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LETTER TO THE EDITOR

ON SUFFICIENT CONDITIONS FOR ERGODICITY OF MULTICHANNEL QUEUEING SYSTEMS WITH REPEATED CALLS

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Abstract

We propose a simple and efficient method of obtaining sufficient conditions for the existence of a stationary regime for multichannel fully available queueing systems with repeated calls.

ERGODICITY: TELEPHONE SYSTEMS WITH REPEATED CALLS

It is well known that a telephone subscriber who obtains an engaged signal usually repeats the call until the required connection is made. As a result, the flow of calls circulating in a telephone network consists of two parts: the flow of primary calls, which reflects the real needs of the telephone subscribers, and the flow of repeated calls, which is a consequence of the lack of success of previous attempts. The classical model of a telephone system, a queueing system with loss, does not take the structure of this real flow of calls into consideration and therefore cannot be applied in solving a number of practically important problems. A new class of queueing systems, a system with repeated calls, has been introduced for their analysis.

One of the most important problems in this area is obtaining necessary and sufficient conditions for the existence of a stationary regime. Here we consider the case of a multichannel fully available system, previously considered by Cohen [1] and Deul [2]. In Cohen's paper an extra 'truncated' system with repeated calls of $M/M/c/m$ type was used, which led to cumbersome arguments. Deul applied a simple and natural method which consisted of considering the embedded Markov chain. For the classification of its states, as is usual in queueing theory, Deul used Mustafa's criterion: for an irreducible and aperiodic Markov chain $\{Z_n\}$ with state space S, a sufficient condition for ergodicity is the existence of a non-negative function $f(s)$, $s \in S$ and $\varepsilon > 0$ such that the mean drift $x_s = E\{f(Z_{n+1}) \mid Z_n = s\} - f(s) < \infty$ for all $s \in S$ and $x_s < -\varepsilon$ for all $s \in S$ (except perhaps a finite number). However, in [2] the structure of the test function $f(s)$ is very complicated and it is difficult to apply it to the more complex systems with repeated calls. We introduce here a test function of a very simple type, i.e. a linear combination of coordinates of a vector Markov process, which describes the functioning of a real

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system. This gives us a simple way of investigating various queueing systems with repeated calls. We illustrate the method by an example of the basic, fundamental multichannel model with repeated calls of $M/M/c/\infty$ type. Let λ be the intensity of the arrival process of primary calls, μ the intensity of repetition, $\nu = 1$ the service intensity, and c the number of channels. (For more details see [2].)

The functioning of the system is described by means of a Markov process of two variables $\xi(t) = (C(t), N(t))$, where $C(t)$ is the number of busy channels, $N(t)$ the number of subscribers who repeat earlier blocked calls, $C(t) = 0, 1, \cdots, c$; $N((t) = 0, 1, 2, \cdots$. Let $\tau_1, \tau_2, \cdots$ be the sequence of moments in time when the process $\xi(t)$ changes its states, and $\{Z_n\} = \{\xi(\tau_n + 0)\}$ the embedded Markov chain. We show that the condition $\lambda < c$ is sufficient for the ergodicity of the chain $\{Z_n\}$, and consequently for the initial process $\xi(t)$. If $\lambda < c$, then there exists a number a such that $\lambda/c < a < 1$. Consider the test function $f(i, j) = ai + j$ and use Mustafa's criterion to prove the ergodicity of the chain $\{Z_n\}$. If $0 \leqq i \leqq c - 1$, then the mean drift of the test function is $x_{ij} = \{\lambda a - ai + j\mu(a - 1)\}/\{\lambda + i + j\mu\}$. Consequently it is always true that $x_{ij} < \infty$. Also, $x_{ij} \to a - 1 < 0$ as $j \to \infty$. This implies that there exists an N_i such that if $j \geqq N_i$ then $x_{ij} < -(1 - a)/2$. If $i = c$ then $x_{cj} = (\lambda - ac)/(\lambda + c) < 0$ for all j. If $\varepsilon = \min\{(1 - a)/2, (ac - \lambda)/(\lambda + c)\}$ then all $x_{ij} < -\varepsilon$, except perhaps for a finite number, namely x_{ij} for $0 \leqq i \leqq c - 1$, $0 \leqq j \leqq N_i - 1$. Thus our chain $\{Z_n\}$ is obviously irreducible and aperiodic, and therefore ergodic.

This method can also be applied in the case of complicated systems with repeated calls. For example, assume that in the above system we have one more Poisson arrival process of calls with intensity α; the calls may wait after blocking and then immediately begin service. (This is a type of head-of-the-line priority discipline.) The functioning of the system is described by means of a process $\xi(t) = ((C(t), N(t))$, where $C(t)$ is the number of busy channels and the priority queue length and $N(t)$, as above, is the number of sources of repeated calls. (Note that now $0 \leqq C(t) < \infty$, differing from the previous case where $0 \leqq C(t) \leqq c$.) For the mean drift x_{ij} of the test function $f(i, j) = ai + j$ of the same kind as above, we have $x_{ij} = \{(\alpha + \lambda)a - ai + j\mu(a - 1)\}/(\alpha + \lambda + j\mu + i)$ if $0 \leqq i \leqq c - 1$ and $x_{ij} = (\alpha a + \lambda - ca)/(\alpha + \lambda + c)$ if $c \leqq i < \infty$. If $0 \leqq i \leqq c - 1$ then $x_{ij} \to a - 1$ as $j \to \infty$. Thus the conditions of Mustafa's criterion will be fulfilled if, simultaneously, $\alpha a + \lambda - ca < 0$ and $a - 1 < 0$, i.e. $\lambda/(c - \alpha) < a < 1$. Such a choice of a is possible if $\alpha + \lambda < c$. This is therefore a sufficient condition for the existence of a stationary regime for the system considered here.

References

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