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WEAK L- SPACES

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In this paper, semi-weak L-spaces and weak L-spaces (which are generalisations of Lindelöf spaces) are introduced and studied .

1. INTRODUCTION

The Jordan curve theorem ([4]) is one of the classical theorem of mathematics. Making abstracts of the properties of this theorem, Michael [5] introduced and studied the J-space. *A space X is a J-space if, whenever $\{A, B\}$ is a closed cover of X with $A \cap B$ compact, then A or B is compact.* A compact space is a J-space, but not conversely. In the definition of the J-space, "A or B is compact" cannot be weakened to "A or B is Lindelöf". In [2], the L-space is introduced and studied which generalised the J-space. *A space X is an L-space if, whenever $\{A, B\}$ is a closed cover of X with $A \cap B$ compact, then A or B is Lindelöf.* J-spaces are L-spaces, but not conversely. The real line $\mathbb{R}$ is such an example.

In this note, we introduce and study semi-weak L-spaces and weak L-spaces which contain the class of L-spaces. This study generalised and enriched Michael's study in [5].

Throughout the note, spaces are Hausdorff. A space X is Lindelöf if every open cover of X has a countable subcover. All maps are continuous. The first uncountable ordinal is denoted by ω_1.

Recall that a map $f : X \to Y$ is *monotone* if all fibres $f^{-1}(y)$ are connected and a map $f : X \to Y$ is *boundary-perfect* ([5]) if f is closed and the boundary of $f^{-1}(y)$ is compact for any $y \in Y$. *The long line* Z is the space $Z = [0, \omega_1) \times [0, 1)$ with the order topology generated by the lexicographical order. Clearly Z is non-Lindelöf, locally compact, countably compact and connected. $Z^* = Z \cup \{\omega_1\}$ is called *the extended long line* (that is, for any $z \in Z, z < \omega_1$ and Z^* with the order topology, equivalently, Z^* is the one-point compactification of Z) (see [7]).

For a subset A of the space X, we reserve ∂A and A° for the boundary and interior of A respectively. $\mathbb{R}$ is the set of all real numbers, $\mathbb{Z}^+$ is the set of all non-negative integers, I is the usual closed unit interval $[0, 1]$, $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geqslant 0\}$ and $\mathbb{R}^- = \{x \in \mathbb{R} : x \leqslant 0\}$. For other terms and symbols see [1].

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2. Properties

Definition 1: A space X is a semi-weak L-space if, whenever A and B are disjoint closed subsets of X with ∂A and ∂B compact, then A or B is Lindelöf.

Definition 2: A space X is a weak L-space if, whenever $\{A, B, K\}$ is a closed cover of X with K compact and $A \cap B = \emptyset$, then A or B is Lindelöf.

Proposition 1.

(1) *A semi-weak L-space X is a weak L-space, but not conversely;*

(2) *Let $A \subset X$ be closed and ∂A compact. If X is a semi-weak L-space, so is A.*

Proof: (1) Let $\{A, B, E\}$ be a closed cover of X with K compact and $A \cap B = \emptyset$. Then the closed subsets ∂A and ∂B of K are compact. Since X is a semi-weak L-space, A or B is Lindelöf. By Example 3, the converse is false.

(2) Let F, B be disjoint closed subsets of A with compact boundaries in A, then F and B are closed in X. Noticing that

$$\partial F = \overline{F} \cap \overline{X - F} \subset \overline{F} \cap (\overline{X - A}) \cup (\overline{F} \cap (\overline{A - F}) \subset \partial(A) \cup (\partial F)_A,$$

where $(\partial F)_A$ is the boundary of F in A, we have that ∂F is compact. Similarly, ∂B is compact. Hence F or B is Lindelöf. □

Clearly, a Lindelöf space is a semi-weak L-space. The Example 1 shows that the converse is not true. Proposition 1(2) is not true for weak L-spaces (see Example 3(3)).

Proposition 2. *Let $\{X_1, X_2\}$ be a closed cover of X with X_2 Lindelöf. If X_1 is a (semi-)weak L-space, so is X.*

Proof: If X_1 is a semi-weak L-space, let A, B be disjoint closed subsets of X with ∂A, ∂B compact. Put $A_1 = A \cap X_1$, $B_1 = B \cap X_1$. Then $A_1 \cap B_1 = \emptyset$ and ∂A_1, ∂B_1 compact and so A_1 or B_1 is Lindelöf. Hence A or B is Lindelöf. Thus X is a semi-weak L-space. If X_1 is a weak L-space, let $\{A, B, E\}$ be a closed cover of X with $A \cap B = \emptyset$ and E compact. Since $\{A \cap X_1, B \cap X_1, E \cap X_1\}$ is a closed cover of X_1, $A \cap X_1$ or $B \cap X_1$ is Lindelöf and thus A or B is Lindelöf. So X is a weak L-space. □

Corollary 1. *Let $X = E \cup O$ with O open in X and $\overline{O}$ compact. If E is a semi-weak L-space, so is X.*

Proof: Note that the closed $A = X \backslash O \subset E$ has a compact boundary in X and thus in E, so A is a semi-weak L-space by Proposition 1(2). The closed cover $\{A, \overline{O}\}$ of X satisfies the condition of Proposition 2, so X is a semi-weak L-space. □

Corollary 2. *Let the closed cover $\{X_1, X_2, K\}$ of X be with $X_1 \cap X_2 = \emptyset$ and K compact. Then the following are equivalent.*

(1) *X is a (semi-)weak L-space;*

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(2) *One of X_1 and X_2 is Lindelöf and the other is a (semi-)weak L-space.*

PROOF: Suppose that X is a weak L-space. (2)$\Rightarrow$ (1) is by Proposition 2. (1)$\Rightarrow$ (2). Suppose (1), and let X_1 be Lindelöf, $\{A, B, W\}$ a closed cover of X_2 with $A \cap B = \emptyset$ and W compact. Then the closed cover $\{A \cup X_1, B, W \cup K\}$ of X satisfies that $A \cup X_1$ or B is Lindelöf. Thus A or B is Lindelöf and (2) holds. Now suppose that X is a semi-weak L-space. Noticing that $\partial(X_1)$, $\partial(X_2) \subset K$ are compact, (1)$\Leftrightarrow$(2) is obvious by Propositions 1 and 2. ∎

PROPOSITION 3. *Let $\{X_1, X_2\}$ be a closed cover of X with $X_1 \cap X_2$ non-Lindelöf. If X_1 and X_2 are weak L-spaces, so is X.*

Proposition 3 is not true for semi-weak L-spaces (see Example 3(1), (2)).

PROPOSITION 4. *The following are equivalent for a space X.*

(1) *X is a semi-weak L-space.*

(2) *If $f : X \to Y$ is boundary-perfect, then $f^{-1}(y)$ is non-Lindelöf for at most one $y \in Y$.*

COROLLARY 3. *If f is a closed map from a paracompact semi-weak L-space X onto a q-space Y, then $f^{-1}(y)$ is non-Lindelöf for at most one $y \in Y$.*

PROOF: This follows from Proposition 4 since every closed map $f : X \to Y$ from a paracompact space X on to a q-space Y is boundary-perfect (see [**6**]). ∎

PROPOSITION 5. *Let $f : X \to Y$ be a perfect map onto Y. If X is a (semi-)weak L-space, so is Y. The converse is not true.*

PROOF: If Y is a weak L-space, let $\{A, B, K\}$ be a closed cover of Y with $A \cap B = \emptyset$ and K compact. Since $\{f^{-1}(A), f^{-1}(B), f^{-1}(K)\}$ is a closed cover of X and $f^{-1}(A) \cap f^{-1}(B) = \emptyset$ and $f^{-1}(K)$ is compact, $f^{-1}(A)$ or $f^{-1}(B)$ is Lindelöf. Hence A or B is Lindelöf. If Y is a semi-weak L-space, let A, B be disjoint closed subsets of Y with compact boundaries. Then $f^{-1}(A) \cap f^{-1}(B) = \emptyset$. Since $\partial\big(f^{-1}(A)\big) \subset f^{-1}(\partial A)$ and $f^{-1}(\partial A)$ is compact, $\partial\big(f^{-1}(A)\big)$ is compact. Similarly, $\partial\big(f^{-1}(B)\big)$ is compact. Thus $f^{-1}(A)$ or $f^{-1}(B)$ is Lindelöf and so A or B is Lindelöf. In Example 2, f is a monotone perfect map and Y is a semi-weak L-space, but X is not a weak L-space. So the converse is false. ∎

In [**5**], the following two classes of spaces are defined and studied.

A space X is ***a semi-weak J-space*** if, whenever A and B are disjoint closed subsets of X with compact boundaries, then A or B is compact. A space X is ***a weak J-space*** if, whenever $\{A, B, K\}$ is a closed cover of X with K compact and $A \cap B = \emptyset$, then A or B is compact.

Clearly, a semi-weak J-space is a semi-weak L-space and a weak J-space is a weak L-space, but the converses are not true (see Theorem 1).

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Proposition 6. ([5]) *Suppose that X is a J-space and $Y = X \cup \{y_0\}$. Then Y is a semi-weak J-space.*

Proposition 7. *Suppose that X is an L-space and $Y = X \cup \{y_0\}$. Then Y is a semi-weak L-space.*

Proof: By modifying the proof of Proposition 6. ∎

Proposition 8. *If X is a connected L-space (a connected J-space), then the quotient space $Q = (X \times I)/(X \times \{1\})$ is a semi-weak L-space (a semi-weak J-space).*

Proof: Denote by y_0 the point $X \times \{1\}$ of Q, then the space Q can be represented as $\big(X \times [0,1)\big) \cup \{y_0\}$.

Suppose that X is a connected L-space. If X is compact, then the projection $f : X \times [0,1) \to [0,1)$ is perfect. For any closed cover $\{A, B\}$ of $X \times [0,1)$ with $A \cap B$ compact, $f(A)$ is closed and Lindelöf since $[0,1)$ is Lindelöf. So $f^{-1}\big(f(A)\big)$ is Lindelöf and thus A is Lindelöf. This shows that $X \times [0,1)$ is an L-space. If X is not compact, then by [**5**, Proposition 2.5], $X \times [0,1)$ is a J-space, hence an L-space. By Proposition 7, Q is a semi-weak L-space.

Suppose that X is a connected J-space. Since $\mathbb{R}^+$ is a J-space ([**5**, Proposition 2.4]), $[0,1)$ is a J-space. By [**5**, Corollary 5.8(d)] the product $X \times [0,1)$ of two connected J-spaces is a J-space. So by Proposition 6, Q is a semi-weak J-space. ∎

It is showed that $J \Rightarrow$ semi-weak $J \Rightarrow$ weak J, but the converses are false; in locally compact spaces, the three properties coincide (see [**5**]).

Theorem 1. *Suppose that X is a space and*

(C) *X is an L-space;*	(c) *X is a J-space;*
(D) *X is a semi-weak L-space;*	(d) *X is a semi-weak J-space;*
(E) *X is a weak L-space;*	(e) *X is a weak J-space.*

Then

(1) (C)⇒(D)⇒(E), (c)⇒(C), (d)⇒(D), (e)⇒(E), *but not conversely;*

(2) *the six properties are not productive (respectively not additive, preserved by quotient maps);*

(3) *if X is locally compact, then* (C)⇔(D)⇔ (E)*;*

(4) *if X is countably compact, then* (C)⇔(c), (D)⇔(d) *and* (E)⇔(e).

Proof: (1) (C)⇒ (D): let A, B be disjoint, closed subsets of X with compact boundaries, then $\{A, \overline{X \setminus A}\}$ is a closed cover of X with $A \cap \overline{X \setminus A}$ compact. By (C), A or B is Lindelöf and thus (D) holds. (D)⇒ (E) is by Proposition 1. (c)⇒ (C), (d)⇒ (D) and (e)⇒ (E) are obvious.

(D)⇏ (C) is by Example 1, (E)⇏ (D) is by Example 3.

The real line $X = \mathbb{R}$ is Lindelöf, so it satisfies (C), (D) and (E). But X is not a weak J-space, thus X does not satisfy (e), (d), (c).

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(2) Not productive: let $X = \{0,1\} \times Z$. Clearly, $\{0,1\}$ is a J-space. The lone line Z is a J-space (in fact, let $\{A,B\}$ be a closed cover of Z with $A \cap B$ compact, then $A \cap B \subset \big[\langle 0,0\rangle, \langle \alpha, 0\rangle\big]$ for some $\alpha \in [0,\omega_1)$ since the compact $A \cap B$ is bounded. Put $K\big[\langle 0,0\rangle, \langle \alpha,0\rangle\big]$. Noticing that $Z \setminus K$ is connected, we have $A \subset K$ or $B \subset K$ and thus A or B is compact because K is compact). Put $A = \{0\} \times Z, B = \{1\} \times Z$. Since Z is not Lindelöf, for the closed cover $\{A, B, \emptyset\}$ of X neither A nor B is Lindelöf, so X is not a weak L-space.

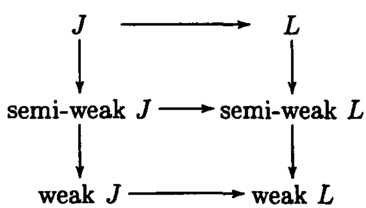
Not additive: The topological sum $Z \oplus Z$ of two J-spaces is not a weak L-space.

Not preserved by the quotient map: the space P in Example 4 is a J-space, but the quotient space Q is not a weak L-space.

(3) Let X be locally compact. By modifying the proof of (e)$\Rightarrow$ (c) in [**5**], we have (E)$\Rightarrow$ (C). Then by (1), (C)$\Leftrightarrow$ (D)$\Leftrightarrow$ (E)

(4) Note that in a countably compact space, Lindelöfness$\Leftrightarrow$ compactness. ∎

To be clear at a glance, we give the following diagram, note that none of the implications is reversible.



3. Examples

Example 1. A semi-weak L-space Y which is not an L-space (so not Lindelöf).

Proof: Let $X = \mathbb{R} \times Z$ and $T = \mathbb{R} \times Z^*$, where $\mathbb{R}$ is the real line, Z the long line and Z^* the extended long line. By [**5**, Proposition 2.5], X is a J-space. The subspace $Y = X \cup \big\{\langle 0, \omega_1\rangle\big\}$ of T is a semi-weak J-space by Proposition 6, so a semi-weak L-space. Put $A = \big\{\langle r,m\rangle \in Y : r \leqslant 0\big\}$ and $B = \big\{\langle r,m\rangle \in Y : r \geqslant 0\big\}$, the $\{A,B\}$ is a closed cover of Y with $A \cap B$ compact, but neither A nor B is not Lindelöf. ∎

Example 2. A space X which is not a weak L-space, whose image Y under a monotone perfect map is a semi-weak J-space (so a semi-weak L-space).

Proof: Let $X = (\mathbb{R} \times Z) \cup \big([-1,1] \times \{\omega_1\}\big)$ be the subspace of $\mathbb{R} \times Z^*$,

$$A = \big\{\langle r,m\rangle \in X : r \leqslant -1\big\},$$
$$B = \big\{\langle r,m\rangle \in X : r \geqslant 1\big\} \text{ and}$$
$$E = [-1,1] \times Z^*.$$

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Then $\{A, B, E\}$ is a closed cover X with $A \cap B = \emptyset$ and E compact, but neither A nor B is Lindelöf. So X is not a weak L-space. Since $\mathbb{R} \times Z$ is a J-space, the subspace $Y = (\mathbb{R} \times Z) \cup \{\langle 0, \omega_1 \rangle\}$ of X is a semi-weak J-space by Proposition 6.

Now we define $f : X \to Y$ as follows. If $\langle r, m \rangle \in A$, then $f(\langle r, m \rangle) = \langle r+1, m \rangle$; if $\langle r, m \rangle \in B$, then $f(\langle r, m \rangle) = \langle r-1, m \rangle$; if $\langle r, m \rangle \in E$, then $f(\langle r, m \rangle) = \langle 0, m \rangle$. It is easy to see that f is a monotone perfect map. ∎

The following example shows that, adding two points to a J-space (respectively an L-space) may not result in a semi-weak J-space (respectively a semi-weak L-space) (compare it with Propositions 6 and 7).

EXAMPLE 3. A weak L-space Y such that

(1) Y has a closed cover $\{Y_1, Y_2\}$ by semi-weak L-spaces Y_1 and Y_2 with $Y_1 \cap Y_2$ non-Lindelöf;

(2) Y is not a semi-weak L-space;

(3) Y has a closed subset F with ∂F compact so that F is not a weak L-space.

PROOF: (1) Put $X = \mathbb{R} \times Z$. Let $Y = X \cup \{\langle -1, \omega_1 \rangle, \langle 1, \omega_1 \rangle\}$ be the subspace of $\mathbb{R} \times Z^*$,

$$Y_1 = (\mathbb{R}^- \times Z) \cup \{\langle -1, \omega_1 \rangle\} \quad \text{and} \quad Y_2 = (\mathbb{R}^+ \times Z) \cup \{\langle 1, \omega_1 \rangle\}.$$

Then $\{Y_1, Y_2\}$ is a closed cover of Y, and $Y_1 \cap Y_2 = \{0\} \times Z$ is not Lindelöf. Since $\mathbb{R}^- \times Z$ and $\mathbb{R}^+ \times Z$ are J-spaces, Y_1, Y_2 are semi-weak J-spaces by Proposition 6 and thus are semi-weak L-spaces, Y is a weak L-space by Propositions 1 and 3.

(2) Put

$$A = \{\langle r, m \rangle \in Y : r \leqslant -1\},$$
$$B = \{\langle r, m \rangle \in Y : r \geqslant 1\}.$$

Then A, B are disjoint, closed subsets of Y with ∂A, ∂B compact, but neither A nor B is Lindelöf. So Y is not a semi-weak L-space.

(3) Put $F = A \cup B$, then F is a closed subset of Y with $\partial F = (\{-1\} \times Z^*) \cup (\{1\} \times Z^*)$ compact, but F is not a weak L-space. ∎

Let $X = \mathbb{R}^2$ be the "bow-tie" space, that is, it has a topology so that a neighbourhood of a point $\langle s, t \rangle \in X$ is the "bow-tie":

$$\{\langle s, t \rangle\} \cup \Big\{\langle s', t' \rangle : 0 < |s - s'| < \varepsilon \ \text{ and } \ \big|(t' - t)/(s' - s)\big| < \delta\Big\},$$

where $\varepsilon > 0$ and $\delta > 0$ can vary (see [**3**]).

EXAMPLE 4. A J-space P whose quotient space Q is not a weak L-space.

PROOF: First we show that the bow-tie space X has a subspace Q which is not a weak L-space. Put

$$C = \{\langle x,y\rangle : x+y < -1, x < -1 \text{ and } y \geqslant 0\} \cup \{\langle -1,0\rangle\},$$
$$D = \{\langle x,y\rangle : x-y > 1, x > 1 \text{ and } y \geqslant 0\} \cup \{\langle 1,0\rangle\} \text{ and}$$
$$E = [-1,1] \times \{0\}.$$

Let $Q = C \cup D \cup E$ be the subspace of X. Then the closed cover $\{C, D, E\}$ of Q is with $C \cap D = \emptyset$ and E compact. Take $x_0 < -1$ and $c < d$ such that the closed non-Lindelöf $\{x_0\} \times [c,d] \subset C$, hence C is not Lindelöf. Similarly, D is not Lindelöf.

Now we show that C and D are connected, and thus Q is connected.

Let us show that C is connected. Assume $C = A_1 \cup B_1$ is with A_1, B_1 closed, $A_1 \cap B_1 = \emptyset$, $A_1 \neq \emptyset$ and $B_1 \neq \emptyset$. For any $y \in \mathbb{R}^+$, since $R_y = \{\langle x,y\rangle : \langle x,y\rangle \in C\}$ is connected, we have $R_y \subset A_1$ or $R_y \subset B_1$. Take $\langle x_1,y_1\rangle \in A_1$, $\langle x_2,y_2\rangle \in B_1$. Then $y_1 \neq y_2$. Without loss of generality, let $y_1 < y_2$. Put

$$H = \{y \in \mathbb{R}^+ : R_y \subset A_1, y < y_2\},$$

then $y_1 \in H$. Let $y_0 = sup\ H$, then $R_{y_0} \subset A_1$ or $R_{y_0} \subset B_1$. If $R_{y_0} \subset A_1$, then $y_0 < y_2$ and for any $y_2 > y > y_0$, $R_y \subset B_1$. So for any $z \in R_{y_0}$, any neighbourhood U_z of z, $U_z \cap R_y \neq \emptyset$ for some $y_2 > y > y_0$. So $U_z \cap B_1 \neq \emptyset$. Since $\overline{B_1} = B_1$, $z \in B_1$ and thus $R_{y_0} \subset A_1 \cap B_1$. A contradiction. If $R_{y_0} \subset B_1$, we can similarly show that $R_{y_0} \subset A_1 \cap B_1$ and a contradiction arises again, thus C is connected. Similarly, D is connected. So Q is connected.

Put $P = Q \times \mathbb{R}$. Then by Proposition 2.5 of [5], P is a J-space. Then the projection $p : P \to Q$ is a quotient map and Q is the quotient space. ∎

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