J. Austral. Math. Soc. (Series A) **47** (1989), 280–299

SUFFICIENT CONDITIONS FOR THE STRONG STABILITY OF THE DIFFERENTIAL EQUATION $[p(D) + f(t)q(D)]y = 0$

A. HOWE

(Received 10 March 1988; revised 21 June 1988)

Communicated by A. J. Pryde

Abstract

A number of sufficient conditions for stability or strong stability, as used in the context of Hamiltonian systems, are found for the differential equation

$$[p(D) + f(t)q(D)]y = 0$$

where the continuous function $f(t)$ is periodic of period ω in t, $D = d/dt$ and $p(s)$, $q(s)$ are real monic polynomials having special properties which allow the differential equation to be transformed into a canonical system of k second order equations.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*) (1985 *Revision*): 34 C 11, 58 F 05.

We consider the differential equation

$$[p(D) + f(t)q(D)]y = 0 \tag{1}$$

where $D = d/dt$, $p(s)$ and $q(s)$ are real monic polynomials of even order with no common factors, and each polynomial has only distinct zeros, all of which lie on the imaginary axis such that the zeros of the two polynomials interlace with each other, above, and of course, below the origin on the imaginary axis. The function $f(t)$ is continuous and periodic of period ω in t.

As we shall be considering pairs of polynomials of the above type, let us denote by $\mathscr{P}$ the set of all pairs of polynomials, $p(s)$ and $q(s)$, which have the above list of properties.

The differential equation (1) occurs in feedback control theory problems. Under the above conditions, the differential equation (1), can be written as a canonical system of k second order equations:

$$\frac{d^2\mathbf{y}}{dt^2} + [P_0 + R(t)]\mathbf{y} = \mathbf{0}, \tag{2}$$

where $\mathbf{y}$ is a $k \times 1$ vector, P_0 and $R(t)$ are real symmetric matrices and $R(t)$ is periodic of period ω in t. System (2) is a special case of the linear Hamiltonian system:

$$J\frac{d\mathbf{x}}{dt} = H(t)\mathbf{x}, \tag{3}$$

with periodic coefficients. Here $H(t)$ is a $2k \times 2k$ real symmetric matrix, continuous and periodic of period ω in t and

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

Sufficient conditions for the differential equation (1) to be strongly stable are found by considering when the equivalent system (2) or the equivalent Hamiltonian system (3) belongs to a domain of stability. Stability tests for (1) were obtained, in this manner, by Brockett [2] and Chiou [3]. The stability criteria of Brockett and Chiou are restricted to particular domains of stability. We will refine the tests of Brockett and Chiou and also give stability criteria which are not restricted to a particular domain of stability.

Here stability is taken to mean boundedness of the solutions and strong stability means that all neighbouring differential equations of the same form, are stable.

We begin by showing that, under the conditions stated above, the differential equation (1) can be written in the form of the system (2).

On account of the interlacing of the zeros of the pair of polynomials $p(s)$ and $q(s) \in \mathscr{P}$, we can write, in the differential equation (1):

$$\begin{aligned} p(D)y &= y^{(2k)} + a_1 y^{(2k-2)} + \cdots + a_{k-1}y'' + a_k y, \\ q(D)y &= y^{(2k-2)} + b_1 y^{(2k-4)} + \cdots + b_{k-2}y'' + b_{k-1}y, \end{aligned} \tag{4}$$

where a_i, $i = 1, 2, \ldots, k$, and b_j, $j = 1, 2, \ldots, k-1$, are real constants. The case of a differential equation (1) with $p(s)$ and $q(s)$ being of the same even order can be easily rewritten as an equation of the same form but with new $p(D)$ and $q(D)$ of the form given by (4), provided that the coefficients of s^{2k-2} in the original $p(s)$ and $q(s)$ are different. Otherwise we will not have interlacing of the zeros.

If we write $x_1 = y$, $x_2 = y''$, $x_3 = y^{(4)}, \ldots, x_k = y^{(2k-2)}$ then the equation

$$p(D)y = v$$

can be written in the form of the system:

$$\frac{d^2\mathbf{x}}{dt^2} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & 1 \\ -a_k & -a_{k-1} & -a_{k-2} & \cdots & -a_1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} v,$$

where $\mathbf{x}$ denotes the column vector $[x_1, x_2, \ldots, x_k]^T$. Now we need only take

$$\begin{aligned} v &= -f(t)q(D)y \\ &= -f(t)[b_{k-1} b_{k-2} \cdots b_1 1]\mathbf{x} \end{aligned}$$

in order to obtain the equivalent system of differential equations for the differential equation (1). The system is

$$(5)\quad \frac{d^2\mathbf{x}}{dt^2} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_k & -a_{k-1} & -a_{k-2} & \cdots & -a_1 \end{bmatrix} \mathbf{x} - f(t) \begin{bmatrix} 0 & & \cdots & & 0 \\ \vdots & & & & \\ 0 & & \cdots & & 0 \\ b_{k-1} & b_{k-2} & \cdots & b_1 & 1 \end{bmatrix} \mathbf{x}.$$

System (5) can be written

$$(6)\qquad \begin{aligned} \frac{d^2\mathbf{x}}{dt^2} &= A\mathbf{x} + Bu \\ y &= C\mathbf{x}, \end{aligned}$$

where

$$u = -f(t)y,$$

and

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_k & -a_{k-1} & -a_{k-2} & \cdots & -a_1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

$$C = [b_{k-1}\ b_{k-2}\ \cdots\ b_1\ 1].$$

System (6), with the second derivative replaced by the first derivative, are state equations having transfer function

$$(7)\qquad G(\tau) = \frac{\hat{q}(\tau)}{\hat{p}(\tau)} = C[\tau I - A]^{-1}B$$

where $\hat{q}(s^2) = q(s)$ and $\hat{p}(s^2) = p(s)$. The matrices $\{A, B, C\}$ are a realization of (7) and its well known that all other realizations $\{\widehat{A}, \widehat{B}, \widehat{C}\}$, with $\widehat{A}$ a $k \times k$ matrix are equivalent under the relationship

$$\widehat{A} = PAP^{-1}, \qquad \widehat{B} = PB, \qquad \widehat{C} = CP^{-1},$$

where P is a real constant nonsingular $k \times k$ matrix.

https://doi.org/10.1017/S1446788700031712 Published online by Cambridge University Press

We note that in terms of the matrices A, B and C, system (5) may be simply written

$$\frac{d^2\mathbf{x}}{dt^2} = A\mathbf{x} - f(t)BC\mathbf{x} \tag{8}$$

and as the matrix of coefficients $A - f(t)BC$ is a companion form matrix, its eigenvalues are zeros of the polynomial $\hat{p}(s) + f(t)\hat{q}(s)$. since $\hat{p}(s^2) = p(s)$ and $\hat{q}(s^2) = q(s)$, we see that if $s_i(t)$ is a zero of $p(s) + f(t)q(s)$ then $s_i(t)^2$ will be a zero of $\hat{p}(s) + f(t)\hat{q}(s)$ and hence an eigenvalue of the matrix $A - f(t)BC$.

The matrix A in (6) and (8) is a companion form matrix with characteristic polynomial $\hat{p}(\tau)$, which, by hypothesis, has distinct real negative roots, say $-\alpha_1^2, -\alpha_2^2, \ldots, -\alpha_k^2$. Therefore the matrix A is diagonalizable. We should note that companion form matrices with eigenvalues of multiplicity greater than one are not diagonalizable. Hence we consider a new realization $\{-P_0, Q, R\}$ of $G(\tau)$ where

$$P_0 = \operatorname{diag}(\alpha_1^2, \alpha_2^2, \ldots, \alpha_k^2). \tag{9}$$

Then system (8) has the form

$$\frac{d^2\mathbf{x}}{dt^2} + [P_0 + f(t)QR]\mathbf{x} = \mathbf{0}$$

and in this system QR is symmetric if and only if

$$QR = \widehat{C}^T\widehat{C} \quad \text{or} \quad QR = \widehat{B}\widehat{B}^T,$$

where $\widehat{C}$ is a $1 \times k$ row vector and $\widehat{B}$ is a $k \times 1$ column vector. Therefore a symmetric realization of $G(\tau)$ is of the form $\{-P_0, \widehat{B}, \widehat{B}^T\}$, giving rise to the system of differential equations

$$\frac{d^2\mathbf{x}}{dt^2} + [P_0 + f(t)\widehat{B}\widehat{B}^T]\mathbf{x} = \mathbf{0}. \tag{10}$$

Also if $\widehat{B} = [\hat{b}_1, \hat{b}_2, \ldots, \hat{b}_k]^T$ then

$$\begin{aligned} G(\tau) &= \frac{\hat{q}(\tau)}{\hat{p}(\tau)} = \widehat{B}^T[\tau I + P_0]^{-1}\widehat{B} \\ &= \frac{\hat{b}_1^2}{\tau + \alpha_1^2} + \frac{\hat{b}_2^2}{\tau + \alpha_2^2} + \cdots + \frac{\hat{b}_k^2}{\tau + \alpha_k^2}. \end{aligned} \tag{11}$$

The derivative of G, given by

$$G'(\tau) = -\frac{\hat{b}_1^2}{(\tau + \alpha_1^2)^2} - \frac{\hat{b}_2^2}{(\tau + \alpha_2^2)^2} - \cdots - \frac{\hat{b}_k^2}{(\tau + \alpha_k^2)^2},$$

is negative for all $\tau \neq -\alpha_i^2$ and as

$$\lim_{\tau \to (-\alpha_i^2)^+} \frac{1}{\tau + \alpha_i^2} = +\infty, \qquad \lim_{\tau \to (-\alpha_i^2)^-} \frac{1}{\tau + \alpha_i^2} = -\infty,$$

https://doi.org/10.1017/S1446788700031712 Published online by Cambridge University Press

we see that G is monotonic between the poles with distinct zeros which interlace with the poles. Hence the differential equation (1) can be written as a canonical system (10) of k second order equations when $p(s)$ and $q(s)$ are a pair of polynomials belonging to the set $\mathscr{P}$.

Since the companion form matrix A in (6) has only distinct eigenvalues, there exists a nonsingular matrix T such that $A = -T^{-1}P_0T$ where P_0 is the diagonal matrix given by (9). Then the substitution

$$\mathbf{w} = T\mathbf{x}$$

will transform system (8) into the system

$$\frac{d^2\mathbf{w}}{dt^2} + [P_0 + f(t)TBCT^{-1}]\mathbf{w} = \mathbf{0}. \tag{12}$$

The similarity transformation matrix T is by no means unique. We take the nonuniqueness of T into account by making the substitution

$$\mathbf{y} = S\mathbf{w}$$

in (12). Here S is any nonsingular matrix such that $S^{-1}P_0S = P_0$. The matrix S is, of course, diagonal and after making the substitution in (11) we obtain the system

$$\frac{d^2\mathbf{y}}{dt^2} + [P_0 + f(t)STBCT^{-1}S^{-1}]\mathbf{y} = \mathbf{0}. \tag{13}$$

From the symmetric property and basic algebraic manipulations we find that

$$STBCT^{-1}S^{-1} = \hat{L}\hat{L}^T$$

where $\hat{L}$ is the $k \times 1$ column vector given by

$$\hat{L} = \frac{1}{\Delta}\begin{bmatrix} s_1(\alpha_2^2 - \alpha_3^2) \\ s_2(\alpha_3^2 - \alpha_4^2) \\ s_3(\alpha_4^2 - \alpha_5^2) \\ \vdots \\ s_{k-2}(\alpha_{k-1}^2 - \alpha_k^2) \\ s_{k-1}(\alpha_k^2 - \alpha_1^2) \\ s_k(\alpha_1^2 - \alpha_2^2) \end{bmatrix}, \tag{14}$$

with

$$\Delta^2 = s_1^2(\alpha_2^2 - \alpha_3^2)^2 + s_2^2(\alpha_3^2 - \alpha_4^2)^2 + \cdots + s_k^2(\alpha_1^2 - \alpha_2^2)^2$$

and the s_i, $i = 1, 2, \ldots, k$, are arbitrary finite real non zero numbers, which are also the diagonal entries of the diagonal matrix S.

From the form of the transfer function G in (11), we see that no matter what finite nonzero values the s_i, $i = 1, 2, \ldots, k$, assume, the structure of the transfer function G remains the same. Therefore differing values of s_i,

https://doi.org/10.1017/S1446788700031712 Published online by Cambridge University Press

$i = 1, 2, \ldots, k$, give rise to differing values of the zeros of G, which also interlace with the poles of G, provided that the s_i, $i = 1, 2, \ldots, k$, are all finite and nonzero. The solutions to the systems (8) and (13) are related to each other by

$$\mathbf{y} = ST\mathbf{x}$$

and as the S and T are finite, we obtain

THEOREM 1. *All differntial equations of the form* (1), *with fixed* $f(t)$ *continuous and periodic of period* ω *in* t, *and fixed* $p(s)$ *but with differing* $q(s)$ *such that the polynomial pairs* $p(s)$ *and* $q(s)$ *always belong to* $\mathscr{P}$, *have the same stability property.*

The above theorem tells us that if the poles of the transfer function G are fixed and provided the zeros and poles of G interlace, we can move the zeros of G around without affecting the stability property of the differential equation (1). That is to say, the differential equation (1) remains stable or unstable when the $q(s)$ is changed in the manner stated above.

Also Theorem 1 tends to suggest that we choose the s_i, $i = 1, 2, \ldots, k$, so that the entries in the matrix $\widehat{B}\widehat{B}^T$ are simple and then consider the stability properties of the corresponding system (13). However choice of particular s_i does not greatly simplify our considerations of the stability properties of (13).

When $k = 2$, system (13) has the simple form

$$\frac{d^2\mathbf{y}}{dt^2} + \left\{ \begin{bmatrix} \alpha_1^2 & 0 \\ 0 & \alpha_2^2 \end{bmatrix} + \frac{f(t)}{s_1^2 + s_2^2} \begin{bmatrix} s_2^2 & -s_1 s_2 \\ -s_1 s_2 & s_1 \end{bmatrix} \right\} \mathbf{y} = \mathbf{0}, \tag{15}$$

which is the equivalent system (13) for the differential equation (1). Here, in (1), we have

$$p(s) = (s^2 + \alpha_1^2)(s^2 + \alpha_2^2),$$

and

$$q(s) = s^2 + a,$$

where

$$a = \frac{\alpha_1^2 + (s_2/s_1)^2 \alpha_2^2}{1 + (s_2/s_1)^2}.$$

We can easily show that a lies between α_1^2 and α_2^2. An obvious choice of s_1, s_2, in the above, would be $s_1 = s_2 = 1$.

We now consider some stability criteria for the differential equation (1) and its equivalent system (13). However, before doing so, we briefly describe what we mean by stability and strong stability for our systems of differential equations.

https://doi.org/10.1017/S1446788700031712 Published online by Cambridge University Press

The Hamiltonian system (3) is said to be stable if each solution $\mathbf{x}(t)$ is bounded for $-\infty < t < \infty$ and is strongly stable if all neighbouring systems of the same form as (3) are stable. Two strongly stable Hamiltonian systems (3) are said to belong to the same domain of stability if and only if they can be continuously deformed into one another without ceasing to be strongly stable and without ceasing to be of the form (3).

Likewise the differential equation (1) and the second order system (2) are said to be stable if their respectively solutions, y and $\mathbf{y}$, are bounded for $-\infty < t < \infty$. We define strong stability for the differential equation (1) and the second order system (2) as meaning that their equivalent Hamiltonian system is strongly stable. Clearly if the equivalent Hamiltonian system to (1) or (2) is stable, then so is the differential equation (1) or the system (2), stable. Therefore strong stability for the differential equation (1) or the system (2) has the meaning that all neighbouring differential equations or systems of the same form as (1) or (2), respectively, are also stable.

Gelfand and Lidskii [4] and Krein [6] have shown that the domains of stability $\mathscr{O}_m^{(\sigma)}$ for the Hamiltonian system (3) are characterised by a signature σ and an integer m $(-\infty < m < \infty)$ which is called the index of the domain of stability.

The signature σ represents the distribution by type, on the upper half unit circle, of the multipliers of any Hamiltonian system (3) in a given domain of stability. There are 2^k different signatures. A multiplier ρ is of positive type or of the first kind if $i(J\mathbf{f}, \mathbf{f}) > 0$ for all vectors $\mathbf{f}$ belonging to the eigenspace of ρ and ρ is of negative type or of the second kind if $i(J\mathbf{f}, \mathbf{f}) < 0$ for all vectors $\mathbf{f}$ belonging to the eigenspace of ρ. A Hamiltonian system (3) is stable if and only if its multipliers have unit modulus and simple elementary divisors. The system (3) is strongly stable if, in addition, there are no repeated multipliers of mixed type.

A real matrix X is said to be symplectic if

$$X^T J X = J.$$

The matrix solution $X(t)$, $0 \leq t \leq \omega$, $X(0) = I$, of the Hamiltonian system (3) lies in the real symplectic group $\mathbf{S}$ and two Hamiltonian systems (3) belong to the same domain of stability if and only if their matrix solutions $X(t)$, $0 \leq t \leq \omega$, $X(0) = I$, can be continuously deformed into each other in $\mathbf{S}$ without ceasing to be strongly stable. Since the real symplectic group $\mathbf{S}$ is homeomorphic to the topological product of the circumference of a circle and a simply-connected topological space, the index of a domain of stability indicates the number of twists and turns which the matrix solution $X(t)$, $0 \leq t \leq \omega$, $X(0) = I$, of any Hamiltonian system in the domain of stability, can make in the real symplectic group $\mathbf{S}$.

There are a number of different formulae for the index of a domain of stability and for each particular formula the index remains invariant under any continuous deformation, of a Hamiltonian system, that preserves strong stability. From Gel′fand and Lidskii [4], a very applicable formula for the index m is given by

$$\sum_{j=1}^{k} \Delta \arg \rho_j(t)\Big|_{t=0}^{\omega} - \sum_{j=1}^{k} \nu_j = 2m\pi, \tag{16}$$

where $\rho_j(\omega) = e^{i\nu_j}$, $-\pi < \nu_j < \pi$ $(j = 1, 2, \ldots, k)$, $\rho_1(t), \rho_2(t), \ldots, \rho_k(t)$ are eigenvalues of positive type of the matrix solution $X(t)$, $0 \le t \le \omega$, $X(0) = I$, of a Hamiltonian system (3) in the domain of stability.

We will now consider a number of stability tests for the differential equation (1) where $f(t)$ is continuous and periodic of period ω in t, $D = d/dt$ and $p(s)$, $q(s)$ are a pair of polynomials belonging to the set $\mathscr{P}$.

The differential equation (1) with the above special conditions, will be denoted by (I).

Krein [5], [6], [8] proved

THEOREM A. *The second order system of differential equations*

$$\frac{d^2\mathbf{y}}{dt^2} + \mu P(t)\mathbf{y} = \mathbf{0} \tag{17}$$

where $\mathbf{y}$ *is a* $k \times 1$ *vector and* $P(t)$ *is a real symmetric matrix, periodic of period* ω *in* t, *such that* $P(t) \ge 0$ $(0 \le t \le \omega)$ *and* $\int_0^\omega P(t)\,dt > 0$, *is strongly stable whenever*

$$0 < \mu < \frac{4}{\omega \int_0^\omega p_M(t)\,dt},$$

where $p_M(t)$ *is the largest eigenvalue of the matrix* $P(t)$.

Krein showed that Theorem A is restricted to systems of differential equations (17) in the domain of stability $\mathscr{O}_0^{(+)}$ where $(+)$ denotes the signature $+ + \cdots + +$.

Brockett [2] used Theorem A with $\mu = 1$ to establish

THEOREM 2. *The differential equation* (1) *with* $f(t) \ge 0$ $(0 \le t \le \omega)$ *is strongly stable when*

$$0 < \int_0^\omega |\lambda(t)|^2\,dt < \frac{4}{\omega},$$

where $\lambda(t)$ *is the zero of* $p(s) + f(t)q(s)$ *which has the largest magnitude.*

https://doi.org/10.1017/S1446788700031712 Published online by Cambridge University Press

The value $|\lambda(t)|^2$ is the largest eigenvalue, $p_M(t)$, of the symmetric matrix of coefficients

$$P(t) = P_0 + f(t)\hat{L}\hat{L}^T \tag{18}$$

of the equivalent system (13). Since the matrix $P(t)$, given by (18), is the sum of two symmetric matrices, it follows (see Marshall and Olkin [9, page 243]) that for $f(t) \geq 0$ $(0 \leq t \leq \omega)$, the largest eigenvalue of $P(t)$ is less than or equal to the sum of the largest eigenvalues of each of the matrices on the right hand side of (18), that is,

$$p_M(t) \leq \alpha^2_{j(\max)} + f(t), \tag{19}$$

where $\alpha^2_{j(\max)}$ is the largest eigenvalue of the matrix P_0, given by (9), and therefore $\pm i\alpha_{j(\max)}$ are zeros of $p(s)$ with the largest magnitude. Hence Theorem 2 can be written

THEOREM 2′. *The differential equation* (1) *with* $f(t) \geq 0$ $(0 \leq t \leq \omega)$ *is strongly stable when*

$$0 < \alpha^2_{j(\max)} + \frac{1}{\omega}\int_0^\omega f(t)\,dt < \frac{4}{\omega^2},$$

where $\pm i\alpha_{j(\max)}$ *are zeros of* $p(s)$ *with the largest magnitude.*

As the matrix P_0, in (18), is similar to the negative of the coefficient matrix A in system (6), we can use Theorem 2.1 of Wolkowicz and Styan [12] to obtain an upper bound for $\alpha^2_{i(\max)}$ in terms of the coefficients of $p(s)$. Then Theorem 2 has the form

THEOREM 2″. *The differential equation* (1) *with* $f(t) \geq 0$ $(0 \leq t \leq \omega)$ *is strongly stable when*

$$0 < a_1 + \{(k-1)^2 a_1^2 - 2k(-1)a_2\}^{1/2} + \frac{k}{\omega}\int_0^\omega f(t)\,dt < \frac{4k}{\omega^2}$$

where a_1 *and* a_2 *are the coefficients of* s^{2k-2} *and* s^{2k-4}, *respectively, in the polynomial* $p(s)$.

The matrix $P(t)$, given by (18), is similar to the negative of the coefficient matrix of system (5) and therefore if we apply Theorem 2.1, of Wolkowicz and Styan [12], to the negative of the coefficient matrix of system (5), we will obtain a bound for $p_M(t)$ in terms of the coefficients of the polynomials $p(s)$ and $q(s)$. Then another form of Theorem 2 would be the following theorem.

THEOREM 2‴. *The differential equation* (1) *with* $f(t) \geq 0$ $(0 \leq t \leq \omega)$ *is strongly stable when*

$$0 < a_1 + \frac{1}{\omega}\int_0^\omega [f(t) + \{(k-1)^2(a_1+f(t))^2 - 2k(k-1)(a_2+b_1 f(t))\}^{1/2}]\,dt < \frac{4k}{\omega^2}$$

where a_1 *and* a_2 *are the coefficients of* s^{2k-2} *and* s^{2k-4}, *respectively, in the polynomial* $p(s)$; *and* b_1 *is the coefficient of* s^{2k-4} *in the polynomial* $q(s)$.

In Theorems 2 to 2‴ we require $f(t) \geq 0$ $(0 \leq t \leq \omega)$. A stability test for the differential equation (1) with $f(t) < 0$ $(0 \leq t \leq \omega)$ is obtained by applying the following theorem of Krein [6]:

THEOREM B. *If* $\int_0^\omega P(t)\,dt > 0$ *and* $P(t) = P^+(t) - P^-(t)$ *where* $P^+(t) \geq 0$ *and* $P^-(t) \geq 0$ *then the first zone of stability of*

$$\frac{d^2\mathbf{y}}{dt^2} + \mu P(t)\mathbf{y} = \mathbf{0}, \tag{17}$$

is not smaller than the first zone of stability of

$$\frac{d^2\mathbf{y}}{dt^2} + \mu P^+(t)\mathbf{y} = \mathbf{0}. \tag{19}$$

Theorem B tells us that if (19) is strongly stable at $\mu = 1$ then so is (17) strongly stable at $\mu = 1$.

With $P(t)$ given by (18) we take

$$P^+(t) = P_0 \quad \text{and} \quad P^-(t) = \kappa(t)\hat{L}\hat{L}^T$$

where $\kappa(t) > 0$ $(0 \leq t \leq \omega)$ and $f(t) = -\kappa(t)$. Then by Theorem A, system (19) is strongly stable when $\mu = 1$ if

$$\alpha^2_{j(\max)} < \frac{4}{\omega^2}. \tag{20}$$

The condition that $\int_0^\omega P(t)\,dt > 0$, gives rise to

$$\int_0^\omega \kappa(t)\,dt\hat{L}\hat{L}^T < \omega P_0$$

and from the form of P_0 we have

$$\int_0^\omega \kappa(t)\,dt\hat{L}\hat{L}^T < \omega\alpha^2_{j(\max)}I. \tag{21}$$

If we choose $\kappa(t)$ such that

$$\int_0^\omega \kappa(t)\,dt < \omega\alpha^2_{j(\max)}, \tag{22}$$

then (21) is immediately satisfied, since the symmetric matrix $\hat{L}\hat{L}^T$ has only eigenvalues 0 and 1.

Hence we have shown that the combined inequalities (20) and (22) are conditions for system (17), with $P(t)$ given by (18) and $f(t) < 0$ $(0 \le t \le \omega)$, to be strongly stable. We have obtained

THEOREM 3. *The differential equation* (1) *with* $f(t) = -\kappa(t)$, $\kappa(t) > 0$ $(0 \le t \le \omega)$ *is strongly stable when*

$$\frac{1}{\omega}\int_0^{\omega} \kappa(t)\,dt < \alpha_{j(\max)}^2 < \frac{4}{\omega^2}$$

where $\pm i\alpha_{j(\max)}$ *are zeros of* $p(s)$ *with the largest magnitude.*

Theorem B also only holds for the domain of stability $\mathcal{O}_0^{(+)}$. Because Theorems A and B are restricted to the domain of stability $\mathcal{O}_0^{(+)}$, the stability tests provided by Theorems 2 and 3 are also restricted to only a certain range of differential equations of type (1).

In a slightly more general situation, Chiou [3] applies the following theorem to the differential equation (1):

THEOREM C. *The second order system of differential equations*

$$\frac{d^2\mathbf{y}}{dt^2} + P(t)\mathbf{y} = \mathbf{0} \tag{23}$$

where $P(t)$ *is a* $k \times k$ *real symmetric matrix, continuous in* t *and periodic of period* ω *in* t, *is strongly stable if*

$$\frac{n^2\pi^2}{\omega^2} I < P(t) < \frac{(n+1)^2\pi^2}{\omega^2} I$$

where n *is some integer greater than or equal to zero.*

Chiou [3] establishes Theorem C as a corollary of a theorem concerning a more general system of equations. Theorem C was originally established by Neigauz and Lidskii [11] where no proof was given. However Theorem C can easily be proved by application of the directional wideness theorem of Yakubovich [14]:

THEOREM D. *The linear Hamiltonian system*

$$J\frac{d\mathbf{x}}{dt} = H(t)\mathbf{x}$$

where $H(t)$ *is a* $2k \times 2k$ *real symmetric matrix, continuous in* t *and periodic of period* ω *in* t, *is strongly stable if*

$$H_1(t) \le H(t) \le H_2(t)$$

where $H_1(t)$ and $H_2(t)$ are $2k \times 2k$ real symmetric matrices, continuous in t and periodic of period ω in t, such that the Hamiltonian system

$$J\frac{d\mathbf{x}}{dt} = H(t,s)\mathbf{x}$$

where $H(t,s) = H_1(t) + s[H_2(t) - H_1(t)]$, is strongly stable for any $0 \leq s \leq 1$.

In fact, from Yakubovich [13], we know that Hill's equation

$$\frac{d^2\xi}{dt^2} + p(t)\xi = 0 \tag{24}$$

where $p(t)$ is continuous and periodic of period ω in t, is strongly stable, in the set of second order linear Hamiltonian systems if

$$\frac{n^2\pi^2}{\omega^2} < p(t) < \frac{(n+1)^2\pi^2}{\omega^2} \tag{25}$$

for some $n = 0, 1, 2, \ldots$. Also under condition (25), Hill's equation (24) belongs to the domain of stability $\mathscr{O}_n^{(\sigma_0)}$ where $\sigma_0 = +$ if n is even and $\sigma_0 = -$ if n is odd.

The stability test given by (25) for Hill's equation (24) is known as Zhukovskii's test (see Zhukovskii [16]).

The system of differential equations

$$\frac{d^2\mathbf{y}}{dt^2} + p(t)\mathbf{y} = \mathbf{0} \tag{26}$$

where $p(t)$ is a continuous function, periodic of period ω in t, is a set of k scalar differential equations (24). Therefore its equivalent Hamiltonian system

$$J\frac{d\mathbf{x}}{dt} = \begin{bmatrix} p(t)I & 0 \\ 0 & I \end{bmatrix}\mathbf{x} \tag{27}$$

will be strongly stable if (25) is satisfied, and will belong to the domain of stability $\mathscr{O}_{kn}^{(\sigma)}$, where $\sigma = (+++\cdots+)$ if n is even and $\sigma = (----\cdots-)$ if n is odd. This follows from a result in Yakubovoch [14]. Yakubovich proves that a Hamiltonian system

$$J\frac{d\mathbf{x}}{dt} = H(t)\mathbf{x},$$

which is separable into k second order Hamiltonian systems

$$J_2\frac{d\mathbf{z}_j}{dt} = H_j(t)\mathbf{z}_j, \qquad (j = 1, 2, \ldots, k), \tag{28}$$

where $J_2 = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ and the $H_j(t)$ are 2×2 real symmetric matrices, continuous in t and periodic of period ω in t, has signature σ and index n given by

$$\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_k), \qquad n = n_1 + n_2 + \cdots + n_k.$$

Here the σ_j and n_j are the respective signatures and indices of the second order Hamiltonian systems (28) and these Hamiltonian systems are arranged in the order given by the decomposition

$$R^T JR = J_2 \dot{+} J_2 \dot{+} \cdots \dot{+} J_2, \quad R^T H(t) R = H_1(t) \dot{+} H_2(t) \dot{+} \cdots \dot{+} H_k(t),$$

where the $\dot{+}$ denotes the direct sum of the 2×2 matrices.

The $2k \times 2k$ orthogonal matrix given by

$$(29) \qquad R = \left[\begin{array}{cccccccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ \hline 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c}0\\0\\0\\0\end{array}}\right\} k \\ \\ \left.\vphantom{\begin{array}{c}0\\0\\0\\0\end{array}}\right\} k \end{array}$$

will transform the Hamiltonian system (27) into k second order Hamiltonian system (28), each of which is equivalent to the Hill's equation (24).

If $p_1(t)$ and $p_2(t)$ are continuous functions, periodic of period ω in t, such that for some $n \geq 0$,

$$\frac{n^2\pi^2}{\omega^2} < p_1(t) < p_2(t) < \frac{(n+1)^2}{\omega^2}\pi^2,$$

then the corresponding Hamiltonian system (27) or the second order system (26) will be strongly stable and they will both belong to the same domain of stability. Also, since

$$\frac{n^2\pi^2}{\omega^2} < (1-s)p_1(t) + sp_2(t) < \frac{(n+1)^2\pi^2}{\omega^2}$$

for $0 \leq s \leq 1$, the Hamiltonian system (27) or the second order system (26) with

$$p(t) = (1-s)p_1(t) + sp_2(t), \qquad (0 \leq s \leq 1),$$

will be strongly stable and belong to the same domain of stability as system (27) with $p(t) = p_1(t)$ or $p(t) = p_2(t)$. Hence by Theorem D, the second order system (23) or the Hamiltonian system (3) with

$$(30) \qquad H(t) = \begin{bmatrix} P(t) & 0 \\ 0 & I \end{bmatrix}$$

where $P(t)$ is a real $k \times k$ symmetric matrix, continuous in t and periodic in t with period ω, will be strongly stable if in addition

$$\begin{bmatrix} p_1(t)I & 0 \\ 0 & I \end{bmatrix} \leq \begin{bmatrix} P(t) & 0 \\ 0 & I \end{bmatrix} \leq \begin{bmatrix} p_2(t)I & 0 \\ 0 & I \end{bmatrix},$$

https://doi.org/10.1017/S1446788700031712 Published online by Cambridge University Press

that is

$$\frac{n^2\pi^2}{\omega^2}I < p_1(t)I \leq P(t) \leq p_2(t)I < \frac{(n+1)^2\pi^2}{\omega^2}I.$$

Furthermore the Hamiltonian system (3) with (30) or the second order system (23) will belong to the same domain of stability as that of the systems (27) or (26), respectively, with $p(t) = p_1(t)$ or $p(t) = p_2(t)$. This domain of stability is of the form $\mathscr{O}_{kn}^{(\sigma)}$ where n is some integer greater than or equal to zero and $\sigma = (+ + + \cdots +)$ if n is even and $\sigma = (- - - - \cdots -)$ if n is odd.

Therefore we have proved Theorem C.

Applying Theorem C to the differential equation (1), Chiou [3] obtained the following stability criteria

THEOREM 4. *The differential equation* (1) *is strongly stable when*

$$\frac{n^2\pi^2}{\omega^2} < |\lambda_j(t)|^2 < \frac{(n+1)^2\pi^2}{\omega^2} \qquad (j = 1, 2, \ldots, 2k) \tag{31}$$

for some integer n, greater than or equal to zero and where the $\lambda_j(t)$, $j = 1, 2, \ldots, 2k$, *are the zeros of* $p(s) + f(t)q(s)$.

The $|\lambda_j(t)|^2$ in (31) are also the eigenvalues of the matrix $P(t)$ given by (18) and therefore (31) has the meaning that

$$0 < \text{spread of } P(t) < \frac{(2n+1)\pi^2}{\omega^2}.$$

The spread of a matrix is defined as the maximum distance between two of its eigenvalues. There are a number of estimates for the spread of a matrix. One of the sharper results is that of Brauer and Mewborn [1], which, when applied to the negative of the matrix of coefficients of the system of differential equations (5) that is the matrix to which $P(t)$ is similar, gives rise to

THEOREM 4′. *The differential equation* (1) *is strongly stable when*

$$0 < \sqrt[4]{\frac{4k}{k+6}} \cdot K_4 < \frac{(2n+1)\pi^2}{\omega^2}$$

for some integer n, greater than or equal to zero, and where

$$\begin{aligned}\frac{kK_4^4}{2} = {} & (k-1)(a_1 + f(t))^4 - 4k(a_1 + f(t))^2(a_2 + f(t)b_1) \\ & + 2(k+6)(a_2 + f(t)b_1)^2 \\ & + 4(k-3)(a_1 + f(t))(a_3 + f(t)b_2 - 4k(a_4 + f(t)b_3).\end{aligned}$$

https://doi.org/10.1017/S1446788700031712 Published online by Cambridge University Press

Here the a_i $(i = 1, 2, 3, 4)$ *are the respective coefficients of* $s^{2(k-i)}$ $(i = 1, 2, 3, 4)$ *in the polynomial* $p(s)$ *and* b_1 *and* b_2 *are the coefficients of* s^{2k-4} *and* s^{2k-6}*, respectively in the polynomial* $q(s)$.

Also from the form of $P(t)$, given by (18), and by use of Weyl's inequalities (see Marshall and Olkin [9, page 243]), we see that, for $f(t) \geq 0$;

$$\text{spread of } P(t) \leq \alpha^2_{j(\max)} - \alpha^2_{j(\min)} + f(t)$$

where $\alpha^2_{j(\max)}$ and $\alpha^2_{j(\min)}$ are respectively the largest and smallest eigenvalues of the matrix P_0 in (18). We therefore have

THEOREM 4″. *The differential equation* (1) *with* $f(t) \geq 0$ $(0 \leq t \leq \omega)$ *is strongly stable when*

$$0 < \alpha^2_{j(\max)} - \alpha^2_{j(\min)} < \frac{(2n+1)\pi^2}{\omega^2} - f(t)$$

for some integer n, greater than or equal to zero and where $\pm i\alpha_{j(\max)}$ *are zeros of* $p(s)$ *with the largest magnitude and* $\pm i\alpha_{j(\min)}$ *are zeros of* $p(s)$ *with the smallest magnitude.*

The difference $\alpha^2_{j(\max)} - \alpha^2_{j(\min)}$ is the spread of the matrix P_0 in (18) or in fact the spread of the negative of the matrix A in the system of differential equations (6). We can apply the estimate for the spread of a matrix given by Mirsky [10] or Wokowicz [12] to obtain

THEOREM 4‴. *The differential equation* (1) *with* $f(t) \geq 0$ $(0 \leq t \leq \omega)$ *is strongly stable when*

$$0 < \left\{2\left(1 - \frac{1}{k}\right)a_1^2 - 4a_2\right\}^{1/2} < \frac{(2n+1)\pi^2}{\omega^2} - f(t)$$

for some integer n, greater than or equal to zero and where a_1 *and* a_2 *are the coefficients of* s^{2k-2} *and* s^{2k-4}*, respectively, in the polynomial* $p(s)$.

The stability tests given by the Theorems 4 are restricted to only a certain range of differential equations of the type (1), since Theorem C, the theorem on which the Theorems 4 are based, holds only for domains of stability $\mathscr{O}^{(\sigma)}_{kn}$

https://doi.org/10.1017/S1446788700031712 Published online by Cambridge University Press

where n is some integer greater than or equal to zero and $\sigma = (+++\cdots+)$ if n is even and $\sigma = (---\cdots-)$ if n is odd.

A more general theorem to apply to the differential equation (1) is that of Yakubovich [14]:

THEOREM E. *Consider the system of differential equations*

$$J'\frac{d\mathbf{x}}{dt} = H(t)\mathbf{x} \tag{32}$$

where $H(t)$ is a real continuous symmetric $2k \times 2k$ matrix, periodic of period ω in t such that

$$H(t) = H'(t) + H_0(t),$$
$$H'(t) = h_1(t)I_2 \dotplus h_2(t)I_2 \dotplus \cdots \dotplus h_k(t)I_2,$$
$$J' = J_2 \dotplus J_2 \dotplus \cdots \dotplus J_2.$$

Here $J_2 = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ and I_2 denotes the 2×2 unit matrix. Let $h_1^{(0)}(t)$, $h_2^{(0)}(t)$ be the minimum and maximum eigenvalues, respectively, of $H_0(t)$, or any functions such that

$$h_1^{(0)}(t)I \le H_0(t) \le h_2^{(0)}(t)I.$$

Let

$$\tilde{h}_j^{(0)} = \int_0^\omega h_j^{(0)}(t)\,dt, \qquad j = 1, 2.$$

Suppose that for certain integers m_{pq}

$$2m_{pq}\pi < \tilde{h}_p + \tilde{h}_q + 2\tilde{h}_1^{(0)} \le \tilde{h}_p + \tilde{h}_q + 2\tilde{h}_2^{(0)}$$
$$< 2(m_{pq}+1)\pi$$

for all $p, q = 1, 2, \ldots, k$, where

$$\tilde{h}_\kappa = \int_0^\omega h_\kappa(t)\,dt, \qquad (\kappa = 1, 2, \ldots, k).$$

Then (32) is strongly stable.

Theorem E holds in every domain of stability $\mathscr{O}_m^{(\sigma)}$.

The differential equation (1) is equivalent to the second order system of differential equations (13) which in turn can be written in the form of the Hamiltonian system

$$J\frac{d\mathbf{z}}{dt} = \begin{bmatrix} P_0 + f(t)\widehat{L}\widehat{L}^T & 0 \\ 0 & I \end{bmatrix}\mathbf{z}. \tag{33}$$

https://doi.org/10.1017/S1446788700031712 Published online by Cambridge University Press

The substitution

$$\mathbf{z} = \begin{bmatrix} P^{-1/4} & 0 \\ 0 & P_0^{1/4} \end{bmatrix} R\mathbf{x},$$

where R is the orthogonal matrix given by (29), reduces the Hamiltonian system (33) to the form (32) with

$$H'(t) = \alpha_1 I_2 \dot{+} \alpha_2 I_2 \dot{+} \cdots \dot{+} \alpha_k I_2$$

and

$$H_0(t) = R^T \begin{bmatrix} f(t) P_0^{-1/4} \widehat{L} \widehat{L}^T P_0^{-1/4} & 0 \\ 0 & 0 \end{bmatrix} R = f(t) \widehat{K} \widehat{K}^T,$$

where

$$\widehat{K} = R^T \begin{bmatrix} P^{-1/4} \widehat{L} \\ 0 \end{bmatrix}$$

is a $2k \times 1$ column vector. From its form, the symmetric matrix $H_0(t)$ must have eigenvalues $f(t), 0, 0, \ldots, 0$. The maximum and minimum eigenvalues of $H_0(t)$ are, respectively,

$$\lambda_{(\max)}(t) = \max[f(t), 0] = \tfrac{1}{2}[f(t) + |f(t)|]$$

and

$$\lambda_{(\min)}(t) = \min[f(t), 0] = \tfrac{1}{2}[f(t) - |f(t)|],$$

for $0 \leq t \leq \omega$. Therefore Theorem E gives us the following stability test for the differential equation (1):

Theorem 5. *The differential equation* (1) *is strongly stable if there exist nonnegative integers m_{pq} such that*

$$2\pi m_{pq} < (\alpha_p + \alpha_q)\omega + \int_0^\omega [f(t) - |f(t)|]\, dt \leq (\alpha_p + \alpha_q)\omega + \int_0^\omega [f(t) + |f(t)|]\, dt < 2\pi(m_{pq} + 1)$$

for all $p, q = 1, 2, \ldots, k$ and where $\pm i\alpha_p, \pm i\alpha_q$ $(p, q = 1, 2, \ldots, k)$ are zeros of $p(s)$.

Another new test for strong stability of the differential equation (1) can be obtained by application of the following theorem from Neigauz and Lidskii [11]:

Theorem F. *The second order system of differential equations*

$$\frac{d^2\mathbf{y}}{dt^2} + [P_0 + R(t)]\mathbf{y} = \mathbf{0},$$

https://doi.org/10.1017/S1446788700031712 Published online by Cambridge University Press

where $R(t)$ is nonnegative definite, continuous in t and periodic of period ω in t, is strongly stable if the second order system of differential equations

$$\frac{d^2\mathbf{y}}{dt^2} + [P_0 + r(t)I]\mathbf{y} = \mathbf{0}$$

is strongly stable for any periodic function $r(t)$, continuous in t and periodic of period ω in t, such that $0 \le r(t) \le \|R\|(t)\|$. (For each fixed t, the norm $\|R(t)\|$ of the matrix $R(t)$ is defined as $\max_i |\lambda_i(t)|$, where the $\lambda_i(t)$, $i = 1, 2, \ldots, k$, are the eigenvalues of the matrix $R(t)$.)

The differential equation (1) is equivalent to the second order system of differential equations (13). By Theorem F, the system of differential equations (13) with $f(t) > 0$ is strongly stable if

$$\frac{d^2\mathbf{y}}{dt^2} + [P_0 + f(t)I]\mathbf{y} = \mathbf{0} \tag{34}$$

is strongly stable. Now the system (34) is simply a set of k scalar equations

$$y'' + [\alpha_j^2 + f(t)]y = 0, \qquad j = 1, 2, \ldots, k, \tag{35}$$

and by Zhukovskii's test (25), each of the differential equations in (35) will be strongly stable if

$$\frac{n_j^2\pi^2}{\omega^2} < \alpha_j^2 + f(t) < \frac{(n_j+1)^2\pi^2}{\omega^2}, \qquad j = 1, 2, \ldots, k, \tag{36}$$

where n_j is some integer greater than or equal to zero. Under the conditions (36), the differential equations (35) will belong to the respective domains of stability $\mathscr{O}_{n_j}^{(\sigma_j)}$, $j = 1, 2, \ldots, k$, where $\sigma_j = +$ if n_j is even and $\sigma_j = -$ if n_j is odd. Therefore in order to avoid multipliers of different types coinciding we could either take all the n_j even or all the n_j odd and then (35) or its equivalent Hamiltonian system will be strongly stable and belong to the domain of stability $\mathscr{O}_n^{(\sigma)}$ where $n = n_1 + n_2 + \cdots + n_k$ and $\sigma = (+ + + \cdots +)$ if all the n_j, $j = 1, 2, \ldots, k$, are even and $\sigma = (- - - \cdots -)$ if all n_j, $j = 1, 2, \ldots, k$, are odd.

Thus we have obtained the following theorem for the differential equation (1):

THEOREM 6. *The differential equation* (1) *with $f(t) > 0$ is strongly stable if*

$$\frac{n_j^2\pi^2}{\omega^2} < \alpha_j^2 + f(t) < \frac{(n_j+1)^2\pi^2}{\omega^2}, \qquad j = 1, 2, \ldots, k,$$

where n_j, $j = 1, 2, \ldots, k$, are nonnegative integers which are either all even or all odd and $\pm i\alpha_j$, $j = 1, 2, \ldots, k$, are zeros of $p(s)$.

The stability test given by Theorem 6 is more general than that of Theorems 2 and 4 but is not as general as that of Theorem 5.

Most of the materials from the Russian journals quoted can be found in the English translation of Yakubovich and Starzhinskii [15].

References

[1] Alfred Brauer and A. C. Mewborn, 'The greatest distance between two characteristic roots of a matrix,' *Duke Math. J.* **26** (1959), 653–661.

[2] R. W. Brockett, 'Lie algebras and Lie groups in control theory,' *Geometric methods in system theory*, edited by D. Q. Mayne and R. W. Brockett, (Reidel, Dordrecht, Holland, 1973).

[3] K. L. Chiou, 'The geometry of indefinite J-spaces and stability behaviour of linear systems,' *J. Differential Equations* **28** (1978), 104–123.

[4] I. M. Gel'fand and V. B. Lidskii, 'On the structure of stability domains of linear canonical systems of differential equations with periodic coefficients,' *Uspehi Mat. Nauk* **10**, No. 1 (63) (1955), 3–40 (Russian); *Amer. Math. Soc. Transl.* (2) **8** (1958), 143–181.

[5] M. G. Krein, 'A generalization of some investigations of A. M. Lyapunov on linear differential equations with periodic coefficients.' *Dokl. Akad. Nauk SSSR* **73** (1950), 445–448 (Russian).

[6] M. G. Krein, 'Fundamental aspects of the theory of λ-zones of stability for a canonical system of linear differential equations with periodic coefficients,' *Pamyati A. A. Andronova*, Izdat Akad. Nauk SSSR, Moscow, 1955, pp. 413–498 (Russian). Translated in M. G. Krein, *Topics in Differential and Integral Equations and Operator Theory: Advances and Applications*, Vol. 7, Birkhauser-Verlag, Basel, 1983, pp. 1–106.

[7] M. G. Krein, 'On criteria for stability and boundedness of solutions of periodic canonical systems', *Prikl. Mat. Meh.* **19**, No. 6 (1955), 641–680 (Russian).

[8] M. G. Krein, 'Introduction to the geometry of indefinite J-spaces and to the theory of operators in those spaces', Second Math. Summer School, Part I, Naukova Dumka, Kiev, 1965, pp. 15–92 (Russian); *Amer. Math. Soc. Transl.* (2) **93** (1970), 177–200.

[9] A. W. Marshall and I. Olkin, *Inequalities: Theory of majorization and its applications*, (Academic Press, New York, 1979).

[10] L. Mirsky, 'The spread of a matrix', *Mathematika* **3** (1956), 127–130.

[11] M. G. Neigauz and V. B. Lidskii, 'On the boundedness of the solutions of linear systems of differential equations with periodic coefficients.' *Dokl. Akad. Nauk SSSR* (*N.S.*) **77** (1951), 189–192 (Russian).

[12] Henry Wolkowicz and George P. H. Styan, 'Bounds for eigenvalues using traces,' *Linear Algebra and Appl.* **29** (1980), 471–506.

[13] V. A. Yakubovich, 'Questions of the stability of solutions of a system of two linear differential equations of canonical form with periodic coefficients', *Mat. Sb.* (*N.S.*) **37** (1955), 21–68 (Russian).

[14] V. A. Yakubovich, 'On convex properties of stability regions of linear Hamiltonian systems of differential equations with periodic coefficients,' *Vestnik Leningrad Univ. Math.* **17** (1962), 61–86 (Russian).

[15] V. A. Yakubovich and V. M. Starzhinskii, *Linear differential equations with periodic coefficients.* (Nauka, Moscow, 1972) (Russian). Transl. (Keter Publishing House Jerusalem Ltd., 1975).
[16] N. E. Zhukovskii, 'Finiteness conditions for integrals of the equation $d^2y/dx^2 + py = 0$,' *Mat. Sb.* **16**, No. 3 (1882), 582–591 (Russian).

Department of Mathematics
Faculty of Science
Australian National University
P. O. Box 4
Canberra, ACT 2601
Australia