POINCARE THETA SERIES AND SINGULAR SETS OF SCHOTTKY GROUPS

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Introduction

In the theory of automorphic functions for a properly discontinuous group G of linear transformations, the Poincaré theta series plays an essential role, since the convergence problem of the series occupies an important part of the theory. This problem was treated by many mathematicians such as Poincaré, Burnside [2], Fricke [4], Myrberg [6], [7] and others. Poincaré proved that the $(-2m)$-dimensional Poincaré theta series always converges if m is a positive integer greater than 2, and Burnside treated the problem and conjectured that (-2)-dimensional Poincaré theta series always converges if G is a Schottky group. This conjecture was solved negatively by Myrberg. As is shown later (Theorem A), the convergence of Poincaré theta series gives an information on a metrical property of the singular set of the group.

In this paper, we shall investigate the convergence problem in the case of a Schottky group and the metrical property of the singular set of the group from the viewpoint of their connection. In § **1**, we prove Theorem A which states some equivalent propositions for the convergence of Poincaré theta series. This theorem gives a relation between the convergence of the series and Hausdorff measure of the singular set of the group. In § **2**, a Schottky group of Ford type [3] is treated by using isometric circles. We have some criteria (Theorem B) for the convergence of Poincaré theta series for such a group and we reprove Myrberg's result which gives the negative answer to Burnside's conjecture. In § **3**, we give an example of a Schottky group whose singular set has positive 1-dimensional measure (Theorem C) and an analogue of Schottky's result. It seems that results in § **3** suggest us a close relation between the convergence problem and the metrical property of the singular set of the group.

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§ 1. Convergence Problem of Poincaré Theta Series

1. Let B_0 be a domain bounded by $2p$ circles $\{H_i, H_i'\}_{i=1}^{p}$ in the complex z-plane which are disjoint from each other. We suppose that B_0 contains the point at infinity. Let S_i be a hyperbolic or a loxodromic transformation

$$(1) \qquad S_i(z) = \frac{\alpha_i z + \beta_i}{\gamma_i z + \delta_i}, \quad (\alpha_i\delta_i - \beta_i\gamma_i = 1),$$

which transforms the outside of H_i onto the inside of H_i'. We denote by S_i^{-1} the inverse transformation of S_i. Obviously S_i^{-1} transforms the outside of H_i' onto the inside of H_i. In general, we denote by ST the transformation obtained by composition of transformations S and T, that is,

$$ST(z) = S(T(z)).$$

We put $SS = S^2$ and $S^{\lambda} = SS^{\lambda-1}$ inductively for any integer $\lambda(>1)$. For a negative integer λ, S^{λ} denotes $(S^{-1})^{|\lambda|}$.

Consider the totality G of all linear transformations in the from

$$(2) \qquad S = S_{i_k}^{\lambda_k} S_{i_{k-1}}^{\lambda_{k-1}} \cdots S_{i_1}^{\lambda_1}, \text{ viz., } S(z) = S_{i_k}^{\lambda_k}(S_{i_{k-1}}^{\lambda_{k-1}}(\cdots (S_{i_1}^{\lambda_1}(z)) \cdots))$$

together with the identical transformation, where λ_j are integers and $i_j \neq i_{j-1}$ for $j = 2, \ldots, k$. As is easily seen, G is a group and B_0 is a fundamental domain of G which is called a Schottky group generated by $S_1, \ldots, S_p$ (Maurer [5], Myrberg [6]).

2. We consider the Schottky group G generated by S_i $(i = 1, \ldots, p)$. Any element S different from the identity of G has the form (2). We call the sum

$$m = \sum_{j=1}^{k} |\lambda_j|$$

grade of S. The image $S(B_0)$ of the fundamental domain B_0 of G by S $(\in G)$ with grade m $(\neq 0)$ is bounded by $2p$ circles $S(H_i)$ and $S(H_i')$ $(i = 1, 2, \ldots, p)$, the one $C^{(m-1)}$ of which is contained in the boundary of the image of B_0 under some T $(\in G)$ with grade $m-1$. For simplicity, we say that $2p-1$ boundary circles of $S(B_0)$ different from $C^{(m-1)}$ are circles of grade m. Circles $\{H_i, H_i'\}_{i=1}^{p}$, which bound B_0, are of grade zero. The number of circles of grade

m is obviously equal to $2p(2p-1)^m$.

Denote by D_m the $2p(2p-1)^m$-ply connected domain bounded by the whole circles of grade m. Evidently $\{D_m\}$ $(m=0, 1, \ldots)$ is a monotone increasing sequence of domains. The complementary set D_m^c of D_m with respect to the extended z-plane consists of $2p(2p-1)^m$ mutually disjoint closed discs. These closed discs are called discs of grade m. The set $E=\bigcap_{m=1}^{\infty} D_m^c$ is perfect and nowhere dense. We call E the singular set of G. The group G is properly discontinuous in the complementary set of E. It is well known that, in the case $p \geqq 2$, the logarithmic capacity of E is positive (Myrberg [8]) and that the 2-dimensional measure of E is equal to zero (Sario [10], Tsuji [12]).

3. Let $H(z)$ be a rational function none of whose poles is contained in the singular set E of the Schottky group G generated by S_i $(i=1, \ldots, p)$ in (1). Denote by $z_j=(a_jz+b_j)/(c_jz+d_j)$ $(j=0, 1, \ldots)$ all the elements of G. The identical transformation of G is denoted by z_0.

Consider the series

$$\Theta_\nu(z)=\sum_{j=0}^{\infty} H(z_j)(c_jz+d_j)^{-\nu}, \tag{3}$$

where ν is a positive integer. This is a so-called $(-\nu)$-dimensional Poincaré theta series.

Let D be the complementary domain of the set E and D' be a relatively closed subdomain of D. Since the point $-d_j/c_j$ $(j \neq 0)$ is the image of infinity by the inverse transformation z_j^{-1} of z_j $(j \neq 0)$ and since G is properly discontinuous in D, there are only finitely many points $-d_j/c_j$ $(j \neq 0)$ in D'. Denote by D'' a non-empty subdomain of D$'$ obtained by deleting suitable neighborhoods of points $-d_j/c_j$ and ∞.

Let e_i $(i=1, \ldots, k_1)$ and f_i $(i=1, \ldots, k_2)$ be poles and zeros of $H(z)$ in D respectively and let U_i and V_i be neighborhoods of e_i and f_i respectively such that $|H(z)|=M_1$ on the boundary of U_i and $|H(z)|=M_2>0$ on the boundary of V_i, where $M_1>M_2$. By using the proper discontinuity of G and by taking M_1 sufficiently large and M_2 sufficiently small, we may assume that $D^*=\overline{D''-\bigcup_{S\in G}\bigcup_{i=1}^{k_1} S(U_i)}$ and $D^{**}=D^*-\bigcup_{S\in G}\bigcup_{i=1}^{k_2} S(V_i)$ are not empty. In D^{**}, each term $H(z_j)(c_jz+d_j)^{-\nu}$ of (3) has no zero and no pole and

$$M_2<|H(z_j)|<M_1.$$

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If $\sum_{j=0}^{\infty} |(c_jz+d_j)|^{-\nu}$ converges uniformly in D'', then $\Theta_\nu(z)$ converges absolutely and uniformly in D^*, since

$$|H(z_j)|\,|(c_jz+d_j)|^{-\nu} < M_1|(c_jz+d_j)|^{-\nu}$$

in D^*.

Next, suppose that $\Theta_\nu(z)$ converges absolutely and uniformly in D^*. Since

$$M_2|(c_jz+d_j)|^{-\nu} < |H(z_j)|\,|(c_jz+d_j)|^{-\nu}$$

in $D^{**}(\subset D^*)$, the series $\sum_{j=0}^{\infty}(c_jz+d_j)^{-\nu}$ also converges absolutely and uniformly in D^{**}. Take a point z' in any $S(V_i)$ or in any $S(U_i)$ which has points in common with D''. We can take a sufficiently large number K such that, for any z'' in D^{**} and for all points $-d_j/c_j$ which are exterior to D',

$$|z'+d_j/c_j|^{-\nu} < K|z''+d_j/c_j|^{-\nu}.$$

Therefore, if $\sum_{j=0}^{\infty}|c_jz+d_j|^{-\nu}$ converges uniformly in D^{**}, it converges uniformly in D''.

Hence we have

THEOREM 1. *The $(-\nu)$-dimensional Poincaré theta series*

$$\Theta_\nu(z) = \sum_{j=0}^{\infty} H(z_j)(c_jz+d_j)^{-\nu},\ z_j = \frac{a_jz+b_j}{c_jz+d_j} \in G$$

converges absolutely and uniformly in D^* if and only if the series

$$\sum_{j=0}^{\infty}(c_jz+d_j)^{-\nu}$$

converges absolutely and uniformly in D''.

Poincaré showed that $\sum_{j=0}^{\infty}(c_jz+d_j)^{-\nu}$ converges in D'' when ν is an even integer greater than 2. Hence, we have

Theorem of Poincaré. The $(-\nu)$-dimensional Poincaré theta series converges absolutely and uniformly in D^* if ν is an even integer greater than 2.

Burnside [2] proved that, in the case of Fuchsian groups of the first class or of Schottky groups with some restriction, the series $\Theta_2(z)$ converges in D^* and conjectured that the (-2)-dimensional Poincaré theta series for a Schottky group always converges absolutely and uniformly in D^*. Fricke [4] investigated the convergence problem of $\Theta_1(z)$ in the case of Fuchsian group of the first

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class. Myrberg [6] showed that the (-2)-dimensional Poincaré theta series $\Theta_2(z)$ with respect to Schottky groups does not always converge absolutely and uniformly in D^* and gave a negative answer to Burnside' conjecture. The Myrberg theorem will be reproved in No. 14 (See Theorem 9).

4. We put

$$P_\nu(z)=\sum_{j=0}^{\infty}|c_jz+d_j|^{-\nu},$$

where ν is a positive number. We call $P_\nu(z)$ the $(-\nu)$-dimensional P-series. Petersson [9] showed that if G is a Fuchsian group, $P_\nu(z)$ converges for $\nu>2$, and that if G is a Fuchsian group of the second class, $P_\nu(z)$ diverges for $\nu<2$.

Let $d(>0)$ be the minimum distance from the boundary of D' to points $-d_j/c_j$ which are poles of $z_j=S_j(z)$ $(\in G)$ and are exterior to $\overline{D}'$. Obviously we have

$$|z+d_j/c_j|\geqq d \qquad (z\in D')$$

for these indices j, so we get

$$\sum_{j\neq 0}{}' |c_jz+d_j|^{-\nu}\leqq d^{-\nu}\sum_{j\neq 0}{}'|c_j|^{-\nu} \qquad (\nu>0),$$

where $\sum'$ denotes the sum of terms which exclude a finite number of $-d_j/c_j$ contained in D'. Since $\sum_{j\neq 0}{}'|c_j|^{-\nu}\leqq\sum_{j=1}^{\infty}|c_j|^{-\nu}$, the series $\sum_{j\neq 0}{}'|c_jz+d_j|^{-\nu}$ converges uniformly in D', provided that $\sum_{j=1}^{\infty}|c_j|^{-\nu}$ is convergent. Hence, if $\sum_{j=1}^{\infty}|c_j|^{-\nu}$ converges, then $P_\nu(z)$ converges uniformly in D''.

Let z be any point of D''. Since we can describe a circle C with center at the origin and radius ρ so that the point z and $\{H_i, H_i'\}_{i=1}^{p}$ lie inside C, we have

$$\sum_{j=1}^{\infty}|c_jz+d_j|^{-\nu}>\left(\frac{1}{2\rho}\right)^{\nu}\sum_{j=1}^{\infty}|c_j|^{-\nu}, \qquad (\nu>0).$$

Therefore, we see that $\sum_{j=0}^{\infty}|c_jz+d_j|^{-\nu}$ diverges in D'', provided that the series $\sum_{j=1}^{\infty}|c_j|^{-\nu}$ diverges.

Thus we obtain the following

THEOREM 2. *The series $P_\nu(z)$ converges uniformly in D'' if and only if the series*

$$(4) \qquad \sum_{j=1}^{\infty}|c_j|^{-\nu}, \qquad (\nu>0)$$

converges.

5. Let

$$S^{(m)}:\ z' = S^{(m)}(z) = \frac{az+b}{cz+d}$$

be a transformation of grade m in G. Then, the radius r_0 of a boundary circle C of $S^{(m)}(B_0)$ is given by

$$2\pi r_0 = \int_H \left|\frac{dS(z)}{dz}\right| |dz| = \int_H \frac{|dz|}{|cz+d|^2},$$

where H is a suitable one in $\{H_i, H_i'\}_{i=1}^{p}$ which $S^{(m)}$ carries to C. Hence, we have

$$2\pi r_0 = \frac{1}{|c|^2}\int_H \frac{|dz|}{|z+(d/c)|^2}.$$

Again we note that the point $-d/c$ is outside of B_0. If we put

$$\varDelta = \max_{z\in H}|z+(d/c)| \quad \text{and} \quad \delta = \min_{z\in H}|z+(d/c)|,$$

then

$$\frac{r}{\varDelta^2}\cdot\frac{1}{|c|^2} \leqq r_0 \leqq \frac{r}{\delta^2}\cdot\frac{1}{|c|^2}, \tag{5}$$

where r is the radius of H.

Such inequalities hold for all circles of grade $m-1$. Hence, there exist two positive constants $k(G)$ and $K(G)$ such that

$$k(G)\sum_{j=1}^{\infty}|c_j|^{-\nu} \leqq \sum_{m=1}^{\infty} l_m^{(\nu)} \leqq K(G)\sum_{j=1}^{\infty}|c_j|^{-\nu}, \qquad (\nu>0).$$

Here, $l_m^{(\nu)}$ is the sum of terms $(r^{(m-1)})^{\nu/2}$ obtained for radii $r^{(m-1)}$ of all circles of grade $m-1$ and $z_j = (a_j z + b_j)/(c_j z + d_j)$, $(j = 1, 2, \ldots)$ are all transformations of G different from the identity. In fact, we may take $k(G)$ as the minimum of $(r/\varDelta^2)^{\nu/2}$ and $K(G)$ as the maximum of $(r/\delta^2)^{\nu/2}$, when H runs in $\{H_i, H_i'\}_{i=1}^{P}$ and $S^{(m)}$ $(m>1)$ varies in G.

Accordingly, we have the following

THEOREM 3. *The series* $\sum_{j=1}^{\infty}|c_j|^{-\nu}$ *converges if and only if* $\sum_{m=1}^{\infty} l_m^{(\nu)}$ *converges.*

Combining Theorems 1, 2 and 3, we have

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THEOREM A. *Let ν be a positive integer. The following four propositions are equivalent to each other:* (i) *The* $(-\nu)$*-dimensional Poincaré theta series* $\Theta_\nu(z)$ *converges absolutely and uniformly in* D^*. (ii) *The* $(-\nu)$*-dimensional P-series* $P_\nu(z)$ *converges uniformly in* D''. (iii) *The series* $\sum_{j=1}^{\infty}|c_j|^{-\nu}$ *converges.* (iv) *The series* $\sum_{m=1}^{\infty} l_m^{(\nu)}$ *converges.*

6. It is evident that, if $\lim_{m\to\infty} l_m^{(\nu)}=0$, then the singular set of G is of $\frac{\nu}{2}$-dimensional measure zero. Hence, from the above theorem, we get

COROLLARY. *If any one of the conditions* (i), (ii), (iii) *and* (iv) *in Theorem A is valid, then the singular set of G is of* $\frac{\nu}{2}$*-dimensional measure zero.*

Noting Poincaré's theorem stated in No. 3, we can easily see that the singular set of a Schottky group is of area zero.

As to the convergence of the sequence $\{l_m^{(\nu)}\}_{m=1}^{\infty}$, we obtain the following which yields a result of the author [1].

PROPOSITION 1. *Let* $r_i^{(m)}$ $(i=1, \ldots, 2p-1)$ *be radii of circles of grade m lying inside a circle of grade* $m-1$ *and with radius* $r^{(m-1)}$. *If there exists a positive constant c* (<1) *such that*

$$\sum_{i=1}^{2p-1}(r_i^{(m)})^{\nu/2}<c(r^{(m-1)})^{\nu/2}$$

for every circle of grade $m-1$, *then* $\lim_{m\to\infty} l_m^{(\nu)}=0$.

As to the divergence of the series $\sum_{m=1}^{\infty} l_m^{(\nu)}$, we get

PROPOSITION 2. *Take any one H of circles* $\{H_i, H_i'\}_{i=1}^{p}$ *and fix this H. Denote by* $C_j^{(m-1)}$ $(j=1, \ldots, 2p(2p-1)^{m-1})$ *the circle with radius* $r_j^{(m-1)}$ *of grade* $m-1$ *inside H. For all* $C_j^{(m-1)}$, *suppose that the sum of radii of circles of grade m inside* $C_j^{(m-1)}$ *is not less than* $r_j^{(m-1)}$. *Then* $\sum_{m=1}^{\infty} l_m^{(\nu)}$ *diverges.*

§ 2. Schottky Group of Ford Type

7. First we shall state the concept of isometric circles of linear transformations due to Ford [3] and some important properties of them.

For a linear transformation of the form

$$T(z)=\frac{az+b}{cz+d},\quad ad-bc=1,\quad c\neq 0,$$

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the circle I: $|cz+d|=1$ is called the isometric circle of the transformation. The radius of I equals $1/|c|$.

(I) By a transformation lengths and areas inside its isometric circle are increased in magnitude, and lengths and areas outside the isometric circle are decreased in magnitude. A transformation carries its isometric circle into the isometric circle of the inverse transformation. The radii of the isometric circles of a transformation and its inverse are equal.

Let G be a properly discontinuous group of linear transformations. We suppose that, if an element of G transforms the point at infinity into itself, then the element is the identity of G. Consider two arbitrary transformations of G

$$T: \ T(z)=\frac{az+b}{cz+d}, \quad ad-bc=1, \ c\neq 0,$$

and

$$S: \ S(z)=\frac{\alpha z+\beta}{\gamma z+\delta}, \quad \alpha\delta-\beta\gamma=1, \ \gamma\neq 0.$$

For a moment we assume that $S\neq T^{-1}$. The isometric circle of $ST=S(T(z))$ is the circle

$$|(\gamma a+\delta c)z+\gamma b+\delta d|=1.$$

Denote by I_S, I'_S, I_T, I'_T and I_{ST} isometric circles of S, S^{-1}, T, T^{-1} and ST, respectively. Let g_S, g'_S, g_T, g'_T and g_{ST} be their centers, and let R_S, R_T and R_{ST} be redii of I_S, I_T and I_{ST}.

As to these values, relations

$$R_{ST}=\frac{1}{|\gamma a+\delta c|}=\frac{R_S\cdot R_T}{|g'_T-g_S|}, \tag{6}$$

and

$$|g_{ST}-g_T|=\frac{R_{ST}\cdot R_T}{R_S}=\frac{R_T^2}{|g'_T-g_S|} \tag{7}$$

hold.

From this we can see the following:

(II) The radii of isometric circles of the group G are bounded and the number of isometric circles with radii exceeding a given positive quantity is finite.

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As to the location of isometric circles, we have the following from (I).

(III) If I_S and I'_T are exterior to each other, then I_{ST} is contained in I_T. If I_S and I'_T are tangent externally, then I_{ST} lies in I_T and is tangent internally.

8. Now consider a group G whose elements are linear transformations

$$z_j = \frac{a_j z + b_j}{c_j z + d_j}, \quad a_j d_j - b_j c_j = 1, \quad j = 0, 1, \ldots, \quad (c_j \neq 0, \ j \geqq 1),$$

where z_0 is the identical transformation. From the definition of isometric circles follows:

THEOREM 4. *The convergence of the series* $\sum_{j=1}^{\infty} |c_j|^{-\nu}$ *is equivalent to the convergence of the sum of* ν*-th powers of radii of isometric circles for all the elements of* G.

Take two positive numbers ν and ν' such that $\nu < \nu'$. Since $|c_j|$ are greater than 1 except a finite number of $|c_j|$ from the property (II), we have

$$|c_j|^{-\nu'} < |c_j|^{-\nu}$$

for almost all j. This yields

THEOREM 5. *Let* $0 < \nu < \nu'$. *If* $\sum_{j=1}^{\infty} |c_j|^{-\nu}$ *converges, then* $\sum_{j=1}^{\infty} |c_j|^{-\nu'}$ $(0 < \nu < \nu')$ *also converges.*

9. From the viewpoint of combination groups, a Schottky group is constructed as follows: Let $\{H_i, H'_i\}_{i=1}^{p}$ be p pairs of circles external to each other. Let S_i be a linear transformation (loxodromic or hyperbolic) carrying the exterior of H_i onto the interior of H'_i. The transformation S_i generates a cyclic group G_i, for which the domain F_i exterior to H_i and H'_i is a fundamental domain. By combining G_i $(i = 1, 2, \ldots, p)$, we have a Schottky group G whose fundamental domain B_0 is bounded by circles $\{H_i, H'_i\}_{i=1}^{p}$, that is, $B_0 = \bigcap_{i=1}^{p} F_i$. (cf. Ford [3])

Now, we consider the case when H_i and H'_i are isometric circles of S_i and S_i^{-1} $(i = 1, \ldots, p)$. In such a case, we shall call G a Schottky group of Ford type.

10. Consider a domain bounded by $2p$ mutually disjoint circles $\{H_i, H'_i\}_{i=1}^{p}$ $(p \geqq 2)$. Denote these circles by

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$$H_i: \ |z-q_i|=r_i, \ H_i': \ |z-q_i'|=r_i'. \ (i=1, \ldots, p).$$

Let S_i be a linear transformation which maps the outside of H_i onto the inside of H_i' and let G be the Schottky group generated by S_i $(i=1, \ldots, p)$. The domain bounded by $\{H_i, H_i'\}_{i=1}^{p}$ is a fundamental domain of G. In the case when $\{H_i, H_i'\}$ and S_i $(i=1, \ldots, p)$ satisfy some conditions, we may take the domain bounded by isometric circles $\{I_i, I_i'\}$ of $\{S_i, S_i^{-1}\}$ as the fundamental domain of G.

Suppose that S_i has the form

$$S_i: \ z_i=S_i(z)=\frac{\mu\{q_i'z-(q_iq_i'+r_ir_i'e^{i\theta})\}}{\mu(z-q_i)}, \quad (i=1, \ldots, p),$$

where $\mu=e^{-i(\theta/2)}/\sqrt{r_ir_i'}$ and θ is real.

From the definition, isometric circles I_i and I_i' of generators S_i and their inverse S_i^{-1} are

$$I_i: \ |z-q_i|=\sqrt{r_ir_i'}=R_i \text{ and } I_i': \ |z-q_i'|=\sqrt{r_ir_i'}=R_i, \quad (i=1, \ldots, p)$$

respectively. It follows from properties of isometric circles that, if the $2p$ circles $\{I_i, I_i'\}_{i=1}^{p}$ are exterior to each other, then the outside of these circles can be considered as a fundamental domain of G.

11. Hereafter we consider a Schottky group G of Ford type. We may suppose that the fundamental domain B_0 of G is bounded by isometric circles $\{I_i, I_i'\}$ of generators $S_1, \ldots, S_p$ and their inverses.

If the grade of the transformation in G is m, its isometric circle is called an isometric circle of grade m.

Take a transformation $S^{(m)}$ of grade m, which is represented by the form

$$S^{(m)}=T_mT_{m-1} \ldots T_2T_1, \tag{8}$$

where T_k $(1 \leqq k \leqq m)$ is a generator or its inverse and $T_k \neq T_{k-1}^{-1}$. We apply the transformation $S^{(m)}$ to a point z_0 of B_0. Since z_0 is exterior to I_{T_1}, T_1 carries z_0 into z_1 inside $I_{T_1}'=I_{T_1^{-1}}$. Since I_{T_1}' is exterior to I_{T_2} from $T_2 \neq T_1^{-1}$, z_1 is exterior to I_{T_2} and T_2 carries z_1 into z_2 inside I_{T_2}' and so on. At each step, lengths in the neighborhood of the point are decreased. Hence $S^{(k)}=T_kT_{k-1} \ldots T_2T_1$, $(1 \leqq k \leqq m)$ transforms z_0 with decrease of lengths, from which we see by (I) in No. 7 that z_0 is outside $I_{S^{(k)}}$. Since every point in B_0 is also outside $I_{S^{(2)}}=I_{T_2T_1}$, we see the circle $I_{S^{(2)}}$ is contained in $I_{S^{(1)}}=I_{T_1}$. By analogous arguments,

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the circle $I_{S^{(m)}}$ is contained in $I_{S^{(m-1)}}$.

After all, we see that the isometric circles of

$$T_1,\ T_2T_1,\ T_3T_2T_1,\ \ldots,\ T_mT_{m-1}\cdots T_2T_1$$

form a sequence such that each circle encloses the circle which follows it and an isometric circle of grade $m-1$ $(m>1)$ surrounds $2p-1$ isometric circles of grade m. The number of the isometric circles with grade m is obviously equal to $2p(2p-1)^m$.

Denote by $R_{S^{(m)}}$ the radius of the isometric circle $I_{S^{(m)}}$. Since $S^{(m)}$ can be written in the form (8), we obtain from (7)

$$R_{S^{(m)}} = R_{T_m\cdot S^{(m-1)}} = \frac{R_{T_m}\cdot R_{S^{(m-1)}}}{|g'_{S^{(m-1)}} - g_{T_m}|}. \tag{9}$$

where $S^{(m-1)} = T_{m-1}\ldots T_1$. Repeating this procedure and putting $S^{(k)} = T_k \ldots T_2T_1$, we get

$$R_{S^{(m)}} = R_{T_m}\prod_{k=1}^{m-1}\frac{R_{T_k}}{|g'_{S^{(k)}} - g_{T_{k+1}}|}.$$

If we suppose that all radii of isometric circles of generators $S_1, \ldots, S_p$ of G are equal to R, then we have finally

$$R_{S^{(m)}} = R^m\prod_{k=1}^{m-1}\frac{1}{|g'_{S^{(k)}} - g_{T_{k+1}}|}. \tag{10}$$

From

$$g'_{S^{(k)}} - g_{T_{k+1}} = g'_{S^{(k)}} - g'_{T_k} + g'_{T_k} - g_{T_{k+1}} = g'_{S^{(k)}} - g_{T_k^{-1}} + g_{T_k^{-1}} - g_{T_{k+1}},$$

follows

$$|g_{T_k^{-1}} - g_{T_{k+1}}| - |g'_{S^{(k)}} - g_{T_k^{-1}}| \leqq |g'_{S^{(k)}} - g_{T_{k+1}}| \leqq |g_{T_{k}^{-1}} - g_{T_{k+1}}| + |g'_{S^{(k)}} - g_{T_k^{-1}}|. \tag{11}$$

Using (7), we have

$$|g'_{S^{(k)}} - g_{T_k^{-1}}| = |g_{(S^{(k)})^{-1}} - g_{T_k^{-1}}| = |g_{(S^{(k-1)})^{-1}T_k^{-1}} - g_{T_k^{-1}}| = \frac{R_{(S^{(k-1)})^{-1}\cdot T_k^{-1}}\cdot R_{T_k^{-1}}}{R_{(S^{(k-1)})^{-1}}},$$

which implies from (I)

$$|g'_{S^{(k)}} - g_{T_k^{-1}}| = \frac{R_{S^{(k)}}}{R_{S^{(k-1)}}}\cdot R. \tag{12}$$

We put

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$$(13)\qquad \begin{aligned}\sigma &= \max_{\substack{1\leqq i,j,k\leqq p\\ i\neq j}} \{\mathrm{dis}(I_i, I_j), \mathrm{dis}(I'_i, I'_j), \mathrm{dis}(I_i, I'_k)\}\\ \tau &= \min_{\substack{1\leqq i,j,k\leqq p\\ i\neq j}} \{\mathrm{dis}(I_i, I_j), \mathrm{dis}(I'_i, I'_j), \mathrm{dis}(I_i, I'_k)\},\end{aligned}$$

where $\mathrm{dis}(I_i, I_j)$ denotes the distance between I_i and I_j. Then we can prove the following

THEOREM 6. *The radius $R_{S^{(m)}}$ of the isometric circle of grade m corresponding to $S^{(m)}$ ($\in G$) satisfies the inequality*

$$\left\{\frac{R(R+\tau)}{(2R+\sigma)(R+\tau)+R^2}\right\}^{m-1}\cdot R < R_{S^{(m)}} < \left(\frac{R}{R+\tau}\right)^{m-1}\cdot R.$$

Proof. Since $S^{(m)}$ is written in the form $T_m S^{(m-1)} = T_m T_{m-1} S^{(m-2)}$, where $T_m \neq T_{m-1}^{-1}$, two points $g'_{S^{(m-1)}}$ and g_{T_m} are contained inside of the different boundary circles of B_0.

(i) Since

$$|g'_{S^{(m-1)}} - g_{T_m}| > R+\tau,$$

we see

$$(14)\qquad \frac{R_{S^{(m)}}}{R_{S^{(m-1)}}} < \frac{R}{R+\tau}.$$

Repeating this process, we have

$$R_{S^{(m)}} < \left(\frac{R}{R+\tau}\right)^{m-1}\cdot R.$$

(ii) We obtain from (9) and (11)

$$\frac{R_{S^{(m)}}}{R_{S^{(m-1)}}} \geqq \frac{R}{|g'_{T_{m-1}} - g_{T_m}| + |g'_{S^{(m-1)}} - g'_{T_{m-1}}|}.$$

Obviously $|g'_{T_{m-1}} - g_{T_m}| \leqq 2R+\sigma$ and from (12) and (14),

$$|g'_{S^{(m-1)}} - g'_{T_{m-1}}| = \frac{R_{S^{(m-1)}}}{R_{S^{(m-2)}}}\cdot R < \frac{R^2}{R+\tau}.$$

Hence

$$\frac{R_{S^{(m)}}}{R_{S^{(m-1)}}} > \frac{R}{(2R+\sigma)+(R^2/(R+\tau))} = \frac{R+\tau}{(2R+\sigma)(R+\tau)+R^2}R.$$

By the same argument as in (i),

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$$\left\{\frac{R(R+\tau)}{(2R+\rho)(R+\tau)+R^2}\right\}^{m-1}\cdot R<R_{S(m)}.$$

12. Denote by L_m the sum of radii of $2p(2p-1)^m$ isometric circles of grade m.

Suppose that $\tau\geqq R$. Then we have

$$|g'_{T_k}-g_{T_{k+1}}|-\frac{1}{2}R\leqq|g'_{S(k)}-g_{T_{k+1}}|\leqq|g'_{T_k}-g_{T_{k+1}}|+\frac{1}{2}R$$

from (11), (12) and (14). This and (10) yield

$$R^m\prod_{k=1}^{m-1}\frac{1}{|g'_{T_k}-g_{T_{k+1}}|+(1/2)R}\leqq R_{S(m)}\leqq R^m\prod_{k=1}^{m-1}\frac{1}{|g'_{T_k}-g_{T_{k+1}}|-(1/2)R},$$

which gives

$$\left(\frac{R}{2R+\sigma+(1/2)R}\right)^{m-1}\cdot R\leqq R_{S(m)}\leqq\left(\frac{R}{2R+\tau-(1/2)R}\right)^{m-1}R.$$

As L_m is the sum of radii of all isometric circles of grade m, we see

$$2p(2p-1)\left\{\frac{(2p-1)R}{(5/2)R+\sigma}\right\}^{m-1}\cdot R\leqq L_m\leqq 2p(2p-1)\left\{\frac{(2p-1)R}{(3/2)R+\tau}\right\}^{m-1}\cdot R$$

Thus we have following two theorems.

THEOREM 7. *If*

$$\tau>(2p-(5/2))R,$$

then $\sum_{m=1}^{\infty}L_m$ *converges.*

THEOREM 8. *If* $\tau\leqq R$ *and*

$$\sigma\leqq(2p-(7/2))R,$$

then L_m *is not less than* $2p(2p-1)R$ *for any* m.

From Theorem A together with Theorems 5, 7 and 8, we have the following

THEOREM B. *Let G be a Schottky group of Ford type with the fundamental domain* B_0 *whose* $2p$ *boundary circles are the isometsic circles* $\{I_{S_i}, I_{S_i^{-1}}\}_{i=1}^{p}$ *with all equal radii R of generators of G and their inverses and let* τ *and* σ *be defined by* (13).

(I) *If*

$$\tau>\left(2p-\frac{5}{2}\right)R,\qquad(p\geqq 2),$$

then the (-1)*-dimensional Poincaré theta series with respect to* G *converges absolutely and uniformly in* D^*. *Accordingly, the singular set* E *of* G *is of* $\frac{1}{2}$*-dimensional measure zero.*

(II) *If*

$$R \leqq \tau \leqq \sigma \leqq \left(2p - \frac{7}{2}\right)R, \qquad (p \geqq 2),$$

then the (-1)*-dimensional Poincaré theta series does not converge absolutely and uniformly in* D^*.

Remark. It is not difficult to see the existence of the case satisfying the condition in (II) of Theorem B.

Draw a circle C with center at the origin whose radius is slightly smaller than 25/4. We can describe 12 circles with unit radius whose centers are on C and two circles with unit radius in C so that the mutual distances between any two of these 14 circles are greater than 1. The domain bounded by these 14 circles with unit radius satisfies the condition.

13. In No. 12 we have shown that there exists a case where the (-1)-dimensional Poincaré theta series does not converge. Now, by using isometric circles we treat an analogous problem for the (-2)-dimensional Poincaré theta series, which Myrberg [6], [7] discussed without using isometric circles. If we can show the existence of the Schottky group such that $\sum_{j=1}^{\infty} |c_j|^{-2}$ diverges, we see from Theorem A that the (-2)-dimensional Poincaré theta series with respect to the Schottky group does not converge absolutely and uniformly in D^*.

First take a positive number ε $(0<\varepsilon<1)$ and draw six circles

$$C_{1,j}: \quad |z - 2e^{i(\pi/3)j}| = 1 - \varepsilon, \qquad (j = 0, 1, \ldots, 5).$$

Next, draw 12 circles $C_{2,j}$ $(j = 0, 1, \ldots, 11)$

$$C_{2,j}: \begin{cases} |z - 4e^{i(\pi/6)j}| = 1 - \varepsilon, \text{ for even } j, \\ |z - 2\sqrt{3}\, e^{i(\pi/6)j}| = 1 - \varepsilon, \text{ for odd } j. \end{cases}$$

Further draw 18 circles $C_{3,j}$ $(j = 0, 1, \ldots, 17)$

$$C_{3,j}: \begin{cases} |z - 6e^{i(\pi/9)j}| = 1 - \varepsilon, \text{ if } j \equiv 0 \pmod 3 \\ |z - 2\sqrt{7}\, e^{i(\pi/9)j}| = 1 - \varepsilon, \text{ if } j \equiv 1 \text{ or } 2 \pmod 3. \end{cases}$$

Let $S_{1,n}$ $(0 \leqq n \leqq 2)$ be the hyperbolic transformation which maps the outside

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of $C_{1,n}$ onto the inside of $C_{1,n+3}$ and $S_{2,n}$ $(0 \leqq n \leqq 5)$ be the one which maps the outside of $C_{2,n}$ onto the inside of $C_{2,n+6}$ and let $S_{3,n}$ $(0 \leqq n \leqq 8)$ be the one which maps the outside of $C_{3,n}$ onto the inside of $C_{3,n+9}$. Denote by $S_{i,n}^{-1}$ $(i=1, 2, 3)$ the inverse of $S_{i,n}$. Obviously $\{S_{i,n}\}$ generate a Schottky group G of Ford type. The fundamental domain B_0 is a domain bounded by these 36 circles $C_{i,j}$.

Take a transformation $S^{(m)}$ $(\in G)$ of grade m (>0). Then $S^{(m)}$ is written in the form

$$S^{(m)} = T_m S^{(m-1)} = T_m T_{m-1} S^{(m-2)},$$

where T_m or T_{m-1} is a generator or the inverse of some generator of G.

The center $g'_{S^{(m-1)}} = g_{(S^{(m-2)})^{-1}T_{m-1}^{-1}}$ of the isometric circle $I'_{S^{(m-1)}}$ is contained in the isometric circle $I'_{T_{m-1}}$ of T_{m-1}^{-1} different from the isometric circle I_{T_m} of T_m which can be verified by using $T_m \neq T_{m-1}^{-1}$. Since $R_{T_m} = 1 - \varepsilon$, it follows from (9) that

$$R^2_{S^{(m)}} = \frac{(1-\varepsilon)^2}{|g'_{S^{(m-1)}} - g_{T_m}|^2} \cdot R^2_{S^{(m-1)}}.$$

Fixing $S^{(m-1)}$ and summing up these equalities for isometric circles $I_{S^{(m)}}$ contained in $I_{S^{(m-1)}}$, we get

$$\sum_{T_m} R^2_{S^{(m)}} = R^2_{S^{(m-1)}} \cdot (1-\varepsilon)^2 \sum_{T_m} \frac{1}{|g'_{S^{(m-1)}} - g_{T_m}|^2}.$$

Assuming that

(15) $$\sum_{T_m} \frac{1}{|g'_{S^{(m-1)}} - g_{T_m}|^2} \geqq \frac{1}{(1-\varepsilon)^2},$$

we obtain

(16) $$\sum_{T_m} R^2_{S^{(m)}} \geqq R^2_{S^{(m-1)}}.$$

Denote by $L_m^{(2)}$ the sum of squares of radii of the isometric circles of grade m. The inequality (16) implies that

$$L_m^{(2)} \geqq L_{m-1}^{(2)}.$$

14. We shall prove that the inequality (15) holds for sufficiently small ε. There are three cases: (i) T_{m-1}^{-1} is an $S_{1,n}$ $(0 \leqq n \leqq 2)$ or its inverse, (ii) T_{m-1}^{-1} is an $S_{2,n}$ $(0 \leqq n \leqq 5)$ or its inverse, (iii) T_{m-1}^{-1} is an $S_{3,n}$ $(0 \leqq n \leqq 8)$ or its inverse.

(I) First let us consider the case (i). Since B_0 is symmetric with respect to the origin, it suffices to prove (15) in the case $T_{m-1}^{-1}=S_{1,0}$. Then the point $g'_{S(m-1)}$ lies inside $I'_{T_{m-1}}=C_{1,0}$ and the point g_{T_m} in (15) is the center of a circle among $C_{1,j}$ $(1\leqq j\leqq 5)$, $C_{2,j}$ $(0\leqq j\leqq 11)$ and $C_{3,j}$ $(0\leqq j\leqq 17)$. Thus, in this case,

$$\sum_{T_m}\frac{1}{|g'_{S(m-1)}-g_{T_m}|^2}>\frac{5}{3^2}+\frac{12}{5^2}+\frac{11}{7^2}+\frac{7}{9^2}>1.34.$$

(II) Next, let us consider the case (ii). It suffices to prove (15) in the case $T_{m-1}^{-1}=S_{2,0}$ and $S_{2,1}$. Similarly, in the case (I), we get

$$\sum_{T_m}\frac{1}{|g'_{S(m-1)}-g_{T_m}|^2}>\frac{6}{3^2}+\frac{6}{5^2}+\frac{9}{7^2}+\frac{7}{9^2}+\frac{7}{11^2}>1.23,\qquad (\text{for } T_{m-1}^{-1}=S_{2,0})$$

and

$$\sum_{T_m}\frac{1}{|g'_{S(m-1)}-g_{T_m}|^2}>\frac{6}{3^2}+\frac{8}{5^2}+\frac{8}{7^2}+\frac{9}{9^2}+\frac{4}{11^2}>1.29,\qquad (\text{for } T_{m-1}^{-1}=S_{2,1}).$$

(III) Finally, let us consider the case (iii). It suffices to prove (15) in the case $T_{m-1}^{-1}=S_{3,0}$ and $S_{3,1}$. Similarly, we get, for $S_{3,1}$, the following inequality

$$\sum_{T_m}\frac{1}{|g'_{S(m-1)}-g_{T_m}|^2}>\frac{4}{3^2}+\left(\frac{3}{5^2}+\frac{3}{(2\sqrt{3}+1)^2}\right)+\left(\frac{2}{7^2}+\frac{4}{(2\sqrt{7}+1)^2}\right)$$
$$+\left(\frac{2}{9^2}+\frac{4}{(2\sqrt{13}+1)^2}+\frac{2}{(4\sqrt{3}+1)^2}\right)+\left(\frac{1}{11^2}+\frac{3}{(2\sqrt{19}+1)^2}+\frac{3}{(2\sqrt{21}+1)^2}\right)$$
$$+\left(\frac{1}{(6\sqrt{3}+1)^2}+\frac{2}{(4\sqrt{7}+1)^2}+\frac{1}{(2\sqrt{31}+1)^2}\right)>1.07.$$

For $S_{3,0}$, we need some consideration about the range of $g'_{S(m-1)}$. The hyperbolic transformation $S_{3,0}^{-1}$, which maps the outside of $C_{3,9}$ onto the inside of $C_{3,0}$, is composed by an inversion with respect to $C_{3,9}$ and a reflection with respect to the imaginary axis. Therefore the images of circles $C_{i,j}$ $(\neq C_{3,9})$ by $S_{3,0}^{-1}$ are contained in the domain F bounded by the left semi-circle of $C_{3,0}$ and its diameter orthogonal to the real axis. Since the centers of isometric circles are equivalent to infinity, which is an interior point of B_0, $g'_{S(m-1)}$ are contained in the images of circles $C_{i,j}$ $(\neq C_{3,9})$ by $S_{3,0}^{-1}$ and hence also in the domain F.

Therefore, we get, for $S_{3,0}$, the following inequality

$$\sum_{T_m}\frac{1}{|g'_{S(m-1)}-g_{T_m}|^2}>\left(\frac{1}{5}+\frac{2}{5+2\sqrt{3}}\right)+\left(\frac{1}{17}+\frac{1}{17+4\sqrt{3}}+\frac{2}{13+2\sqrt{3}}\right)$$
$$+\left(\frac{2}{7^2}+\frac{4}{(2\sqrt{7}+1)^2}\right)+\left(\frac{1}{9^2}+\frac{2}{(4\sqrt{3}+1)^2}+\frac{4}{(6\sqrt{2}+1)^2}\right)+\left(\frac{1}{11^2}+\right.$$

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$$+\frac{4}{(2\sqrt{19}+1)^2}+\frac{2}{(2\sqrt{21}+1)^2}\Big)+\Big(\frac{1}{13^2}+\frac{2}{(6\sqrt{3}+1)^2}+\frac{2}{(4\sqrt{7}+1)^2}+$$
$$+\frac{2}{(2\sqrt{31}+1)^2}\Big)>1.05.$$

Thus, in all cases, we have the inequality (15) by taking ε sufficiently small. Therefore, in our example, $L_m^{(2)} \geqq L_{m-1}^{(2)}$ for all m (>1). Thus the following theorem is obtained from Theorems A and 4.

THEOREM 9. *Let B_0 be the domain bounded by the above* 36 *circles. Suppose that the generators $S_{i,n}$ of a Schottky group G of Ford type are constructed as stated above. If ε is taken sufficiently small, the* (-2)*-dimensional Poincaré theta series with respect to G does not converge absolutely and uniformly in D^*.*

§ 3. Measure of Singular Sets of Schottky Groups

15. Given a set $\mathscr{E}$ of points in the z-plane and a positive number δ, we denote by $I(\delta, \mathscr{E})$ a family of a countable number of closed discs U of diameter $l_U \leqq \delta$ such that every point of $\mathscr{E}$ is an interior point of at least one U.

We call the quantity

$$\Lambda^{\eta}\mathscr{E} = \lim_{\delta\to 0}\,[\inf_{\{I(\delta,\mathscr{E})\}} \sum_{U\in I(\delta,\mathscr{E})} l_U^{\eta}\,]$$

the η-dimensional measure of $\mathscr{E}$.

16. Let us consider a Schottky group G generated by p linear transformation in (1).

For two circles with radii R and r in the z-plane, the quantity

$$\frac{(R^2+r^2-\rho^2)^2}{4R^2r^2}-1=K$$

is invariant under any linear transformation, where ρ is the distance of centers of above two circles. There are three cases: (i) K is zero, if they are tangent, (ii) negative, if they intersect themselves and (iii) otherwise, positive. In the third case we obtain

$$R^2+r^2-\rho^2 = \pm 2Rr\sqrt{1+K}, \tag{17}$$

where plus sign is used in the case where a circle is contained in the inside of the other and minus sign in the other case.

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Denoting by $r_j^{(m-1)}$ and by $r_i^{(m)}$ $(i=1, \ldots, 2p-1)$ the radius of the outer boundary circle $C_j^{(m-1)}$ and the radii of $2p-1$ inner boundary circles $C_i^{(m)}$ $(i=1, \ldots, 2p-1)$ of the image B_m of the fundamental domain B_0 by a transformation $S^{(m)}$ $(\in G)$ with grade m, we have

$$(r_j^{(m-1)})^2+(r_i^{(m)})^2-(\rho_i^{(m)})^2=2r_j^{(m-1)}r_i^{(m)}\sqrt{1+K}, \qquad (i=1, \ldots, 2p-1),$$

where $\rho_i^{(m)}$ denotes the distance from the center $C_j^{(m-1)}$ to the center of $C_i^{(m)}$. From this we see that there exists a constant k depending only on B_0 such that

$$\left(\frac{r_i^{(m)}}{r_j^{(m-1)}}\right)^2-2\left(\frac{r_i^{(m)}}{r_j^{(m-1)}}\right)\sqrt{1+k}+1\geqq\left(\frac{\rho_i^{(m)}}{r_j^{(m-1)}}\right)^2.$$

whence follows

$$(18) \qquad r_i^{(m)}\leqq\frac{1}{\sqrt{1+k}+\sqrt{k}}r_j^{(m-1)}, \qquad (i=1, \ldots, 2p-1).$$

Next we shall give an estimate $r_i^{(m)}/r_j^{(m-1)}$ from below. We put

$$S^{(m)}(z)=\frac{az+b}{cz+d}, \qquad (ad-bc=1).$$

From (5), we have

$$k(G)|c|^{-2}\leqq r_i^{(m)}$$

and

$$K(G)|c|^{-2}\geqq r_j^{(m-1)},$$

which imply

$$(19) \qquad \frac{1}{\mu(G)}r_j^{(m-1)}\leqq r_i^{(m)},$$

where $\mu(G)=K(G)/k(G)$ is dependent of B_0.

From (18) and (19) we obtain the following

THEOREM 10. *There exist positive constants* K_0 (<1) *and* k_0 *depending only on* B_0 *such that*

$$k_0r_j^{(m-1)}\leqq r_i^{(m)}\leqq K_0r_j^{(m-1)} \qquad (i=1, \ldots, 2p-1),$$

where $r_j^{(m-1)}$ *and* $r_i^{(m)}(i=1, \ldots, 2p-1)$ *are the radius of the outer boundary circle* $C_j^{(m-1)}$ *and the radii of* $2p-1$ *inner boundary circles* $C_i^{(m)}$ $(i=1, \ldots, 2p-1)$ *of* $B_m=S^{(m)}(B_0)$, $(S^{(m)}\in G)$.

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17. Denote by $\mathfrak{F}_{n_0}$ the family of all closed discs bounded by circles of grade n ($\geqq n_0$). It is easy to see that $\mathfrak{F}_{n_0}$ is a covering of the singular set of G and that the diameter of any disc of $\mathfrak{F}_{n_0}$ is less than a given δ (>0) for a sufficiently large n_0. This fact is verified by Theorem 10.

Consider a family $I(\delta, E)$ of coverings of E stated in No. 15. Since E is compact, the set E is covered by a finite number of discs $\mathfrak{C}_1, \ldots, \mathfrak{C}_k$ of a covering of E in $I(\delta, E)$. Take an arbitrary $\mathfrak{C}_i$ among these k circles and let l_i ($\leqq \delta/2$) be the radius of $\mathfrak{C}_i$.

Let δ be sufficiently small. For a $\mathfrak{C}_i$ fixed, we can find closed discs ${}^iC^{(m_1)}$, $\ldots$, ${}^iC^{(m_{N(i)})}$ in $\bigcup_{n=1}^{\infty} \mathfrak{F}_n$ satisfying following conditions:

(i) The radius ${}^ir^{(m_j)}$ of ${}^iC^{(m_j)}$ ($1 \leqq j \leqq N(i)$) of grade m_j is larger than l_i;

(ii) There exists at least one circle of grade m_j+1 lying inside the boundary of ${}^iC^{(m_j)}$, meeting $\mathfrak{C}_i$ and of radius ${}^ir^{(m_j+1)}$ not greater than l_i;

(iii) $\bigcup_{j=1}^{N(i)} {}^iC^{(m_j)} \supset \mathfrak{C}_i \cap E.$

It is easy to see that there exists a constant κ indepent of i such that $N(i) \leqq \kappa$. We can prove $\kappa = 5$ by some geometrical consideration.

By the preceding theorem,

$$k_0 {}^ir^{(m_j)} \leqq {}^ir^{(m_j+1)} \leqq l_i < {}^ir^{(m_j)}.$$

Construct such discs $\{{}^iC^{(m_j)}\}$ for every $\mathfrak{C}_i$ ($i = 1, \ldots, k$). Then it is obvious that $\bigcup_{i=1}^{k} \bigcup_{j=1}^{N(i)} {}^iC^{(m_j)} \supset E$ and

$$\sum_{i=1}^{k} \sum_{j=1}^{N(i)} ({}^ir^{(m_j)})^{\eta} \leqq \kappa k_0^{-\eta} \sum_{i=1}^{k} l_i^{\eta}.$$

Thus we have

THEOREM 11. *Let $\mathfrak{F}_{n_0}^{\delta/k_0}$ be a covering of E constructed by discs in $\mathfrak{F}_{n_0}$ whose radii are not greater than $\delta/2k_0$ and let r_C be the radius of a disc C in $\mathfrak{F}_{n_0}^{\delta/k_0}$. Then it holds*

$$L^{\eta}E = \lim_{\delta \to 0} \inf_{\{\mathfrak{F}_{n_0}^{\delta/k_0}\}} \sum_{C \in \mathfrak{F}_{n_0}^{\delta/k_0}} (2r_C)^{\eta} \leqq \kappa \left(\frac{k_0}{2}\right)^{-\eta} \Lambda^{\eta} E. \tag{20}$$

By Theorem 11, we can prove

THEOREM 12. *Given a Schottky group G, if*

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(21) $$\sum_{T_m} (R_{S^{(m)}})^\nu \geqq (R_{S^{(m-1)}})^\nu, \qquad (0<\nu<4,\ S^{(m)} = T_m S^{(m-1)})$$

for radius $R_{S^{(m-1)}}$ *of any isometric circles* $I_{S^{(m-1)}}$ *of grade* $m-1$ *and radii* $R_{S^{(m)}}$ *of the* $2p-1$ *isometric circles* $I_{S^{(m)}}$ *of grade* m *contained in* $I_{S^{(m-1)}}$, *then the* $\frac{\nu}{2}$ *-dimensional measure of the singular set* E *of* G *is positive.*

Proof. Take a covering $\mathfrak{F}_{n_0}^{\delta/k_0}$ of E constructed by a finite number of discs $C^{(m_1)}, \ldots, C^{(m_N)}$. We assume that $C^{(m_j)}$ is bounded by a circle with grade m_j. Denote by $r^{(m_j)}$ the radius of $C^{(m_j)}$. Then, from (5),

$$\sum_{j=1}^{N} (r^{(m_j)})^{\nu/2} \geqq k(G) \sum_{j=1}^{N} (R^{(m_j)})^\nu,$$

where $R^{(m_j)}$ is the radius of the isometric circle of the transformation $S^{(m_j)}$, of the group, mapping B_0 onto $S^{(m_j)}(B_0)$. Using the properties (III) of isometric circles stated in No. 7 and noting our assumption (21), we get

$$\sum_{j=1}^{N} (R^{(m_j)})^\nu \geqq \sum_{S^{(m_0)}} (R_{S^{(m_0)}})^\nu,$$

where $m_0 = \min_{1\leqq j\leqq N} m_j$ and the summation in the right hand side is taken over all transformations in G with grade m_0. By a similar argument, we see

$$\sum_{S^{(m_0)}} (R_{S^{(m_0)}})^\nu \geqq \sum_{S^{(0)}} (R_{S^{(0)}})^\nu,$$

where $\sum_{S^{(0)}}$ denotes the sum with respect to all generators and their inverses. Here the quantity in the right hand side is a positive constant. Thus, for any covering $\mathfrak{F}_{n_0}^{\delta/k_0}$ of E, we have

$$\sum_{j=1}^{N} (r^{(m_j)})^{\nu/2} \geqq k(G) \sum_{S^{(0)}} (R_{S^{(0)}})^\nu.$$

Putting $\eta = \frac{\nu}{2}$ in (20), we can prove our theorem from the above inequality and Theorem 11.

Noting the example of a Schottky group of Ford type given in No. 13, we have the following theorem from Theorem 12.

THEOREM C. *There exists a Schottky group whose singular set has positive 1-dimensional measure.*

Remark. (i) In the case $p \geqq 3$, Myrberg [6] gave the example where $l_m^{(2)}$

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is monotone increasing. Hence we see by Theorem 11 that the singular sets of such Schottky groups have positive 1-dimensional measure.

(ii) In our example of a Schottky group of Ford type stated in No. 13, if we take a sufficiently small number δ (>0), the condition (20) is satisfied for $\nu=2+\delta$. So the $\left(1+\frac{\delta}{2}\right)$-dimensional measure of the singular set E of this group is positive. Hence E is not a Painlevé null set.

18. In [1] we obtained a sufficient condition for the 1-dimensional measure of the singular set E of a Schottky group to be zero. Here we shall give a more precise form as an application of Proposition 1.

For radii $r_j^{(m-1)}$ of the outer boundary circle $C_j^{(m-1)}$ and radii $r_i^{(m)}$ of the $2p-1$ inner boundary circles $C_i^{(m)}$ $(i=1, \ldots, 2p-1)$ of an image $B_m=S^{(m)}(B_0)$ by $S^{(m)}$, we have proved the inequality (18). Summing up (18) with respect to i from 1 to $2p-1$, we obtain

$$\sum_{i=1}^{2p-1} r_i^{(m)} \leqq \frac{2p-1}{\sqrt{1+k}+\sqrt{k}} r_j^{(m-1)}.$$

Under the assumption

$$\frac{2p-1}{\sqrt{1+k}+\sqrt{k}}<1,$$

the 1-dimensional measure of E is zero from Proposition 1. Hence we have

THEOREM 13. *If*

$$k>\left\{\frac{2p(p-1)}{2p-1}\right\}^2, \qquad (p\geqq 2), \tag{22}$$

the 1-*dimensional measure of the singular set* E *of a Schottky group is zero.*

19. Let us compare our condition with the condition of Schottky [11]: Suppose that $2p-3$ circles $C_1, C_2, \ldots, C_{2p-3}$ can be described so that each C_j is disjoint from each other, C_1 contains two circles of $2p$ boundary circles $\{H_i, H_i'\}_{i=1}^p$ of B_0, C_1 and C_2 surround a domain together with a circle of $\{H_i, H_i'\}_{i=1}^p$ and so on, and finally there are two circles of $\{H_i, H_i'\}_{i=1}^p$ outside C_{2p-3}. Then $\sum_{m=1}^{\infty} \sum_{i=1}^{2p(2p-1)^m} r_i^{(m)}<\infty$. This Schottky theorem implies that, under the same assumption as in the above, the 1-dimensional measure of E is equal to zero. Schottky's condition is geometric, but our condition is quantitative.

If $p=2$, we obtain $k>16/9$ from (22). If we assume that B_0 is bounded by four circles with unit radius, the mutual distances between any two circles are greater than 0.309 . . . by (17). In such a domain the condition of Schottky is always satisfied. But in the case of four circles with unequal radii, there are many examples which satisfy our condition but do not satisfy Schottky's.

Consider two pairs of circles H_1, H_1' and H_2, H_2' with radii 1 and 1/15 respectively. We take the mutual distance between H_1 and H_1' is slightly greater than 0.309 . . . . We see that in general the mutual distance between circles C and C^* with radii 1 and $1/N$ respectively is greater than $k/2(N+1)$ by (17). For $N=15$, it is greater than 0.055 . . . . Draw two common tangents L_1 and L_2 between H_1 and H_1', and let the point of intersection be the origin and further draw H_2 and H_2' near enough the origin such that they intersect L_1 and L_2, and the distance of $\{H_1, H_1'\}$ from $\{H_2, H_2'\}$ is greater than 0.055 . . . . Obviously such a domain B_0 does not satisfy the condition of Schottky.

Remark. (i) In the case of $p=3$, even if we assume that B_0 is bounded by six circles with unit radius, there are many examples which satisfy our condition but do not satisfy Schottky's.

(ii) Our theorem is not necessarily an extension of the Schottky theorem. Because it is easy to get the fundamental domains, which do not satisfy our condition but do Schottky's.

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