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ON THE RELATION BETWEEN $[f, g]$ AND A_λ SUMMABILITY

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1. First we will briefly define the $[f, g]$ and A_λ summability methods. Let $K = \{w : |w| < 1\}$. T. H. Gronwall [**3**] introduced a general class of summability methods each of which involves a pair of functions f and g with the following properties. The function $z = f(w)$ is analytic on $\bar{K}\backslash\{1\}$, continuous and univalent on $\bar{K}$, with $f(0) = 0$, $f(1) = 1$, $|f(w)| < 1$ if $w \in K$. The inverse function $w = f^{-1}(z)$ is analytic on $f(K)\backslash\{1\}$, and at $z = 1$

$$(1.1) \qquad w = f^{-1}(z) = 1 - (1-z)^\gamma[a + a_1(1-z) + \ldots]$$

where $\gamma \geqq 1$, $a > 0$, and the quantity in brackets is a power series in $1 - z$ with positive radius of convergence. The function g is of the form $g(w) = (1-w)^{-\alpha} + k(w)$ where $\alpha > 0$, $k(w)$ is analytic in $\bar{K}$, and we assume $g(w) \neq 0$ for $w \in K$. The function g has the expansion $g(w) = \sum b_n w^n$, $b_n \neq 0$, $n = 0, 1, \ldots$. (Any sum $\sum$ without limits means $\sum_0^\infty$.)

The Gronwall or $[f, g]$ transform of a series $\sum u_n$ is the sequence U_n defined implicitly by the formal power series identity

$$(1.2) \qquad g(w)\sum u_n[f(w)]^n = \sum b_n U_n w^n.$$

If $\lim U_n = s$ then $\sum u_n$ is said to be $[f, g]$ summable to s.

When f and g satisfy Gronwall's conditions the resulting summability method is regular. $[f, g]$ includes as special cases the Cesaro, Euler-Knopp and de la Vallee Poussin methods, and in [**2**] it is proved that a generalized Cesaro method due to D. Borwein is also essentially an $[f, g]$ method.

An interesting class of $[f, g]$ means is given by taking $g(w) = (1-w)^{-\alpha}$, $\alpha > 0$ and

$$(1.3) \qquad f(w) = \frac{a[1-(1-w)^\beta]}{a + (1-a)(1-w)^\beta}, \quad 0 < a \leqslant 1, 0 < \beta \leqslant 1.$$

If we take $\alpha = \beta = 1$ we get the Euler-Knopp means while $\alpha = \beta = a = 1/2$ and $\beta = a = 1$, $\alpha \geqq 1$ give respectively de la Vallee Poussin and $(C, \alpha - 1)$ summability.

The identity (1.2) implies a relation of the form

$$U_n = \sum_{k=0}^{n} a_{nk} s_k$$

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where the s_k are the partial sums of $\sum u_k$ and the a_{nk} depend on the power series coefficients of f and g. It is usually very difficult to obtain a tractable expression for the matrix elements a_{nk}. Consequently, information about general $[f, g]$ methods must normally be obtained by function-theoretic techniques using the properties of f and g and the fundamental identity (1.2).

D. Borwein [**1**] defined the A_λ family of summability methods. If $\lambda > -1$ then the sequence s_n is A_λ summable to s if

$$\lim_{x\to 1} (1 - x)^{\lambda+1} \sum \binom{n+\lambda}{n} s_n x^n = s.$$

We will replace the real variable x by a complex variable z. Given a sequence s_n and $\lambda > -1$, set

$$(1.4) \qquad \phi_\lambda(z) = (1 - z)^{\lambda+1} \sum \binom{n+\lambda}{n} s_n z^n.$$

If $\phi_\lambda(z) \to s$ as $z \to 1$ within a Stolz angle then we say s_n is A_λ summable to s. Borwein proved that $A_\mu \subset A_\lambda$ if $-1 < \lambda < \mu$ and that $s_n \to s(A_\lambda)$ if and only if $s_{n+1} \to s(A_\lambda)$. These results hold true for the complex definition of A_λ summability.

If s_n is the sequence of partial sums of $\sum u_n$ then (1.2) is equivalent to

$$(1.5) \qquad g(w)[1 - f(w)] \sum s_n[f(w)]^n = \sum b_n U_n w^n,$$

and from (1.4) this last can be written

$$(1.6) \qquad g(w)\phi_0[f(w)] = \sum b_n U_n w^n.$$

We will use (1.6) throughout this note.

2. D. Borwein obtained the inclusion $(C, \alpha) \subset A_\lambda$ for $\lambda > -1$ and $\alpha > -1$. That is, any sequence s_n that is (C, α) summable to s for some $\alpha > -1$ is also A_λ summable to s for every $\lambda > -1$. On the other hand, T. H. Gronwall proved that if s_n is $[f, g]$ summable to s then $\phi_0[f(w)] \to s$ as $w \to 1$ within a Stolz angle. We will give a qualified extension of these results to an inclusion theorem for $[f, g]$ and A_λ methods.

If we define $Q(w)$ by

$$(2.1) \qquad f(w) = 1 - (1 - w)^\beta Q(w)$$

where $\beta = 1/\gamma$ (γ is the exponent in (1.1)) then $Q(w)$ is analytic in $|w| < 1$ and by (1.1) it follows that if $w \to 1$ inside a Stolz angle, then $Q(w) \to Q(1) \neq 0$.

Let $B(w) = \beta Q - (1 - w)Q'$. By differentiating (2.1) it follows that

$$\frac{1}{f'} = \frac{(1 - w)^{-\beta+1}}{B(w)}.$$

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We will require of our functions f that if $w \to 1$ inside a Stolz angle then

(2.2) $$Q^{(n)}(w) = O[(1 - w)^{-n}], \quad n = 1, 2, \ldots$$

and

(2.3) $$\liminf_{w \to 1,\ w \in \mathrm{St}(1)} |B(w)| > 0.$$

In (2.3), St(1) means a Stolz angle at 1.

Of course if (2.2) is true then $B(w) = O(1)$, but the further requirement (2.3) that $B(w)$ be bounded away from zero as $w \to 1$ inside a Stolz angle will be important. We note that the functions defined by (1.3) satisfy these conditions. For these functions we have

$$Q(w) = [a + (1 - a)(1 - w)^\beta]^{-1}.$$

In the hope of preventing confusion we make a few remarks about notation. Firstly we will frequently suppress the independent variable. For example we often write $\phi(f)$ instead of $\phi[f(w)]$. Secondly it should be pointed out that $\phi^{(n)}(f)$ denotes the nth derivative of ϕ evaluated at $f(w)$, while $[\phi(f)]^{(n)}$ means the nth derivative of $\phi \circ f$.

In this section we will prove

THEOREM 1. *Suppose that s_n is $[f, g]$ summable to s and that* (2.2) *and* (2.3) *are satisfied. If $\lambda > -1$ then $\phi_\lambda[f(w)] \to s$ as $w \to 1$ within a Stolz angle.*

Theorem 1 gives a qualified extension of both Borwein's and Gronwall's results. The theorem also holds for Euler-Knopp and de la Vallee Poussin means since these are special $[f, g]$ means of the type (1.3) as mentioned in the introduction, and the functions (1.3) satisfy the conditions of the theorem.

The proof of Theorem 1 is accomplished by a series of lemmas.

LEMMA 1. *If* (2.2) *holds and p, q are any two positive integers then $(1 - w)^p B^{(p)} B^q = O(1)$ as $w \to 1$ inside a Stolz angle.*

Proof. Firstly if $w \to 1$ in a Stolz angle then $Q = O(1)$ and $(1 - w)Q' = O(1)$ so $B(w) = O(1)$. An easy induction gives

$$B^{(p)} = (\beta - p)Q^{(p)} + (1 - w)Q^{(p+1)}$$

and hence

$$(1 - w)^p B^{(p)} = (\beta - p)(1 - w)^p Q^{(p)} + (1 - w)^{p+1} Q^{(p+1)} = O(1).$$

LEMMA 2. *If* (2.2) *and* (2.3) *hold then for every $n = 1, 2, \ldots,$*

$$\left(\frac{1}{f'}\right)^{(n)} = O[(1 - w)^{-\beta - n + 1}]$$

as $w \to 1$ inside a Stolz angle.

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Proof. The proof is by induction. We have

$$(2.4)\qquad \frac{1}{f'} = \frac{(1-w)^{-\beta+1}}{B(w)}.$$

Differentiating this equation we find

$$(2.5)\qquad \left(\frac{1}{f'}\right)' = \frac{(1-w)^{-\beta}[(1-\beta)B - (1-w)B']}{B^2}.$$

Now by (2.3) B is bounded away from zero and is $O(1)$. Also $(1-w)B' = O(1)$ by Lemma 1, so Lemma 2 holds when $n = 1$. Now if we differentiate (2.4) n times by the quotient rule for derivatives we will arrive at an expression of the form

$$\left(\frac{1}{f'}\right)^{(n)} = \frac{D_n(w)}{B^{2^n}}$$

where $D_n(w)$ involves powers of B along with derivatives of various orders up to n of the functions $(1-w)^{-\beta+1}$ and B.

We will make this statement precise by proving that

$$(2.6)\qquad \left(\frac{1}{f'}\right)^{(n)} = \frac{(1-w)^{-\beta-n+1}G_n(w)}{B^{2^n}}$$

where $G_n(w)$ is a finite sum of certain terms $G_{ni}(w)$ that are sums and products of B and $(1-w)^pB^{(p)}$ for values of $p \leqq n$. That is,

$$(2.7)\qquad G_n = \sum_i G_{ni}(w)$$

and each G_{ni} is of the form

$$(2.8)\qquad G_{ni} = C_{ni}B^j\,(1-w)^qB^{(q)}\ldots(1-w)^rB^{(r)}.$$

The number of terms in (2.7) depends on n but the exact dependence is not important here. In (2.8) C_{ni} is a constant and the indices $j, q, \ldots, r$ depend on i with $0 \leqq q \leqq r \leqq n$, but again the exact dependence does not concern us. Quantities of the form (2.8) are $O(1)$ by Lemma 1 and hence $G_n = O(1)$. We proceed then to prove the relations (2.6), (2.7) and (2.8) by induction.

Firstly, looking at the expression we found in (2.5) we see that equations (2.6) through (2.8) hold when $n = 1$. Suppose that (2.6) through (2.8) hold when $n = k$. Differentiating and writing $m = 2^k$ for convenience, we find

$$(2.9)\qquad \left(\frac{1}{f'}\right)^{(k+1)} = \frac{(1-w)^{-\beta-k}}{B^{2m}}[(\beta+k-1)B^mG_k + (1-w)G_k'B^m - m(1-w)B'B^{m-1}G_k].$$

The quantity inside the brackets is G_{k+1}. The first and third terms inside the bracket are again of the form (2.7) since G_k is, and they are $O(1)$. For the middle term we need only observe that

$$(1-w)[(1-w)^pB^{(p)}B^q]' = -p(1-w)^pB^{(p)}B^q + (1-w)^{p+1}B^{(p+1)}B^q + q(1-w)^pB^{(p)}(1-w)B'B^{q-1}$$

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and hence $(1 - w)G_k'$ is again of the form (2.7) and is $O(1)$. Thus G_{k+1} is as in (2.7) and is $O(1)$ as $w \to 1$ inside a Stolz angle. This proves the lemma since the denominator in (2.6) is bounded away from zero by hypothesis.

LEMMA 3. *If* (2.2) *is satisfied then for every* $n = 1, 2, \ldots$,

$$[f(1-f)]^{(n)} = O[(1-w)^{\beta-n}].$$

Proof. First we have

$$f^{(n)} = [1 - (1-w)^\beta Q]^{(n)} = \sum_{k=0}^{n} c_{\beta k}\binom{n}{k}(1-w)^{\beta-k}Q^{(n-k)}$$

where $c_{\beta 0} = 1$, $c_{\beta k} = \beta(\beta - 1)\ldots(\beta - k + 1)$. Then rewriting this last expression we have

$$f^{(n)} = (1-w)^{\beta-n}\sum_{k=0}^{n} c_{\beta k}\binom{n}{k}(1-w)^{n-k}Q^{(n-k)} = O[(1-w)^{\beta-n}].$$

The last step follows from (2.2). Now

$$[f(1-f)]^{(n)} = f^{(n)} - (f^2)^{(n)} = f^{(n)} - 2(ff')^{(n-1)},$$

$$\begin{aligned}(ff')^{(n-1)} &= \sum_{k=0}^{n-1}\binom{n-1}{k}f^{(k)}(f')^{(n-1-k)} = \sum_{k=0}^{n-1}\binom{n-1}{k}f^{(k)}f^{(n-k)}\\ &= \sum_{k=0}^{n-1}\binom{n-1}{k}(1+w)^{\beta-k}(1-w)^{\beta-n+k}O(1)\\ &= O[(1-w)^{\beta-n}].\end{aligned}$$

This proves Lemma 3.

We define the function $P(w)$ by $P(w) = f(1-f)/\lambda f'$, $\lambda \neq 0$. $P(w)$ is analytic in $|w| < 1$ because f' does not vanish there ($f(w)$ is univalent).

LEMMA 4. *If* $w \to 1$ *inside a Stolz angle and* (2.2), (2.3) *hold then for* $m = 1, 2, \ldots$

(2.10) $\quad P^{m-1}P^{(m)} = O(1)$.

Also, independently of (2.2) *and* (2.3) *we have*

(2.11) $\quad (1/g)^{(m)} = O[(1-w)^{\alpha-m}]$.

Proof.

$$P^{(m)} = \frac{1}{\lambda}\sum_{k=0}^{m}\binom{m}{k}[f(1-f)]^{(k)}(1/f')^{(m-k)}.$$

But by Lemmas 2 and 3 we have

$$[f(1-f)]^{(k)} = O[(1-w)^{\beta-k}]$$
$$(1/f')^{(m-k)} = O[(1-w)^{-\beta-m+k+1}].$$

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Hence $P^{(m)} = O[(1-w)^{-m+1}]$. This proves (2.10) since $P = O[(1-w)]$. The proof of (2.11) is not difficult and we omit it.

LEMMA 5. *Set* $H(w) = \sum b_m U_m w^m$. *If* $U_m \to 0$ *then* $(1-w)^n(1/g)^{(n-k)}H^{(k)} = o(1)$ *as* $w \to 1$ *inside a Stolz angle.*

Proof. Given $\epsilon > 0$ choose M so that $|U_m| < \epsilon$ for $m \geqq M$. Then letting $c_{mk} = m(m-1)\ldots(m-k+1)$,

$$|H^{(k)}| \leqslant \left|\sum_{m=k}^{M-1} c_{mk}b_m U_m w^{m-k}\right| + \epsilon \sum_{m=k}^{\infty} c_{mk}|b_m||w|^{m-k}$$

$$= c_k\epsilon|1-|w||^{-\alpha-k} + O(1) \quad \text{as } w \to 1$$

where $c_0 = 1$, $c_k = \alpha(\alpha+1)\ldots(\alpha+k-1)$. Using (2.11) we then have

$$|1-w|^n|(1/g)^{(n-k)}||H^{(k)}| \leqslant c_k\epsilon\left|\frac{1-w}{1-|w|}\right|^{\alpha+k} + o(1).$$

This proves the lemma since $(1-w)/(1-|w|) = O(1)$ as $w \to 1$ inside a Stolz angle.

LEMMA 6. *If* $U_m \to 0$ *and* (2.2) *and* (2.3) *are satisfied then* $P^n[\phi_0(f)]^{(n)} = o(1)$ *for each* $n = 1, 2, \ldots$ *as* $w \to 1$ *inside a Stolz angle.*

Proof. We need only show that $(1-w)^n[\phi_0(f)]^{(n)} = o(1)$. From (1.6) it follows that

$$(1-w)^n[\phi_0(f)]^{(n)} = (1-w)^n \sum_{k=0}^{n} \binom{n}{k}(1/g)^{(n-k)}H^{(k)}.$$

The result follows by Lemma 5.

Proof of Theorem 1. We may take $s = 0$. The case $\lambda = 0$ follows from Gronwall's theorem. If we prove Theorem 1 for each $\lambda = 1, 2 \ldots$ then the result will hold for all $\lambda > -1$ because $A_\lambda \subset A_\mu$ when $-1 < \mu < \lambda$. We suppose then that λ is a natural number. It is not difficult to prove that if λ is a natural number then

(2.12) $\quad \phi_\lambda(f) = \phi_{\lambda-1}(f) + P[\phi_{\lambda-1}(f)]'$,

where P is the function in Lemma 4. It is clear that if we apply (2.12) λ times to ϕ_λ we will reduce ϕ_λ to an expression involving P, ϕ_0, and derivatives of P and ϕ_0. More precisely we will prove by induction on λ that

(2.13) $\quad \phi_\lambda(f) = \phi_0(f) + \Delta_\lambda(w)$,

where $\Delta_\lambda(w)$ is a linear combination of terms having the general forms

(2.14) $\quad [p^{k-1}p^{(k)}]\ldots[p^{n-1}p^{(n)}]\cdot(P')^m\cdot\{P^q[\phi_0(f)]^{(q)}\}$

or

(2.15) $\quad P^q[\phi_0(f)]^{(q)}$.

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Beginning the induction with $\lambda = 1$ we have from (2.12) that

$$\phi_1(f) = \phi_0(f) + P[\phi_0(f)]'.$$

Thus for $\lambda = 1$ we have $\Delta_1(w) = P[\phi_0(f)]'$ which is of the form (2.15). Supposing that our claim is true for $\lambda = r$ we prove it also holds when $\lambda = r + 1$. Applying (2.12) and the induction hypothesis to ϕ_{r+1} we have

$$\begin{aligned}\phi_{r+1}(f) &= \phi_r(f) + P[\phi_r(f)]' \\ &= \phi_0(f) + \Delta_r(w) + P[\phi_0(f) + \Delta_r(w)]' \\ &= \phi_0(f) + \Delta_r(w) + P[\phi_0(f)]' + P[\Delta_r(w)]'.\end{aligned}$$

Now $\Delta_r(w)$ is a linear combination of terms like (2.14) or (2.15) by hypothesis, and $P[\phi_0(f)]'$ is of the form (2.15). We need only show that $P[\Delta_r(w)]'$ is a linear combination of terms like (2.14) or (2.15). An easy but somewhat tedious computation which we omit shows that if we differentiate either (2.14) or (2.15) and then multiply this derivative by P we get a linear combination of terms of the same type. Hence $P[\Delta_r(w)]'$ is of the desired form and so is

$$\Delta_{r+1}(w) = \Delta_r(w) + P[\phi_0(f)]' + P[\Delta_r(w)]'.$$

This completes the induction.

Now by Lemmas 4 and 6, terms like (2.14) and (2.15) are $o(1)$ as $w \to 1$ inside a Stolz angle, hence $\Delta_\lambda(w) = o(1)$. Furthermore, $\phi_0(f) = o(1)$ by Gronwall's theorem. Hence $\phi_\lambda(f) = o(1)$ by (2.13) and the theorem is proved.

3. Let σ_n be the (C, α) mean of s_n, $\alpha > 0$. Otto Szasz [**5**] proved that if s_n is Abel summable then σ_n is also Abel summable. We will extend this result to the complex valued A_λ methods and some $[f, g]$ methods.

THEOREM 2. *Let s_n be A_λ summable to s for some non-negative integer λ. Suppose that $\alpha > 1$, that U_n is the $[f, (1 - w)^{-\alpha}]$ transform of s_n, and that* (2.2) *is satisfied. Then U_n is A_λ summable to s.*

We need various preliminary results before we prove Theorem 2. Let $\psi_\lambda(z)$ be the A_λ transform of U_n. That is,

$$(3.1) \qquad \psi_\lambda(z) = (1 - z)^{\lambda+1} \sum \binom{n + \lambda}{n} U_n z^n.$$

First we will derive an integral formula for ψ_λ. Let C be the path in the complex plane defined by $C: |w| = R$, $|z| < R < 1$. Then from (1.6), since $g(w)$, since $g(w) = (1 - w)^{-\alpha}$ here,

$$(3.2) \qquad U_n = \frac{1}{2\pi i b_n} \int_C \frac{\phi_0[f(w)](1 - w)^{-\lambda} dw}{w^{n+1}}.$$

Then from (3.1) and (3.2) we get after changing the order of integration and summation

$$(3.3)\qquad \psi_\lambda(z) = \frac{(1-z)^{\lambda+1}}{2\pi i}\int_C \frac{\phi_0[f(w)](1-w)^{-\alpha}}{w}\sum \binom{n+\lambda}{n}\frac{1}{b_n}\left(\frac{z}{w}\right)^n dw.$$

Now $b_n = \binom{n+\alpha-1}{n}$ so that

$$\frac{1}{b_n} = (\alpha-1)\int_0^1 r^n(1-r)^{\alpha-2}dr.$$

Consequently,

$$\sum\binom{n+\lambda}{n}\frac{1}{b_n}\left(\frac{z}{w}\right)^n = (\alpha-1)\int_0^1(1-r)^{\alpha-2}\sum\binom{n+\lambda}{n}\left(\frac{rz}{w}\right)^n dr$$

$$= (\alpha-1)\int_0^1(1-r)^{\alpha-2}w^{\lambda+1}(w-rz)^{-\lambda-1}dr.$$

Substituting this last expression back into (3.3) and interchanging the integrals we get

$$(3.4)\qquad \psi_\lambda(z) = (\alpha-1)(1-z)^{\lambda+1}\int_0^1(1-r)^{\alpha-2}I_\lambda(rz)dr$$

where

$$(3.5)\qquad I_\lambda(rz) = \frac{1}{2\pi i}\int_C \phi_0[f(w)](1-w)^{-\alpha}w^\lambda(w-rz)^{-\lambda-1}dw.$$

Now let $\lambda > -1$ be an integer. Since rz lies inside C for every r, $0 \leqq r \leqq 1$, the integral $I_\lambda(rz)$ can be evaluated by residues. That is,

$$(\lambda!)I_\lambda(w) = \{\phi_0[f(w)](1-w)^{-\alpha}w^\lambda\}^{(\lambda)}.$$

We turn now to the problem of evaluating $I_\lambda(rz)$. Let $\phi_{\lambda,k}$ denote the A_λ transform of the sequence s_{n+k}, $n = 0, 1, \ldots$. That is

$$\phi_{\lambda,k}(z) = (1-z)^{\lambda+1}\sum\binom{n+\lambda}{n}s_{n+k}z^n.$$

$\phi_{\lambda 0}$ is of course identical to ϕ_λ. From a result of Borwein $\phi_{\lambda,k}(z)\to s$ if and only if $\phi_\lambda(z)\to s$. It is easy to show that the $\phi_{\lambda,k}$ satisfy

$$(3.6)\qquad \phi_{\lambda,k}'(z) = (\lambda+1)[\phi_{\lambda+1,k+1}(z)-\phi_{\lambda,k}(z)]/(1-z).$$

LEMMA 7. *If* $\lambda > -1$ *is an integer and* (2.2) *is satisfied then*

$$(3.7)\qquad I_\lambda(w) = (1-w)^{-\alpha-\lambda}\sum_{j=0}^{\lambda}A_j(w)\phi_{j,j}[f(w)]$$

where $A_j(w) = O(1)$ *as* $w\to 1$ *inside a Stolz angle.*

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Proof. We will show that the $A_j(w)$ are of the form $G_j(w)/Q^\lambda(w)$ where the $G_j(w)$ are sums and products of w, $(1 - w)$, and $[(1 - w)^m Q^{(m)}]$, $m = 0, 1, \ldots, \lambda$.

If we show this then $A_j(w) = O(1)$ by (2.2). We will proceed by induction. The lemma clearly holds for $\lambda = 0$. We will also directly compute $I_1(w)$ in order to illustrate the statement of the lemma. For $\lambda = 1$ we have

$$I_1(w) = \phi_0'(f)f'(1 - w)^{-\alpha}w + \alpha\phi_0(f)w(1 - w)^{-\alpha-1} + \phi_0(f)(1 - w)^{-\alpha}.$$

Since $f = 1 - (1 - w)^\beta Q(w)$ we have by (3.6).

$$\phi_0'(f) = 2(1 - w)^{-\beta}[\phi_{1,1}(f) - \phi_0(f)]/Q.$$

Replacing the above in $I_1(w)$ and simplifying we get

$$I_1(w) = (1 - w)^{-\alpha-1}\{2[\phi_{1,1}(f) - \phi_0(f)][\beta Q - (1 - w)Q']w/Q + \alpha\phi_0(f)w + (1 - w)\phi_0(f)\}.$$

Hence the lemma is also true for $\lambda = 1$. Suppose it is true for $\lambda = n$, then for $\lambda = n + 1$ we have

$$\begin{aligned}(n + 1)!I_{n+1}(w) &= [\phi_0(f)(1 - w)^{-\alpha}w^{n+1}]^{(n+1)} \\ &= [\phi_0(f)(1 - w)^{-\alpha}w^n]^{(n)} \\ &\quad + w[\phi_0(f)(1 - w)^{-\alpha}w^n]^{(n+1)}.\end{aligned}$$

By the induction hypothesis it suffices to consider the last term, and we have

$$\begin{aligned}[\phi_0(f)(1 - w)^{-\alpha}w^n]^{(n+1)} &= n!I_n' \\ &= (1 - w)^{-\alpha-n-1}\sum_{j=0}^{n}[(n + \alpha)A_j(w)\phi_{j,j}(f) \\ &\quad + (1 - w)A_j'(w)\phi_{j,j}(f) \\ &\quad + (1 - w)A_j(w)\phi_{j,j}'(f)f'].\end{aligned}$$

We need only consider the second and third terms in the above sum. For the second term it suffices to observe that

$$(1 - w)[(1 - w)^m Q^{(m)}]' = m(1 - w)^m Q^{(m)} + (1 - w)^{m+1}Q^{(m+1)},$$

and consequently $(1 - w)A_j'(w)$ is of the proper form. Turning now to the third term we apply (3.6) to $\phi_{j,j}'$, calculate f' and $1 - f$ from (2.1) and we get

$$\begin{aligned}(1 - w)A_j(w)\phi_{j,j}'(f)f' &= A_j(w)(j + 1)(1 - w)[\phi_{j+1,j+1}(f) \\ &\quad - \phi_{j,j}(f)]f'/(1 - f) \\ &= A_j(w)(j + 1)[\beta Q - (1 - w)Q'] \\ &\quad \times [\phi_{j+1,j+1}(f) - \phi_{j,j}(f)]/Q.\end{aligned}$$

Consequently, the third term is also of the proper form. This proves the lemma.

Proof of Theorem 2. We may take $s = 0$. We have from (3.4) and (3.7) that

$$\psi_\lambda(z) = (\alpha - 1)(1 - z)^{\lambda+1} \sum_{j=0}^{\lambda} \int_0^1 (1 - r)^{\alpha-2}(1 - rz)^{-\alpha-\lambda} \times A_j(rz)\phi_{j,j}[f(rz)]dr.$$

Make a change of variable $y = (1 - r)/(1 - rz)$ in the integral above and for convenience let $w = (1 - y)z/(1 - yz)$. Then

$$\int_0^1 (1 - r)^{\alpha-2}(1 - rz)^{-\alpha-\lambda}A_j(rz)\phi_{j,j}[f(rz)]dr = -(1 - z)^{-\lambda-1} \int_0^1 y^{\alpha-2}(1 - yz)^\lambda A_j(w)\phi_{j,j}[f(w)]dy.$$

Consequently,

$$\psi_\lambda(z) = (1 - \alpha) \sum_{j=0}^{\lambda} \int_0^1 y^{\alpha-2}(1 - yz)^\lambda A_j(w)\phi_{j,j}[f(w)]dy. \tag{3.8}$$

Let $s(\theta, \delta)$ denote the sector $|z - 1| < \delta$, $|\arg(1 - z)| < \theta$, $\theta < \pi/2$. For small δ, the transformation $w = (1 - y)z/(1 - yz)$, $0 \leqq y \leqq 1$, maps $s(\theta, \delta)$ into a region $R \subset \{|w| < 1\}$ in which $A_j(w)$ and $\phi_{j,j}[f(w)]$ are bounded. Hence for any $\epsilon > 0$ we may choose η, $0 < \eta < 1$, such that if $z \in s(\theta, \delta)$ then

$$\left| \int_\eta^1 y^{\alpha-2}(1 - yz)^\lambda A_j(w)\phi_{j,j}[f(w)]dy \right| < \epsilon. \tag{3.9}$$

If $z \to 1$ within the angle $|\arg(1 - z)| < \theta$, and $0 \leqq y \leqq \eta$, then $w \to 1$ inside the same angle, and there exists (as a consequence of (1.1)) ϕ, $0 < \phi \leqq \pi/2$ such that $f(w) \to 1$ inside $|\arg[1 - f(w)]| < \phi$. Hence if $z \to 1$ inside $|\arg(1 - z)| < \theta$ then $\phi_{j,j}[f(w)] \to 0$ when $0 \leqq y \leqq \eta$. This last fact in conjunction with (3.9) and (3.8) imply that $\psi_\lambda(z) \to 0$ as $z \to 1$ in a Stolz angle.

Remark 1. Although we have proved Theorem 2 only for the values $\lambda = 0, 1, 2, \ldots$, it seems reasonable that the result should be true for all $\lambda > -1$

Remark 2. Recently B. Kwee [**4**] proved that any sequence absolutely summable (C, α) is also absolutely summable by the method of de la Vallee Poussin. That is, $|(C, \alpha)| \subset |V|$. Gronwall proved a relation of the form $(C, \alpha) \subset [f, g]$ for certain f (see [**2**]), and in particular that $(C, \alpha) \subset (V)$. The question arises when does the relation $|(C, \alpha)| \subset |[f, g]|$ hold true? Kwee's result is a special case of the question. The author conjectures that if $\beta < 1$ in (1.3) then $|(C, \alpha)| \subset |[f, g]|$ for each f in (1.3), but he has been unable to prove this. It seems likely that this relation holds for a very large class of $[f, g]$ means.

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