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THE AXISYMMETRIC CASE IN HYDROMAGNETICS

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ABSTRACT

Recent work at the Yerkes Observatory has been concerned with the study of configurations in which the magnetic and velocity fields possess a common axis of symmetry. In those cases where the density ρ may be assumed constant, it has proved advantageous to employ a representation suggested by Lüst and Schlüter [1]: in cylindrical co-ordinates (ϖ, ϕ, z) let

$$\mathbf{h}=\frac{\mathbf{H}}{\sqrt{4\pi\rho}}=\varpi T\hat{\phi}+\operatorname{curl}(\varpi P\hat{\phi}),$$

and

$$\mathbf{v}=\varpi V\hat{\phi}+\operatorname{curl}(\varpi U\hat{\phi}),$$

where $\hat{\phi}$ is a unit vector and T, P, V, and U are independent of the azimuthal angle ϕ. The hydrodynamic equation

$$\rho\frac{\partial\mathbf{v}}{dt}+\rho\mathbf{v}.\operatorname{grad}\mathbf{v}=-\operatorname{grad}p+\rho\operatorname{grad}V+\frac{1}{4\pi}(\operatorname{curl}\mathbf{H})\times\mathbf{H},$$

may then be replaced by the pair of equations (cf. Chandrasekhar [2])

$$\varpi^3\frac{\partial V}{\partial t}=\frac{\partial(\varpi^2 T, \varpi^2 P)}{\partial(z,\varpi)}-\frac{\partial(\varpi^2 V, \varpi^2 U)}{\partial(z,\varpi)}, \tag{1}$$

and

$$\varpi\,\Delta_5\frac{\partial U}{\partial t}-\frac{\partial(\Delta_5 P, \varpi^2 P)}{\partial(z,\varpi)}+\frac{\partial(\Delta_5 U, \varpi^2 U)}{\partial(z,\varpi)}=\varpi\frac{\partial T^2}{\partial z}-\varpi\frac{\partial V^2}{\partial z}, \tag{2}$$

where Δ_5 is the Laplacian operator in 5 dimensions. The equation for the magnetic field,

$$\frac{\partial\mathbf{H}}{\partial t}=\operatorname{curl}(\mathbf{v}\times\mathbf{H})-\frac{1}{4\pi\sigma}\operatorname{curl}\operatorname{curl}\mathbf{H},$$

may similarly be replaced by the pair of equations

$$\varpi^3\,\Delta_5 P-\varpi^3\frac{\partial P}{\partial t}=\frac{\partial(\varpi^2 P, \varpi^2 U)}{\partial(z,\varpi)}, \tag{3}$$

and

$$\varpi\,\Delta_5 T-\varpi\frac{\partial T}{\partial t}=\frac{\partial(T, \varpi^2 U)}{\partial(z,\varpi)}-\frac{\partial(V, \varpi^2 P)}{\partial(z,\varpi)}. \tag{4}$$

Three classes of problems have been studied with the aid of these equations: first, the equilibrium of a mass of fluid of infinite electrical conductivity; second, the decay time of a magnetic field in the presence of fluid motions; third, the stability of static equilibrium configurations. Problems in the first category are governed by equations (1) and (2), with $\partial/\partial t \equiv 0$. If in addition the velocity is zero, these equations possess the two general integrals (cf. Chandrasekhar and Prendergast [3])

$$\varpi^2 T = F(\varpi^2 P),$$

and

$$\Delta_5 P = -T \frac{d}{d(\varpi^2 P)} (\varpi^2 T) + \Phi(\varpi^2 P), \tag{5}$$

where F and Φ are arbitrary. Eq. (5) with $\Phi \equiv 0$, and $F(\varpi^2 P) = \alpha\varpi^2 P$, $\alpha =$ constant gives rise to a class of force-free fields, one example of which was studied by Lüst and Schlüter. The general solution for this case has been given by Chandrasekhar [4]. It has also been shown by Prendergast [5] that the case $F(\varpi^2 P) = \alpha\varpi^2 P$, $\Phi = \kappa$ determines a spherical equilibrium configuration in which the magnetic forces do not vanish. More recently an equilibrium solution embodying a toroidal velocity field has been obtained by Sykes [6].

For an assigned velocity field (V, U) Eqs. (3) and (4) govern the decay of a magnetic field in a fluid conductor. Cowling's [7] theorem on the impossibility of a self-excited axisymmetric dynamo with a purely poloidal field follows from Eq. (3); the analogous result for a toroidal dynamo has been proved from Eq. (4) by Backus and Chandrasekhar [8]. The problem of the prolongation of decay times by means of suitable patterns of fluid motion has been investigated by Chandrasekhar [9], who has obtained results which are valid for small velocities. An upper limit to the decay time of the external field of a sphere in which arbitrarily strong fluid motions are present has recently been derived by Backus [10]. He finds that in no case can the decay time be greater than four times that which would prevail in the absence of internal motions.

Stability problems have been treated using the linearized versions of Eqs. (1) to (4). The stability of Prendergast's equilibrium sphere has been investigated by this method and a variational principle has been obtained for the frequencies of oscillation. The stability of a force-free field in a compressible medium has been studied by S. K. Trehan [11], using an extension of the same technique.

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Spitzer: As I shall report on Wednesday, I have obtained some exact results for the axisymmetric case which seem to suggest that fluid motions have relatively minor effect on the decay rate of a dipole field. In view of this result I would like to question how certain you are of the convergence of the series which you obtain in your deductions. Is it not possible that the higher modes would seriously modify the results?

Prendergast: Yes, this is quite true. The convergence of this series is a rather difficult question. It seems to be established that one can prolong the decay time in the axisymmetric case by a factor of 4 at least. Whether one can prolong it by a factor of 15 is extremely doubtful.

Blackett: I want to make two remarks about the use of the results of the rock magnetic data. If the reversely magnetized rocks do indicate a reverse field then the fluid dynamo producing the field must have properties like a flip-flop circuit with two positions of stability characterized by equal currents in opposite directions. The currents remain constant for a time of the order of 1 million years and then suddenly change over a period of some 10^4 years. However, one must not be too certain as yet that the earth's field has reversed. The possibilities of physical-chemical reversal are so numerous and complex that it is difficult to rule them out for certain. The crucial test of tracing a given reversal at a given moment of geological time over the major continents has not yet been achieved. Till this has been done, I think it would be wise to assume that reversals of the earth's field, though probable, have not finally been proved.

Prendergast: It is, of course, realized that there may be some resemblance with a flip-flop circuit. The discussion of the decay time here leans on considerations which are valid also when one has a flip-flop. The decay may go on for some time until the field gets weak, and then something happens, i.e. in the form of a reversal. On the other hand, one has not as yet found any mechanism which describes the reversal.

Schlüter: A special case of the static equilibrium models considered in the paper presented here is that where the magnetic field does not act at all on the matter ('force-free' magnetic fields). The corresponding equations have been generally solved by R. Lüst and A. Schlüter, if axial symmetry and proportionality between electrical current and magnetic field are supposed. If one restricts the geometry even more, namely to the symmetry of an infinite cylinder (all lines of force and of electric current form co-axial helices), then also the non-linear case can be solved generally. This model of a field can serve to demonstrate that force-free fields exist, which can transport along an axis of symmetry angular momentum around this axis.

Mestel: I wish to emphasize the seriousness of the assumption made here and in other recent papers that the material is incompressible. The equations of hydrostatic equilibrium then restrict the magnetic field to a certain shape; otherwise mass motions are set up. A similar result holds if the density is not constant but a function of pressure only. However, in a real non-degenerate star, an arbitrary magnetic field can be balanced by non-spherical variations in temperature as well as in pressure. In a convectively stable region the pressure field reacts on the meridional magnetic field not through the equation of support, but through the energy equation. As long as the magnetic field is not too large, this reaction is small, and so the star can support an arbitrary imposed field. Hence the results for barotropic and perfect-gas stars are *qualitatively* quite different, and so the barotropic or incompressible approximation is misleading. Whether the temperature field can adjust itself to support an arbitrary magnetic field depends on the rate at which the field is imposed. A proto-star condensing in the Kelvin–Helmholtz time scale, for example, will automatically adjust its density and temperature fields to balance the magnetic force. On the other hand a rapidly imposed field will lead to density variations accompanied by adiabatic temperature changes, so that the resulting pressure field will not in general balance the magnetic force. The resulting mass-motions and the distortion of its field are difficult to compute; but certainly assumptions of incompressibility or barotropy are again misleading.