(Q)|}{(\overline{PQ})^\alpha}$$

where l.u.b. is over $P \neq Q$ in G, and all the derivatives of order k.

1.3. In $C_{m+\alpha}(G)$, the norms may be defined as follows

$$\|g\|_0 = \max_G |g(x_1, x_2)|$$

$$\|g\|_{k+\alpha} = \sum_{i=0}^{k} \|D^i g\|_0 + H_\alpha^G(D^k g), \qquad k = 0, 1, \ldots, m.$$

In [1], it is proved that, if the right hand side and the coefficients of equation (1.1) are of class $C_{m+\alpha}(\bar{G})$, and if the boundary Γ of G can be represented parametrically by functions of class $C_{m+2+\alpha}$, and if the boundary function ϕ belongs to $C_{m+2+\alpha}(\Gamma)$, then any solution of the Dirichlet problem

$$\sum_{i,j=1}^{2} a_{ij}(x_1, x_2) \frac{\partial^2 u}{\partial x_i \, \partial x_j} + \sum_{i=1}^{2} a_i(x_1, x_2) \frac{\partial u}{\partial x_i} + a(x_1, x_2)u = f(x_1, x_2) \tag{1.2}$$

$$u = \phi \quad \text{on} \quad \Gamma \tag{1.3}$$

belongs to $C_{m+2+\alpha}(\bar{G})$. If the boundary Γ contains some corners, and the open arcs between these corners are of class $C_{m+2+\alpha}$, then the solution $u(x_1, x_2)$ belongs to $C_{m+2+\alpha}(G_1) \cap C_0(\bar{G})$ where G_1 is any compact subdomain of $\bar{G}$ with positive distances from the corners.

In a rectangle $G = \{(x_1, x_2),\ 0<x_1<a,\ 0<x_2<b\}$, Nikolskii [11] studied the problem

$$\Delta u \equiv \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = 0 \quad \text{in} \quad G \tag{1.4}$$

$$u = \phi \quad \text{on} \quad \Gamma. \tag{1.5}$$

He gave necessary and sufficient conditions for the solution to belong to $C_{m+2+\alpha}(\bar{G})$. These conditions are that on the open intervals $0<x_1<a$, $0<x_2<b$ the boundary function ϕ belongs to $C_{m+2+\alpha}$ and at the four corners it satisfies certain compatibility conditions. These results were generalized in [3] by Fufaev to the case of Poisson equation. Fufaev also proved that if the angles at the corners are of values π/q, $q = 2, 3, \ldots$ then the solution of the Dirichlet

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problem

$$\Delta u = f(x_1, x_2) \quad \text{in} \quad G \tag{1.6}$$

$$u = 0 \quad \text{on} \quad \Gamma \tag{1.7}$$

will belong to $C_{m+2+\alpha}(\bar{G})$ if and only if $f(x_1, x_2)\varepsilon C_{m+\alpha}(\bar{G})$ and at the corners $f(x_1, x_2)$ satisfies certain agreement conditions (see also [13]). In domains with angles π/q, $q=2, 3, \ldots$ Volkov [14] studied mixed boundary value problems for the Laplace equation, and gave the necessary and sufficient conditions for the solution to belong to $C_{m+2+\alpha}(\bar{G})$. If the angle ω on the boundary is not of the form π/q, $q=2, 3, \ldots$ then the solution of problem (1.4)–(1.5) may not belong to $C_{m+2+\alpha}(\bar{G})$. Fufaev [3] proved that if π/ω is not an integer, then the smaller the angle ω the smoother in $\bar{G}$ is the solution u of problem (1.4)–(1.5). He proved that if $(\pi/\omega) > m+2+\alpha$, then $u(x_1, x_2)\varepsilon C_{m+2+\alpha}(\bar{G})$ provided that the boundary function is continuous on the whole boundary Γ and is of class $C_{m+2+\alpha}$ on the open intervals between the corners. In this paper we generalize this result to the case of problem (1.2)–(1.3). We first prove this result in a circular sector then we use this to prove the theorem in the general case.

§2. **The problem in a circular sector.** Consider the sector $\Omega_\sigma = \{(r, \theta), r \le \sigma < 1, \beta < \theta < \omega + \beta\}$ where (r, θ) the polar coordinates of the point $x = (x_1, x_2)$ and $(\pi/\omega) > m+2+\alpha$ and $\beta > 0$ satisfies $\pi/(\omega+2\beta) > m+2+\alpha$. Suppose that the function $w(x_1, x_2)$ satisfies inside Ω_σ the elliptic equation

$$Lw \equiv \sum_{i,j=1}^{2} b_{ij}(x) \frac{\partial^2 w}{\partial x_i \, \partial x_j} + \sum_{i=1}^{2} b_i(x) \frac{\partial w}{\partial x_i} + b(x)w = g(x) \tag{2.1}$$

where $g(x)$, $b_{ij}(x)$, $b_i(x)$ and $b(x)$ belong to $C_{m+\alpha}(\bar{\Omega}_\sigma)$ and $b_{ij}(0,0) = \delta_{ij}$ the kronecker delta, $i, j = 1, 2$. Suppose also that $w(x)$ is bounded in $\bar{\Omega}_\sigma$ and that its boundary value $\Psi(r, \theta)$ on the two lines $\theta = \beta$ and $\theta = \omega + \beta$ belongs to $C_{m+2+\alpha}$, $0 \le r < \sigma$, and

$$\Psi^{(k)}(0, \theta) \equiv \left. \frac{d^k \Psi}{dr^k} \right|_{r=0} = 0, \qquad k = 0, 1, \ldots, m+2 \tag{2.2}$$

We also assume that $g(x)$ satisfies

$$D^k g(0,0) \equiv \left. \frac{\partial^k g}{\partial x_1^{k_1} \, \partial x_2^{k-k_1}} \right|_{(0,0)} = 0, \qquad k = 0, 1, \ldots, m \tag{2.3}$$

for all $k_1 = 0, 1, \ldots, k$.

Theorem 1. *There exists a number r_0, $0 < 2r_0 < \sigma$ such that in Ω_{2r_0} the bounded solution $w(x)$ of the problem*

$$Lw = g(x) \quad \textit{in} \quad \Omega_\sigma \tag{2.4}$$

$$w = \Psi \quad \textit{on} \quad \Gamma \tag{2.5}$$

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where g and ψ fulfill (2.2) *and* (2.3), *satisfies the inequality*

$$|w| \leq Mr^{m+2+\alpha} \tag{2.6}$$

In the course of the proof we will need the following two lemmas.

LEMMA 1 [7]. *Let* Ω *be a finite domain with diameter r, and let L be a linear second order elliptic operator. Let* $v \in C_2(\Omega)$ *and* $Lv \geq 0$ (≤ 0) *inside* Ω, *while* $v \leq 0(\geq 0)$ *on the boundary of* Ω. *There exists* $r_0 > 0$, *such that if* $r \leq r_0$, *then* $v \leq 0(\geq 0)$ *will be satisfied inside* Ω.

LEMMA 2 [1]. *Let* Γ' *be a portion of the boundary* Γ *of a domain* Ω, *and* $\Gamma' \in C_{m+2+\alpha}$. *Let* Ω' *be a subdomain of* Ω *with the property that the intersection of the boundary of* Ω' *with* Γ *lies in the interior of* Γ', *then any bounded solution* $w(x)$ *of the problem* (2.4)–(2.5) *in* Ω *will satisfy Schauder's inequality*

$$\|w\|^{\Omega'}_{m+2+\alpha} \leq \delta(\|w\|^{\Omega}_0 + \|g\|^{\Omega}_{m+\alpha} + \|\Psi\|^{\Gamma'}_{m+2+\alpha}) \tag{2.7}$$

where $\delta > 0$ *is a finite number independent of w.*

We now prove Theorem 1.

Proof. Since on the two lines $\theta = \beta$ and $\theta = \omega + \beta$ we have $\Psi \in C_{m+2+\alpha}$ then from (2.2) we get

$$|\Psi^{(m+2)}(r, \theta)| = |\Psi^{(m+2)}(r, \theta) - \Psi^{(m+2)}(0, \theta)| \leq K_{m+2} r^{\alpha}$$

$$|\Psi^{(m+1)}(r, \theta)| \leq \int_0^r |\Psi^{(m+2)}(r, \theta)| \, d\theta \leq K_{m+1} r^{1+\alpha}$$

And generally for any $k = 0, 1, \ldots, m+2$

$$|\Psi^{(k)}(r, \theta)| \leq K_k r^{m+2-k+\alpha} \tag{2.8}$$

Similary for $g(x_1, x_2)$

$$|D^k g(x_1, x_2)| \leq H_k r^{m-k+\alpha} \qquad k = 0, 1, \ldots, m \tag{2.9}$$

Consider now the function $v(x) = -Mr^{\nu} \sin \lambda\theta$ where $M > 0$ will be defined later and $\lambda = \pi/(\omega + 2\beta) > m + 2 + \alpha = \nu$. We rewrite the operator Lw as follows

$$Lw = \frac{\partial^2 w}{\partial x_1^2} + \frac{\partial^2 w}{\partial x_2^2} + \sum_{i,j=1}^{2} [b_{ij}(x) - \delta_{ij}] \frac{\partial^2 w}{\partial x_i \, \partial x_j} + \sum_{i=1}^{2} b_i(x) \frac{\partial w}{\partial x_i} + b(x) w \tag{2.10}$$

In particular

$$\begin{aligned} Lv = M(\lambda^2 - \nu^2) r^{m+\alpha} \sin \lambda\theta + M \sum_{i,j=1}^{2} [b_{ij}(x) - \delta_{ij}] H_{ij} r^{m+\alpha} \\ + Mh_1 r^{m+1+\alpha} + Mh_2 r^{m+2+\alpha} \end{aligned} \tag{2.11}$$

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where H_{ij} and h_i are bounded functions, say

$$\sum_{i,j=1}^{2} |H_{ij}| + \sum_{i=1}^{2} |h_i| \le R.$$

Since $b_{ij}(x_1, x_2)$ are continuous functions and $b_{ij}(0,0) = \delta_{ij}$, then for any $\varepsilon > 0$, we can find $r_0 > 0$ such that if $r \le 2r_0$,

$$|b_{ij}(x) - \delta_{ij}| < \varepsilon/4R \qquad i, j = 1, 2.$$

Since for $\beta < \theta < \omega + \beta$, $\sin \lambda\theta \ge \sin \lambda\beta$, then for $r \le 2r_0$, (2.11) gives

$$\begin{aligned} Lv &\ge M[(\lambda^2 - \nu^2)\sin \lambda\beta - \varepsilon]r^{m+\alpha} - 2MRr^{m+1+\alpha} \\ &\ge M[(\lambda^2 - \nu^2)\sin \lambda\beta - \varepsilon - 4Rr_0]r^{m+\alpha}. \end{aligned}$$

We choose $\varepsilon > 0$ sufficiently small, such that $A = (\lambda^2 - \nu^2)\sin \lambda\beta - \varepsilon > 0$ then we take r_0 small enough to make $B = A - 4Rr_0 > 0$ then we set $M > H_0/B$, where H_0 is taken from (2.9). Thus inside Ω_{2r_0} we have $Lv \ge H_0 r^{m+\alpha} \ge g(x)$, i.e. $L(w - v) = Lw - Lv \le 0$.

We now choose M sufficiently large, such that on the boundary of Ω_{2r_0}, $w - v \ge 0$. On the two lines $\theta = \beta$ and $\theta = \omega + \beta$ we have

$$w - v = \Psi(r, \theta) + Mr^{m+2+\alpha} \sin \lambda\beta$$

and from (2.8) we get

(i) $$w - v \ge (M \sin \lambda\beta - K_0) r^{m+2+\alpha}.$$

On the arc $r = 2r_0$ we have

(ii) $$w - v \ge w + M(2r_0)^{m+2+\alpha} \sin \lambda\beta \ge -\|w\|_0^{\Omega_\sigma} + M(2r_0)^{m+2+\alpha} \sin \lambda\beta$$

We now take M sufficiently large so that the right hand sides of (i) and (ii) are nonnegative. Now inside Ω_{2r_0}, $L(w - v) \le 0$ while on its boundary $w - v \ge 0$. We take r_0 sufficiently small, such that in Ω_{2r_0} we can apply lemma 1. Thus in Ω_{2r_0} we now have $w - v \ge 0$, i.e.

$$w \ge -Mr^{m+2+\alpha} \sin \lambda\theta \ge -Mr^{m+2+\alpha}.$$

Similarly, taking r_0 sufficiently small and M sufficiently large we can prove that in Ω_{2r_0}, $w \le Mr^{m+2+\alpha}$ i.e. in Ω_{2r_0}

$$|w| \le Mr^{m+2+\alpha}.$$

This proves the theorem.

THEOREM 2. *Under the assumptions of Theorem 1, the following inequalities hold in* Ω_{r_0}

(2.12) $$|D^k w| \le M_k r^{m+2-k+\alpha}, \qquad k = 0, 1, \ldots, m+2$$

where $D^k w$ *is any derivative of order* k *of* $w(x)$.

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Proof. Consider the domains

$$\Omega_n=\left\{(r,\theta);\frac{r_0}{2^{n+1}}\le r\le\frac{r_0}{2^n};\ \beta\le\theta\le\omega+\beta\right\}$$

$$\Omega_n'=\Omega_{n-1}\cup\Omega_n\cup\Omega_{n+1}.$$

We denote by Γ_n' the straight parts of the boundary of Ω_n'. Any point $(r,\theta)\in\Omega_{r_0}$ belongs to some Ω_n with $\Omega_n'\subset\Omega_{2r_0}$. The transformation

$$x_i=\frac{1}{2^n}y_i \qquad i=1,2 \tag{2.13}$$

transforms Ω_n and Ω_n' to Ω_0 and Ω_0' respectively. In Ω_0' the function $w_0(y)=w_0(y_1,y_2)=w(y_1/2^n,y_2/2^n)$ satisfies the elliptic equation

$$\sum_{i,j=1}^{2}c_{ij}(y)\frac{\partial^2 w_0}{\partial y_i\,\partial y_j}+2^{-n}\sum_{i=1}^{2}c_i(y)\frac{\partial w_0}{\partial y_i}+2^{-2n}c(y)w_0=2^{-2n}g_0(y) \tag{2.14}$$

where $c_{ij}(y)=b_{ij}(y/2^n)$ and similarly the functions $c_i(y)$, $c(y)$ and $g_0(y)$ may be defined in terms of b_i, b and g. The boundary value of $w_0(y)$ is $\Psi_0(\rho,\theta)=\Psi(\rho/2^n,\theta)$; $\rho^2=y_1^2+y_2^2$. In Ω_0 and Ω_0' the Schauder's inequality (2.7) gives

$$\|w_0\|_{m+2+\alpha}^{\Omega_0}\le\delta(\|w_0\|_0^{\Omega_0'}+2^{-2n}\,\|g_0\|_{m+\alpha}^{\Omega_0'}+\|\Psi_0\|_{m+2+\alpha}^{\Gamma_0'}) \tag{2.15}$$

Now $\|w_0\|_0^{\Omega_0}=\|w\|_0^{\Omega_n'}$ and from (2.6) it follows that

$$\|w_0\|_0^{\Omega_0'}\le Mr^{m+2+\alpha}\le T_1\left(\frac{1}{2^n}\right)^{m+2+\alpha}$$

Since $g_0(y)=g(y/2^n)$, then

$$D_0^k g_0(y)=\left(\frac{1}{2^n}\right)^k D^k g(x) \qquad k=0,1,2,\ldots,m$$

where D_0^k is the derivative in the y-plane corresponding to D^k. From that and from (2.9) we get

$$|D_0^k g_0(y)|\le H\left(\frac{1}{2^n}\right)^{m+\alpha} \qquad k=0,1,\ldots,m.$$

If P and Q are any two points in Ω_n' and their images in Ω_0' are P_0 and Q_0, then

$$\frac{D_0^m g_0(P_0)-D_0^m g_0(Q_0)}{(\overline{P_0Q_0})^\alpha}=\left(\frac{1}{2^n}\right)^{m+\alpha}\frac{D^m g(P)-D^m g(Q)}{(\overline{PQ})^\alpha}.$$

From this it follows that

$$H_\alpha^{\Omega_0'}(D_0^m g_0)=\left(\frac{1}{2^n}\right)^{m+\alpha}H_\alpha^{\Omega_n'}(D^m g)$$

Thus

$$\|g_0\|_{m+\alpha}^{\Omega_0'} = \sum_{k=0}^{m} \|D_0^k g_0\|_0^{\Omega_0'} + H_\alpha^{\Omega_0'}(D_0^m g_0) \le T\left(\frac{1}{2^n}\right)^{m+\alpha} \|g\|_{m+\alpha}^{\Omega_n'}$$
$$\le T_2\left(\frac{1}{2^n}\right)^{m+\alpha}$$

Similarly we can show that

$$\|\Psi_0\|_{m+2+\alpha}^{\Gamma_0'} \le T_3\left(\frac{1}{2^n}\right)^{m+2+\alpha}$$

Thus (2.15) yields

$$\|w_0\|_{m+2+\alpha}^{\Omega_0} \le \delta\left[T_1\left(\frac{1}{2^n}\right)^{m+2+\alpha} + T_2\left(\frac{1}{2^n}\right)^{m+2+\alpha} + T_3\left(\frac{1}{2^n}\right)^{m+2+\alpha}\right]$$
$$= \delta_0\left(\frac{1}{2^n}\right)^{m+2+\alpha}$$

Since $|D_0^k w_0| \le \|w_0\|_{m+2+\alpha}^{\Omega_0}$, $k=0,1,\ldots,m+2$ and since $D_0^k w_0 = (1/2^n)^k D^k w$, we get that in Ω_n

$$\left(\frac{1}{2^n}\right)^k |D^k w| \le \delta_0\left(\frac{1}{2^n}\right)^{m+2+\alpha}$$

or equivalently

$$|D^k w| \le \delta_0\left(\frac{1}{2^n}\right)^{m+2-k+\alpha} \le M_k r^{m+2-k+\alpha}, \qquad k=0,1,\ldots,m+2.$$

This proves the theorem.

THEOREM 3. *Under the assumptions of Theorem 1, $w(x_1, x_2) \in C_{m+2+\alpha}(\Omega_{r_0})$.*

Proof. Consider any two points $P(r_1, \theta_1)$ and $Q(r_2, \theta_2)$ in $\bar{\Omega}_{r_0}$, and suppose that $0 \le r_2 \le r_1 \le r_0$. We have two cases to consider

1. $r_2 \le \frac{1}{2}r_1$. In this case $\overline{PQ} \ge \frac{1}{2}r_1$ and

$$|D^k w(P) - D^k w(Q)| \le M_k r_1^{m+2-k+\alpha} + M_k r_2^{m+2-k+\alpha}$$
$$< 2M_k r_1^{m+2-k+\alpha}; \qquad k=0,1,\ldots,m+2$$

and

$$\frac{|D^{m+2}w(P) - D^{m+2}w(Q)|}{(\overline{PQ})^\alpha} \le \frac{2M_{m+2}r_1^\alpha}{(\frac{1}{2}r_1)^\alpha} = \text{const.}$$

2. $r_2 > \frac{1}{2}r_1$. We will prove that $w \in C_{m+2+\alpha}(G_0)$, where $G_0 = \{(r, \theta),$

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$(r_1/2)\le r\le r_1,\ \beta\le\theta\le\omega+\beta\}$. The transformation

$$x_i=\frac{2r_1}{r_0}y_i \qquad i=1,2$$

transforms G_0 to

$$G_1=\left\{(\rho,\theta),\frac{r_0}{4}\le\rho\le\frac{r_0}{2},\beta\le\theta\le\omega+\beta\right\},\rho^2=y_1^2+y_2^2,$$

and transforms P and Q to $P_1(\rho_1,\theta_1)$ and $Q_1(\rho_2,\theta_2)$ where $\rho_1=r_0/2$ and $\rho_2=(r_0r_2/2r_1)>(r_0/4)$. In

$$G_1'=\left\{(\rho,\theta);\frac{r_0}{8}\le\rho\le r_0,\beta\le\theta\le\omega+\beta\right\}$$

the function

$$w_1(y)=w_1(y_1,y_2)=w\left(\frac{2r_1}{r_0}y_1,\frac{2r_1}{r_0}y_2\right),$$

satisfies the elliptic equation

$$\sum_{i,j=1}^{2}d_{ij}(y)\frac{\partial^2 w_1}{\partial y_i\,\partial y_j}+\left(\frac{2r_1}{r_0}\right)\sum_{i=1}^{2}d_i(y)\frac{\partial w_1}{\partial y_i}+\left(\frac{2r_1}{r_0}\right)^2 d(y)w_1=\left(\frac{2r_1}{r_0}\right)^2 g_1(y)$$

where

$$d_{ij}(y)=d_{ij}(y_1,y_2)=b_{ij}\left(\frac{2r_1}{r_0}y_1,\frac{2r_1}{r_0}y_2\right).$$

Similarly the functions $d_i(y)$, $d(y)$ and $g_1(y)$ can be defined. On the straight parts of the boundary of G_1', the boundary value of $W_1(y)$ is $\Psi_1(\rho,\theta)=\Psi[(2r_1/r_0)\rho,\theta]$. Schauder's inequality (2.7) in G_1 and G_1' yields

$$\|w_1\|_{m+2+\alpha}^{G_1}\le\delta\left[\|w_1\|_0^{G_1'}+\left(\frac{2r_1}{r_0}\right)^2\|g_1\|_{m+\alpha}^{G_1'}+\|\Psi_1\|_{m+2+\alpha}^{\Gamma_1'}\right]$$

Exactly as in the proof of Theorem 2 we get

$$\|w_1\|_{m+2+\alpha}^{G_1}\le\delta_0 r_1^{m+2+\alpha}.$$

We note as before that $D_1^k w_1=(2r_1/r_0)^k\ D^k w$ where D_1^k is the derivative corresponding to D^k, and noting that $\|w_1\|_\mu\le\|w_1\|_{m+2+\alpha}$, $0\le\mu\le m+2+\alpha$ we get that in G_0.

$$\left(\frac{2r_1}{r_0}\right)^k|D^k w|\le\delta_1 r_1^{m+2+\alpha}$$

or equivalently

$$|D^k w|\le N_k r_1^{m+2-k+\alpha} \qquad k=0,1,2,\ldots,m+2.$$

Also noting that $H_\alpha^{G_1}(D_1^{m+2}w_1)=(2r_1/r_0)^{m+2+\alpha}H_\alpha^{G_0}(D^{m+2}w)$ we get

$$(2r_1/r_0)^{m+2+\alpha}H_\alpha^{G_0}(D^{m+2}w)\le\delta_1 r_1^{m+2+\alpha}$$

$$\text{i.e)}\quad H_\alpha^{G_0}(D^{m+2}w)\le H.$$

This means that $w\in C_{m+2+\alpha}(G_0)$. The theorem is thus proved.

§3. **The general case.** Let the domain G have a boundary Γ consisting of a piece of a smooth curve of class $C_{m+2+\alpha}$ the ends of which join at the point 0 forming an angle γ, $0<\gamma<2\pi$. In G we consider the first boundary value problem

$$\sum_{i,j=1}^{2} a_{ij}(x)\frac{\partial^2 u}{\partial x_i\,\partial x_j}+\sum_{i=1}^{2} a_i(x)\frac{\partial u}{\partial x_i}+a(x)u=f(x) \tag{3.1}$$

$$u=\phi \quad \text{on} \quad \Gamma \tag{3.2}$$

the right hand side of (3.1) and the coefficients a_{ij}, a_i and a belong to $C_{m+\alpha}(\bar{G})$. The boundary function ϕ is continuous on Γ and is of class $C_{m+2+\alpha}(\Gamma\setminus\{0\})$. We transform the equation

$$\sum_{i,j=0}^{2} a_{ij}(0)\frac{\partial^2 u}{\partial x_i\,\partial x_j}=0 \tag{3.3}$$

to canonical form. The new angle after transformation $\omega(0)$ is independent of the transformation used and is given by

$$\omega(0)=\arctan\frac{[a_{11}(0)a_{22}(0)-a_{12}^2(0)]^{1/2}}{a_{22}(0)\cot\gamma-a_{12}(0)} \tag{3.4}$$

THEOREM 4. *If* $(\pi/\omega(0))>m+2+\alpha$, *then the bounded solution of problem* (3.1)–(3.2) *where* f *belongs to* $C_{m+\alpha}(\bar{G})$ *and* ϕ *belongs to* $C_{m+2+\alpha}(\Gamma\setminus\{0\})\cap C_0(\Gamma)$ *belongs to* $C_{m+2+\alpha}(\bar{G})$.

Proof. It is sufficient to prove the theorem in any neighborhood $G_0\subset\bar{G}$, of the corner 0 since $u(x)\in C_{m+2+\alpha}(\bar{G}\setminus G_0)$. With no loss of generality we take the corner 0 to be at the origin, and assume that the two curves Γ_1 and Γ_2 forming the angle are $x_2=f_1(x_1)$ and $x_1=f_2(x_2)$ where $f_i(x_i)\in C_{m+2+\alpha}(\Gamma_i)$ and $f_2(0)=f_1(0)=f_1'(0)=0$. Consider the following transformation

$$\begin{aligned} y_1&=\frac{1}{\Delta\sqrt{\alpha_{11}}}[\alpha_{12}(x_1-f_2(x_2))+\alpha_{11}(x_2-f_1(x_1))]\\ y_2&=\frac{1}{\sqrt{\alpha_{11}}}[x_1-f_2(x_2)] \end{aligned} \tag{3.5}$$

where

$$\alpha_{11}=a_{11}(0,0)-2f_2'(0)a_{12}(0,0)+f_2'^2(0)a_{22}(0,0)$$
$$\alpha_{12}=a_{22}(0,0)f_2'(0)-a_{12}(0,0)$$
$$\alpha_{22}=a_{22}(0,0)$$
$$\Delta=[a_{11}(0,0)a_{22}(0,0)-a_{12}^2(0,0)]^{1/2}$$

The two curves Γ_2 and Γ_1 will be transformed to $y_2=0$ and $y_2=y_1 \tan\omega$, where $\tan\omega=\Delta/\alpha_{12}$, $(\pi/\omega)>m+2+\alpha$. We now choose $\beta>0$ such that $\pi/(\omega+2\beta)>m+2+\alpha$, then we rotate the axes with this angle using the transformation

$$\begin{aligned} z_1 &= y_1\cos\beta - y_2\sin\beta \\ z_2 &= y_1\sin\beta + y_2\cos\beta. \end{aligned} \tag{3.6}$$

Consider the subdomain G_2 in $\bar{G}$ defined by

$$G_2=\{(x_1, x_2):(x_1, x_2)\in\bar{G},\, x_1^2+x_2^2\le d_1^2,\, d_1>0\}.$$

In the z-plane the domain G_2 will be transformed by (3.5)–(3.6) to a domain Ω_0 bounded by the two lines $\theta=\beta$ and $\theta=\omega+\beta$ and by a curve $r=\sigma(\theta)$, where $\sigma(\theta)\ge\sigma>0$, $\beta\le\theta\le\omega+\beta$ ($r^2=z_1^2+z_2^2$, $\theta=\arctan z_2/z_1$). In $\Omega_\sigma=\{(r,\theta)\in\Omega_0,\ r\le\sigma,\ \beta<\theta<\omega+\beta\}$ the transformed function $U(z)=U(z_1, z_2)=u(x_1, x_2)$ satisfies the elliptic equation

$$LU\equiv\sum_{i,j=1}^{2} b_{ij}(z)\frac{\partial^2 U}{\partial z_i\,\partial z_j}+\sum_{i=1}^{2} b_i(z)\frac{\partial \bar{U}}{\partial z_i}+b(z)U=F(z) \tag{3.7}$$

Where

$$\begin{aligned}
b_{11}(z)&=e_{11}(z)\cos^2\beta-2e_{12}(z)\sin\beta\cos\beta+e_{22}(z)\sin^2\beta\\
b_{12}(z)&=e_{11}(z)\sin\beta\cos\beta+e_{12}(z)\cos 2\beta-e_{22}(z)\sin\beta\cos\beta\\
b_{22}(z)&=e_{11}(z)\sin^2\beta+2e_{12}(z)\sin\beta\cos\beta+e_{22}(z)\cos^2\beta\\
b_1(z)&=e_1(z)\cos\beta-e_2(z)\sin\beta\\
b_2(z)&=e_1(z)\sin\beta+e_2(z)\cos\beta\\
b(z)&=a(x)\\
F(z)&=f(x)
\end{aligned}$$

$$\begin{aligned}
e_{11}(z)=\frac{1}{\Delta^2\alpha_{11}}&[(\alpha_{11}f_1'(x_1)-\alpha_{12})^2a_{11}(x)\\
&-2(\alpha_{11}f_1'(x_1)-\alpha_{12})(\alpha_{11}-\alpha_{12}f_2'(x_2))a_{12}(x)\\
&+(\alpha_{11}-\alpha_{12}f_2'(x_2))^2a_{22}(x)]
\end{aligned}$$

$$\begin{aligned}
e_{12}(z)=\frac{1}{\Delta\alpha_{11}}&[(\alpha_{12}-\alpha_{11}f_1'(x_1))a_{11}(x)\\
&+(\alpha_{11}+\alpha_{11}f_1'(x_1)f_2'(x_2)-2\alpha_{12}f_2'(x_2))a_{12}(x)\\
&-f_2'(x_2)(\alpha_{11}-\alpha_{12}f_2'(x_2))a_{22}(x)]
\end{aligned}$$

$$e_{22}(z)=\frac{1}{\alpha_{11}}[a_{11}(x)-2f_2'(x_2)a_{12}(x)+f_2'^2(x_2)a_{22}(x)]$$

$$e_1 = \frac{1}{\Delta\sqrt{\alpha_{11}}}[a_1(x)(\alpha_{12}-\alpha_{11}f_1'(x_1))+a_2(x)(\alpha_{11}-\alpha_{12}f_2'(x_2)$$
$$-\alpha_{11}f_1''(x_1)a_{11}(x)-\alpha_{12}f_2''(x_2)a_{22}(x)]$$
$$e_2 = \frac{1}{\sqrt{\alpha_{11}}}[a_1(x)-f_2'(x_2)a_2(x)-f_2^{\circ}(x_2)a_{22}(x)]$$
$$b_{11}(z)b_{22}(z)-b_{12}^2(z) = \frac{\sin^4\beta+\cos^4\beta}{\Delta^2}[1-f_1'(x_1)f_2'(x_2)]^2[a_{11}(x)a_{22}(x)-a_{12}^2(x)]$$

The functions b_{ij}, b_i, b and F belong to $C_{m+\alpha}(\bar{\Omega}_\sigma)$ and $b_{ij}(0,0)=\delta_{ij}$ $i,j=1,2$. On the two lines $\theta=\beta$ and $\theta=\omega+\beta$ the boundary values $\Phi(r,\beta)$ and $\Phi(r,\omega+\beta)$ of $U(z)$ belong to $C_{m+2+\alpha}$, $0\le r\le\sigma$ and $\Phi(0,\beta)=\Phi(0,\omega+\beta)$. We denote by Φ^0 and Φ_β^k the values of $\Phi(r,\beta)$ and $(d^k\Phi/dr^k)(r,\beta)$ at $r=0$, $k=1,2,\ldots,m+2$. $\Phi_{\omega+\beta}^k$ is defined similarly. Consider now the function

$$\Theta(z)=\Phi^0+\sum_{k=1}^{m+2}\frac{(z_1\cos\beta+z_2\sin\beta)^k}{k!}\Phi_\beta^k$$
$$+\sum_{k=1}^{m+2}\sum_{l=1}^{k}\frac{(z_1\cos\beta+z_2\sin\beta)^{k-l}(-z_1\sin\beta+z_2\cos\beta)^l}{(k-l)!\,l!}\cdot\frac{\Phi_{\omega+\beta}^k-\Phi_\beta^k\cos^k\omega}{(\sin\omega+\cos\omega)^k-\cos^k\omega}.$$

The function $v(z)=U(z)-\Theta(z)$ satisfies in Ω_σ the equation

$$Lv=h(z)\equiv F-L\Theta(z). \tag{3.8}$$

On the two lines $\theta=\beta$ and $\theta=\omega+\beta$ the boundary value $\Psi(r,\theta)$ of $v(z)$ belongs to $C_{m+2+\alpha}$ and satisfies (2.2). In Lemma 3 we will show that there exists a polynomial $s(z)=s(z_1,z_2)$ vanishing on $\theta=\beta$ and $\theta=\omega+\beta$ such that $w=v-s$ satisfies in Ω_σ equation (2.4) with $g(z)=h(z)-Ls(z)$ satisfying (2.3). Thus all the conditions of Theorem 1 are satisfied, and we conclude that in $\bar{\Omega}_{r_0}$, $2r_0<\sigma$, $w(z)\in C_{m+2+\alpha}$. Since $U(z)=w(z)+\Theta(z)+s(z)$, where $\Theta(z)$ and $s(z)$ are polynomials, then $U(z)\in C_{m+2+\alpha}(\bar{\Omega}_{r_0})$. Returning now to the x-plane and noting that the transformation (3.5) is of class $C_{m+2+\alpha}$, and at the origin it has a Jacobian equal to $-(1/\Delta)$ which is finite and different from zero, we conclude that in a subdomian $\bar{G}_0\subset\bar{G}$; $G_0=\{(x_1,x_2):(x_1,x_2)\in\bar{G},\ x_1^2+x_2^2\ge d_2^2,\ d_2>0\}$, $u(x_1,x_2)\in C_{m+2+\alpha}$. This proves the theorem.

LEMMA 3. *There exists a polynomial $s(z)$ vanishing on the two lines $\theta=\beta$ and $\theta=\omega+\beta$, and is such that the function $w(z)=v(z)-s(z)$ satisfies in Ω_σ the equation $Lw=g(z)\equiv h(z)-Ls(z)$, with $g(z)$ satisfying* (2.3).

Proof. We set $m_1=\tan\beta$ and $m_2=\tan(\omega+\beta)$, $1+m_1m_2\ne 0$ and by $h^{(k_1,k_2)}(0,0)$ we denote

$$\left.\frac{\partial^{k_1+k_2}h}{\partial x_1^{k_1}\,\partial x_2^{h_2}}\right|_{(0,0)},\qquad 1\le k_1+k_2\le m.$$

We will prove the lemma by induction. Consider the function

$$s_0(z)=(z_2-m_1z_1)(z_2-m_2z_1)\frac{h(0,0)}{2(1+m_1m_2)}$$

$s_0(z)$ vanishes on $\theta=\beta$ and $\theta=\omega+\beta$, and

$$Ls_0(z)=h(0,0)+p_1(z)+o(r)$$

where $p_k(z)$ is a homogeneous polynomial in z_1 and z_2 of order k, $k=1,2,\ldots,m$. Consider now the two functions

$$P_{k-1}(z)=(z_2-m_1z_1)(z_2-m_2z_1)T_{k-1}$$
$$P_k=(z_2-m_1z_1)(z_2-m_2z_1)T_k$$

where

$$T_{k-1}(z)=\sum_{k_1+k_2=0}^{k-1}\lambda_{k_1k_2}z_1^{k_1}z_2^{k_2}$$
$$T_k(z)=\sum_{k_1=0}^{k}\lambda_{k_1}z_1^{k_1}z_2^{k-k_1},\qquad k\le m$$

We put

$$s_k(z)=P_{k-1}(z)+P_k(z)$$

Suppose now that the coefficients $\lambda_{k_1k_2}$ in $P_{k-1}(z)$ are already found such that

(3.9) $$LP_{k-1}(z)=\sum_{k_1+k_2=0}^{k-1}\frac{z_1^{k_1}z_2^{k_2}}{k_1!\,k_2!}h^{(k_1,k_2)}(0,0)+p_k(z)+o(r^k)$$

Now we find the coefficients $\lambda_0,\lambda_1,\ldots,\lambda_k$ in P_k such that

(3.10) $$LP_k(z)=\sum_{k_1=0}^{k}\frac{z_1^{k_1}z_2^{k-k_1}}{k_1!\,(k-k_1)!}h^{(k_1,k-k_1)}(0,0)+p_{k+1}(z)-p_k(z)+0(r^{k+1})$$

where $p_k(z)$ is taken from (3.9). Now

$$LP_k(z)=\sum_{i,j=1}^{2}b_{ij}(z)\frac{\partial^2P_k}{\partial z_i\,\partial z_j}+p_{k+1}(z)+o(r^{k+1})$$

and since $b_{ij}(0,0)=\delta_{ij}$ we get

$$LP_k(z)=\frac{\partial^2P_k}{\partial z_1^2}+\frac{\partial^2P_k}{\partial z_2^2}+p_{k+1}(z)+o(r^{k+1}).$$

We now find $\lambda=\{\lambda_0,\lambda_1,\ldots,\lambda_k\}$ from the identity

(3.11) $$\frac{\partial^2P_k}{\partial z_1^2}+\frac{\partial^2P_k}{\partial z_2^2}=\sum_{k_1=0}^{k}\frac{z_1^{k_1}z_2^{k-k_1}}{k_1!\,(k-k_1)!}h^{(k_1,k-k_1)}(0,0)-p_k(z)$$

Equating the coefficients of $z_1^{k_1}z_2^{k-k_1}$, $k_1=0,1,\ldots,k$ on both sides of (3.11) we get the system of algebraic equations

$$A\lambda = B \tag{3.12}$$

If $\det A \neq 0$, then (3.12) is solvable for λ. We now show that $\det A \neq 0$. Suppose contrarily that $\det A = 0$. We find a nonzero solution λ^* of the equation

$$A\lambda = 0 \tag{3.13}$$

then we substitute this solution in $P_k(z)$. $P_k(z)$ is now a homogeneous polynomial of order $k+2$, vanishes on the two lines $z_2 = z_1 \tan\beta$ and $z_2 = z_1 \tan(\omega+\beta)$ and satisfies in Ω_σ the Laplace equation

$$\frac{\partial^2 P_k}{\partial z_1^2} + \frac{\partial^2 P_k}{\partial z_2^2} = 0.$$

Thus $P_k(z)$ vanishes on all the lines making angles $t\omega$ with these two lines, where t is any positive integer. If π/ω is irrational, then the number of these lines is not finite. If $\omega = (p/q)\pi$ where p/q is an irreducible fraction and $p \geq 1$, then the number of these lines is q, $q \geq (q/p) = (\pi/\omega) > m+2+\alpha$. In both cases the number of the different lines on which $P_k(z)$ vanishes is greater than $m+2$. This is a contradiction since $P_k(z)$ is a polynomial of degree $k+2$, $k \leq m$. Thus $\det A \neq 0$, and (3.12) uniquely gives λ. The function $s_m(z)$ satisfies all the requirements of the lemma. This proves the lemma.

REMARKS.

1. If $(\pi/\omega) \leq m+2+\alpha$, then using the same argument we can show that $u(x) \in C_{(\pi/\omega)-\varepsilon}(\bar{G})$, $\varepsilon > 0$ is arbitrary.
2. If there is more than one corner on the boundary then the smoothness of the solution in the neighborhood of each corner may be discussed separately.

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