LATTICE-ORDERED RINGS OF QUOTIENTS

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Introduction. R. E. Johnson **(10)**, Utumi **(18)**, and Findlay and Lambek **(7)** have defined for each ring R a unique maximal "ring of right quotients" Q. When R is a commutative integral domain (in this paper an integral domain need not be commutative) or an Ore domain, then Q is the usual division ring of quotients of R. Moreover, it is well known that in these special cases, if R is totally ordered, then so is Q.

The main purpose of this paper is to study the ring of quotients Q, and in particular its order properties, for certain lattice-ordered rings R. Since very little is known about the structure of general lattice-ordered rings, we shall restrict our attention to lattice-ordered rings which are subdirect sums of totally ordered rings; these are the f-*rings* of Birkhoff and Pierce **(4)**. For the sake of simplicity, but at the expense of some generality, we shall also assume that R has an identity.

As we shall show, the fact that R is an f-ring (even a totally ordered integral domain) does not imply that Q is an f-ring extension of R. If Q is an f-ring extension of R, then R is called a qf-*ring*. Two of our results are devoted to characterizing the qf-rings. The more interesting of these states that if R has a zero singular ideal **(10)**, then R is a qf-ring if and only if for all $a, b \in R$, if $aR \cap bR = 0$, then $a \perp b$. Thus the qf-integral domains are precisely the ordered Ore domains. In general, however, not every qf-ring, even with zero singular ideal, is an Ore ring.

Since not every semi-prime f-ring with the maximum condition for right l-ideals is a qf-ring, the natural f-ring analogue of Goldie's theorem **(8)** for the ring of quotients of a semi-prime noetherian ring is not possible. However, in §6 we obtain an analogue for qf-rings.

If the singular ideal of a ring R is zero, then Q is a regular right self-injective ring. Utumi **(18)** has characterized such rings in terms of cosets of principal right ideals; in §7 we prove a new characterization of regular self-injective f-rings.

1. Preliminaries. Unless explicitly stated otherwise all rings will be assumed to possess an identity element.

We begin this section by recalling a few facts concerning generalized rings of quotients. Further details can be found in **(7, 10, 11, 12,** and **18)**.

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If R is a ring, then there is a right R-module E, called the *injective envelope* of R, characterized by the properties **(6)**:

(1) R_R is an essential submodule of E_R;

(2) E_R is injective.

Given a right ideal D of R and an R-homomorphism $\phi:D \to R$, the injectivity of E insures the existence of an extension $\bar{\phi}:R \to E$ of ϕ. The right ideal D is *dense* in R in case each such ϕ has a unique extension $\bar{\phi} \in \mathrm{Hom}_R(R, E)$. Then one readily proves:

1.1. LEMMA. *For a right ideal D of R the following conditions are equivalent*:

(1) *D is dense in R*;

(2) *for each $h \in \mathrm{Hom}_R(E, E)$, $D \subseteq \ker h$ implies $R \subseteq \ker h$*;

(3) $\{x \in E; xD = 0\} = 0$.

Let Δ denote the set of all dense right ideals of R. Further easily proved properties of Δ include:

(D.1) *each $D \in \Delta$ is essential in R_R*;

(D.2) *Δ is a dual ideal in the lattice of right ideals of R*;

(D.3) *if $a \in E$ and if $D, D' \in \Delta$, then $\{x \in D; ax \in D'\} \in \Delta$*;

(D.4) *if $D \in \Delta$ and if I is a right ideal of R such that $(I{:}x) \in \Delta$ for each $x \in D$, then $I \in \Delta$* (if A, B are subsets of a right R-module M, then $(A{:}B) = \{x \in R; Bx \subseteq A\}$).

1.2. THEOREM (Utumi **18**). *The ring R has a unique, to within isomorphism over R, ring extension Q satisfying*:

(1) *$D_q = \{x \in R; qx \in R\}$ is dense for each $q \in Q$*;

(2) *for each $D \in \Delta$ and each $\phi \in \mathrm{Hom}_R(D, R)$ there is a unique $q \in Q$ such that*

$$\phi(x) = qx \qquad (x \in D).$$

The unique ring extension Q of R assured by Utumi's theorem is called the *maximal ring of right quotients* of R (when no ambiguity is likely, we shall call Q simply the ring of quotients of R). Several characterizations of Q are known; cf. **(12)**. Here it will suffice to observe that with the obvious ring structure,

$$\{x \in E; (R{:}x) \in \Delta\}$$

is isomorphic to Q.

Next we review some material from the theory of lattice-ordered rings. A detailed treatment can be found in the work of Birkhoff and Pierce **(4)**. For more on the structure of f-rings, see D. G. Johnson **(9)**.

A partial ordering $\geqslant$ defined on the underlying set of a ring R is compatible with the ring structure in case for all $x, y, z \in R$,

(i) $x \geqslant y$ implies $x + z \geqslant y + z$;

(ii) $x \geqslant 0$ and $y \geqslant 0$ imply $xy \geqslant 0$.

If $\geqslant$ is a compatible partial ordering on R, then the set P of positive elements clearly satisfy:

$$P \cap (-P) = 0; \qquad P + P \subseteq P; \qquad PP \subseteq P.$$

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Moreover, the correspondence associating with each compatible partial ordering its positive set is one to one.

By a *partially ordered ring* is meant a pair consisting of a ring and a compatible partial order. A *lattice ordered ring* (a *totally ordered ring*) is a partially ordered ring whose partial ordering is a lattice ordering (a total ordering). An f-*ring* is a lattice-ordered ring R such that for all $x, y, z \in R$,

$$x \wedge y = 0 \text{ and } z \geqslant 0 \quad \text{imply } xz \wedge y = zx \wedge y = 0.$$

1.3. THEOREM (Birkhoff and Pierce **4**). *A lattice-ordered ring is an* f-*ring if and only if it is isomorphic to a subdirect sum of totally ordered rings.*

We now recall some notation. Let R be an f-ring and let $a \in R$. Then we set $a^+ = a \vee 0$, $a^- = -(a \wedge 0)$, and $|a| = a \vee (-a)$. It follows that

$$a = a^+ - a^-,$$
$$|a| = a^+ + a^-,$$
$$a^+ \wedge a^- = 0.$$

Two facts that we shall frequently use are that for $a, b \in R$ with $b \geqslant 0$,

$$a^+b = (ab)^+ \text{ and } a^-b = (ab)^-.$$

For $a, b \in R$ we write $a \perp b$ and say that a and b are *orthogonal* in case $|a| \wedge |b| = 0$. If $A \subseteq R$, then

$$A^+ = \{a \in A; a \geqslant 0\},$$
$$A^\perp = \{x \in R; x \perp a \text{ for all } a \in A\}.$$

Again let R be an f-ring. A right ideal I of R is an l-*ideal* in case for each $a \in I$ and each $x \in R$ if $|x| \leqslant |a|$, then $x \in I$. The set $N(R)$ of all nilpotent elements of R is an l-ideal of R called the l-*radical* of R.

1.4. THEOREM (Pierce **17**). *If R is an* f-*ring, then $N(R) = 0$ if and only if R is a subdirect sum of totally ordered integral domains.*

1.5. COROLLARY. *An* f-*ring R has no non-zero nilpotent elements if and only if the ring R is semi-prime.*

1.6. COROLLARY. *Let R be a semi-prime* f-*ring and let $a, b \in R$. Then $ab = 0$ if and only if $a \perp b$.*

2. Uniqueness of order. Let R be an f-ring and let Q be the ring of quotients of R.

2.1. LEMMA. *If D is a dense right ideal of the ring R, then D^+R is also dense in R.*

Proof. Let $x \in E$ with $xD^+ = \{0\}$ and let $a \in R$ with $xa \in R$. Set

$$C = \{d \in D; a^+d, a^-d \in D\}.$$

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Then by (D.2) and (D.3) we have $C \in \Delta$. For each $d \in C^+$

$$(xa)^+d = (xad)^+ = (xa^+d - xa^-d)^+ = 0$$

since $a^+d,\ a^-d \in D^+$. Thus $(xa)^+C^+ = 0$. If $c_1,\ c_2 \in C$, then $(c_1 + c_2)^2,\ c_1^2,\ c_2^2 \in C^+$ so that

$$0 = (xa)^+(c_1 + c_2)^2 = (xa)^+c_1 c_2 + (xa)^+c_2 c_1.$$

In any totally ordered ring, this would force each term to be zero, so using Theorem 1.3 we infer that $(xa)^+c_1 c_2 = 0$ for each pair $c_1, c_2 \in C$. But $C \in \Delta$, whence, by Lemma 1.1, we have $(xa)^+ = 0$. Similarly, $(xa)^- = 0$ and thus $xa = 0$. Therefore since R_R is essential in E, we conclude that $x = 0$, so by Lemma 1.1, $D^+R \in \Delta$.

2.2. Lemma. *Let $q \in Q$ and $D \in \Delta$. If $qD^+ \subseteq R^+$, then $qD_q{}^+ \subseteq R^+$.*

Proof. Let $d \in D_q{}^+$ and set

$$C = \{x \in D;\ dx \in D\}.$$

Then by (D.3), $C \in \Delta$. Since $d \geqslant 0$, $dC^+ \subseteq D^+$, whence

$$(qd)C^+ \subseteq qD^+ \subseteq R^+.$$

Thus $(qd)^-C^+ = 0$. By Lemma 2.1, $C^+R \in \Delta$, so by Lemma 1.1, $(qd)^- = 0$, and therefore $qD_q{}^+ \subseteq R^+$.

Let S be a subring of Q containing R. Then S admits at least one compatible partial order extending that of R, namely that obtained by taking $S^+ = R^+$. Since the property of being the positive set for a compatible partial order is one of finite character, there is at least one maximal partial order for S such that $S^+ \cap R = R^+$.

2.3. Theorem. *Let S be a subring of Q containing the* f*-ring R. Then there is a unique maximal partial ordering for the ring S relative to which $S^+ \cap R = R^+$. In fact, in this ordering*

$$S^+ = \{s \in S;\ sD_s{}^+ \subseteq R^+\}.$$

Proof. Set $P = \{s \in S;\ sD_s{}^+ \subseteq R^+\}$. It will clearly suffice to show that P is the positive set for a compatible partial ordering on S. So first let $s,\ -s \in P$. Then $sD_s{}^+ \subseteq R^+ \cap (-R^+) = 0$, so that $sD_s{}^+ = 0$. Hence from Lemma 1.1 and Lemma 2.1 we infer that $s = 0$. Next let $s, t \in P$. Then

$$(s + t)(D_s \cap D_t)^+ = (s + t)(D_s{}^+ \cap D_t{}^+) \subseteq R^+.$$

Therefore, by (D.2) and Lemma 2.2, $s + t \in P$. Finally, let

$$C = \{x \in D_t;\ tx \in D_s\}.$$

Then by (D.3), $C \in \Delta$. Since we clearly have $stC^+ \subseteq R^+$, it follows from Lemma 2.2 that $st \in P$.

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If S is a ring between R and Q, then the ordering for S described in the last theorem will be called the *canonical ordering* for S. In the remainder of the paper, unless otherwise stated, we shall assume that each such S is equipped with its canonical ordering.

2.4. THEOREM. *Let S be a subring of Q containing the* f*-ring R. If S admits a partial ordering relative to which it is an* f*-ring extension of R, then this partial ordering is the canonical one.*

Proof. Let P be the positive set for such an f-ring ordering on S and let S^+ be the positive set for the canonical ordering. By the maximality of S^+ it will suffice to show that $S^+ \subseteq P$. So let $s \in S^+$ and let

$$D = D_s \cap D_{s^+} \cap D_{s^-}.$$

(All lattice operations are taken relative to P.) If $s \notin P$, then $s^- \neq 0$; thus by Lemmas 1.1 and 2.1, there is an $a \in D^+$ such that $s^-a > 0$. As S is an f-ring relative to P, we have $(s^+a) \perp (s^-a)$. Thus

$$sa = s^+a - s^-a \notin P.$$

But $s \in S^+$ and $a \in D_s^+$, so that $sa \in R^+$, contrary to $R^+ \subseteq P$. Thus $S^+ = P$.

3. qf-rings. If the quotient ring Q is an f-ring in its canonical ordering, then we call the f-ring R a qf-*ring*.

3.1. THEOREM. *An* f*-ring R (with identity) is a* qf*-ring if and only if for each $q \in Q$ and each pair $d_1, d_2 \in D_q^+$,*

(i) $$(qd_1)^+ \wedge (qd_2)^- = 0.$$

(ii) $$d_1 \wedge d_2 = 0 \textit{ implies } (qd_1)^+ \wedge d_2 = 0.$$

Remark: Not every f-ring R, without identity, can be embedded in an f-ring with identity (**9**, Chapter III). Thus this result is less general than possible. However, every left faithful qf-ring, without identity, is, by virtue of (**7**, Proposition 6.2), embeddable in an f-ring with identity. Hence there is no loss of generality in our assumption in §§4–8.

Proof. The necessity of the condition is clear from Theorem 1.3. Conversely, assume the stated condition. We show first that the canonical ordering on Q is a lattice ordering, and for this it will suffice to show that $q \vee 0 \in Q$ for each $q \in Q$ (**3**, p. 215). So let $q \in Q$. Let $d_i \in D_q^+$ and $r_i \in R$ $(i = 1, \ldots, n)$, and suppose that

$$\textstyle\sum_i d_i r_i = 0.$$

By Theorem 1.3, R is a subdirect sum of totally ordered rings, and since the assumed condition implies that the signs of the qd_i all agree co-ordinate-wise,

$$\textstyle\sum_i (qd_i)^+ r_i = 0.$$

Therefore there is an $h \in \mathrm{Hom}_R(D_q{}^+R, R)$ such that

$$h(\textstyle\sum_i d_i r_i) = \sum_i (qd_i)^+ r_i$$

for all $d_i \in D_q{}^+$ and $r_i \in R$ $(i = 1, \ldots, n)$. By Lemma 2.1, $D_q{}^+R \in \Delta$; so by Utumi's theorem there is a $q^* \in Q$ such that

$$q^*x = h(x) \qquad (x \in D_q{}^+R).$$

As $q^*D_q{}^+ \subseteq R^+$, it follows from Lemma 2.2 that $q^* \geqslant 0$. Also for each $d \in D_q{}^+$,

$$(q^* - q)d = (qd)^+ - (qd) = (qd)^- \geqslant 0,$$

so that $q^* \geqslant q$. Now if $p \in Q^+$ such that $p \geqslant q$, then $pd \geqslant 0$ and $pd \geqslant qd$ for all $d \in (D_p \cap D_q)^+$. Therefore

$$(p - q^*)d = pd - (qd)^+ \geqslant 0$$

for all $d \in (D_p \cap D_q)^+$ and so $p \geqslant q^*$. Hence $q^* = q \vee 0$ and the canonical ordering for Q is a lattice ordering.

Finally we show that Q is an f-ring. For this it will suffice (**4**, p. 59, Corollary 1) to prove that if $s \in Q^+$, then multiplication by s is both left and right distributive over joins. So let $p, q \in Q$ and set

$$D = \{x \in R;\ px, qx, (p \vee q)x \in D_s\}.$$

Then by (D.2), (D.3), and Lemma 2.1, $D^+R \in \Delta$. For each $x \in D^+$, $(q - p)^+x \in D_s{}^+$ and so by condition (ii), $s(q-p)^+x \perp s(q-p)^-x$. Therefore, $s(q-p)^+x = [s(q-p)x]^+$, whence

$$\begin{aligned} s(p \vee q)x &= s[p + (q-p)^+]x = spx + [s(q-p)x]^+ \\ &= spx + (sq - sp)^+x = (sp \vee sq)x. \end{aligned}$$

So by Lemma 1.1, $s(p \vee q) = sp \vee sq$. For the other side, let

$$C = \{x \in R;\ sx \in D_p \cap D_q \cap D_{p \vee q}\}.$$

Again $C^+R \in \Delta$ and for each $x \in C^+$,

$$\begin{aligned} (p \vee q)sx &= [p + (q-p)^+]sx = psx + [(q-p)sx]^+ \\ &= psx + [(q-p)s]^+x = (ps \vee qs)x. \end{aligned}$$

So, as before, $(p \vee q)s = ps \vee qs$ and Q is an f-ring.

3.2. COROLLARY. *Every commutative* f-*ring with identity is a* qf-*ring.*

Proof. Let $q \in Q$ and let $d_1, d_2 \in D_q{}^+$. Since R is a commutative f-ring, we have for each $d \in D_q{}^+$,

$$[(qd_1)^+ \wedge (qd_2)^-]d = (qd)^+d_1 \wedge (qd)^-d_2 = 0.$$

As $D_q{}^+R \in \Delta$, this implies that $(qd_1)^+ \wedge (qd_2)^- = 0$. If $d_1 \wedge d_2 = 0$, then for each $d \in D_q{}^+$,

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$$[(qd_1)^+ \wedge d_2]d = (qd)^+d_1 \wedge d_2d = 0,$$

and as before $(qd_1)^+ \wedge d_2 = 0$.

Recall that an f-ring R is *Archimedean* in case for every pair $a, b \in R$ if $na \leqslant b$ for all integers n, then $a = 0$.

3.3. Corollary. *If R is an Archimedean* f-*ring, then R is a* qf-*ring and Q is Archimedean.*

Proof. Since an Archimedean f-ring is commutative (**4**, Theorem 13), the first statement follows from Corollary 3.2. Finally, if $p, q \in Q$ with $np \leqslant q$ for all n, then $npd \leqslant qd$ for each $d \in (D_p \cap D_q)^+$ and each n. But since R is Archimedean, this means that $pd = 0$ for all $d \in (D_p \cap D_q)^+$; hence $p = 0$.

3.4. Theorem. *Let R be a* qf-*ring. If $e \in R$ is a weak order unit, then e is also a weak order unit of Q.*

Proof. Let $q \in Q$ with $e \wedge q = 0$. For each $d \in D_q^+$ we have

$$(e \wedge q)d = e \wedge qd = 0$$

since Q is an f-ring. Since e is a weak order unit in R, this means that $qd = 0$ for all $d \in D_q^+$; thus $q = 0$.

3.5. Corollary. *If R is a totally ordered* qf-*ring, then Q is totally ordered.*

In general, strong order units in qf-rings are not strong order units in the ring of quotients. For example, a hyper-real field is the ring of quotients of the ring of its bounded elements.

4. qf-rings with zero singular ideal. An element x of the ring R is a singular element of R if $(0{:}x)$ is an essential right ideal. The set of all singular elements of R form a two-sided ideal called the *singular ideal* of R (**10**). The rings R with zero singular ideal are precisely those for which Q is regular (in the sense of von Neumann).

4.1. Lemma. *Let R be a* qf-*ring. Then the singular ideal of R is zero if and only if R is semi-prime; that is, if and only if R has no non-zero nilpotent elements.*

Proof. If the singular ideal of R is zero, then Q is a regular f-ring and thus a subdirect sum of totally ordered division rings. Clearly then the subring R has no non-zero nilpotent elements. Conversely, by (1.6), if R is semi-prime and if $x \in R$, then $(0{:}x) = x^\perp$. Thus, as $ax \perp x$ implies $ax = 0$, it is clear that if $(0{:}x)$ is essential in R; then $x = 0$.

The next example shows that in general the l-radical and the singular ideal of an f-ring are not the same.

4.2. *Example.* Let S be the semigroup with identity e and zero element 0 on generators a and ba with $a^2 = 0$. Totally order S by

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$$e > \ldots > (ba)^n > (ba)^{n+1} > \ldots > a > \ldots > a(ba)^n > a(ba)^{n+1} > \ldots > 0.$$

Finally, let R be the semigroup ring on S over the rational field and totally order R lexicographically. The largest annihilator right ideal of R is the inessential right ideal aR; thus R has zero singular ideal. However, the l-radical of R is aR; hence R is not semi-prime. In particular, R is not a qf-ring.

Two further properties of a ring with zero singular ideal that we shall need are, first, that Δ coincides with the set of essential right ideals and, second, that Q_R is injective **(18)**.

4.3. THEOREM. *Let R be an* f*-ring with zero singular ideal. Then R is a* qf*-ring if and only if for each pair $a, b \in R^+$*

$$aR \cap bR = 0 \text{ implies } a \perp b.$$

Proof. (Necessity) Assume R is a qf-ring and let $a, b \in R^+$ with $aR \cap bR = 0$. Then there is an R-homomorphism $h{:}aR \oplus bR \to R$ defined by

$$h(ax + by) = ax - by \qquad (x, y \in R).$$

Since Q_R is injective, there is a $q \in Q$ such that

$$q(ax + by) = ax - by \qquad (x, y \in R).$$

By hypothesis Q is an f-ring, so that $(qa)^+ \perp (qb)^-$. But $(qa)^+ = a$ and $(qb)^- = b$.

(Sufficiency) Assume the given condition for R, let $q \in Q$, and let $d_1, d_2 \in D_q{}^+$. If $d_1 \wedge d_2 = 0$, then clearly $[(qd_1)^+ \wedge d_2]^2 = 0$, whence $(dq_1)^+ \wedge d_2 = 0$. Next set

$$s = (qd_1)^+ \text{ and } t = (qd_2)^-.$$

By Theorem 3.1 it will suffice to show that $s \wedge t = 0$. We establish this via a sequence of numbered steps.

(1) *For each right ideal I* of R, $I + I^\perp \in \Delta$: For if J is a right ideal of R with $J \cap I = 0$, then by hypothesis $J \subseteq I^\perp$. So if $J \cap (I + I^\perp) = 0$, then $J \subseteq I^\perp$ and $J \cap I^\perp = 0$, whence $J = 0$. Thus $I + I^\perp$ is essential in R so that as the singular ideal is zero, $I + I^\perp \in \Delta$.

(2) *For each $d \in D_q$, $qd \in d^{\perp\perp}$.* Since R is semi-prime, it follows from (1.6) that $d^\perp = (0{:}d) \subseteq (0{:}qd)$ and $(0{:}qd)^\perp = (qd)^{\perp\perp}$. Thus

$$qd \in (qd)^{\perp\perp} = (0{:}qd)^\perp \subseteq d^{\perp\perp}.$$

(3) $(d_2 R{:}d_1) \subseteq (0{:}s \wedge t)$. For let $d_1 x = d_2 y$. As $d_1, d_2 \geqslant 0$, we have $d_1|x| = d_2|y|$ so that $(qd_1)^+|x| = (qd_2)^+|y|$. Therefore $s|x| \perp t$. This implies that $(s \wedge t)|x| = 0$ so that $x \in (0{:}s \wedge t)$.

(4) $(d_2 R{:}d_1)^\perp \subseteq (0{:}s \wedge t)$. For let $x \in (d_2 R{:}d_1)^\perp$; we may assume that $x \geqslant 0$. Then $d_1 x \perp d_2 x$; for if not, our hypothesis implies the existence of

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$u, v \in R$ such that $d_1 xu = d_2 xv \neq 0$, whence $xu \in (d_2 R:d_1) \cap (d_2 R:d_1)^{\perp}$ contrary to $xu \neq 0$. From (2) we infer that

$$(sx \wedge tx) \in (d_1 x)^{\perp\perp} \cap (d_2 x)^{\perp\perp}.$$

But as $d_1 x \perp d_2 x$, we have $(d_1 x)^{\perp\perp} \cap (d_2 x)^{\perp\perp} = 0$, whence

$$(s \wedge t)x = sx \wedge tx = 0.$$

Finally (1), (3), and (4) together with Lemma 1.1 yield the desired fact that $s \wedge t = 0$.

5. The classical case. Let R be a ring and let M be the set of all non zero-divisors of R. An over-ring Q_c of R is a *classical ring of quotients* of R in case each element of M is invertible in Q_c and

$$Q_c = \{ad^{-1};\, a \in R \text{ and } d \in M\}.$$

If R has a classical ring of quotients, then it is unique to within isomorphism over R. The ring R is an *Ore ring* when for each $a \in R$ and $d \in M$,

$$(dR:a) \cap M \neq \emptyset.$$

It is well known that the ring R has a classical ring of quotients if and only if it is an Ore ring. Moreover, if R is an Ore ring, then Q_c is a subring of Q **(12)**.

5.1. THEOREM. *Let R be both an* f*-ring and an Ore ring. Then in its canonical order Q_c is an* f*-ring.*

Proof. Since $d^{-1} = d(d^2)^{-1}$ for each $d \in M$, it follows that

$$Q_c = \{ad^{-1};\, a \in R \text{ and } d \in M^+\}.$$

Set

$$P = \{ad^{-1};\, a \in R^+ \text{ and } d \in M^+\}.$$

Then it is a routine matter to show that P is the positive set for a partial ordering on Q_c such that $P \cap R = R^+$. To complete the proof it will suffice, in view of Theorem 2.4, to show that Q_c is an f-ring relative to the ordering given by P. So let $ad^{-1} \in Q_c$, where $d > 0$. Set $(ad^{-1})^* = a^+d^{-1}$. Then

$$(ad^{-1})^* - (ad^{-1}) = (a^+ - a)d^{-1} = a^-d^{-1} \in P,$$

so that (in the ordering given by P) $(ad^{-1})^* \geqslant ad^{-1}, 0$. Next let $bc^{-1} \geqslant ad^{-1}, 0$ where $c \in M^+$. As R is an Ore ring, there exist $h, k \in M$ such that $ch = dk$. As $c, d \geqslant 0$ and $|h|, |k| \in M$, we may assume that $h, k \geqslant 0$. Then

$$(bc^{-1}) - (ad^{-1})^* = (bh)(ch)^{-1} - (a^+k)(dk)^{-1} = (bh - a^+k)(ch)^{-1},$$

and similarly

$$(bc^{-1}) - (ad^{-1}) = (bh - ak)(ch)^{-1}.$$

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Since $bc^{-1} \geqslant ad^{-1}$ in Q_c, we have in R

$$bh - ak \geqslant 0.$$

Thus, in R,

$$bh - a^+k = (bh - ak) + a^-k \geqslant 0,$$

whence $(bh - a^+k)(ch)^{-1} \in P$. That is, $bc^{-1} \geqslant (ad^{-1})^*$. Therefore

$$(ad^{-1})^* = (ad^{-1}) \vee 0$$

and Q_c is an l-ring. Finally, to show that Q_c is an f-ring, it will suffice to show that absolute value is preserved under multiplication **(4**, §9**)**. So let $h, k \in M$ with $dh = bk$. Then

$$\begin{aligned} |ad^{-1}||bc^{-1}| &= (|a|d^{-1})(|b|c^{-1}) = (|a||h|)(c|k|)^{-1} \\ &= |ah||ck|^{-1} = |(ad^{-1})(bc^{-1})|. \end{aligned}$$

Applying our results and Ore's theorem **(16)** to totally ordered integral domains, we obtain:

5.2. Corollary. *Let R be a totally ordered integral domain* (*not necessarily commutative*). *Then the following are equivalent:*

(1) *R is a* qf*-ring;*
(2) *R is an Ore ring;*
(3) *Q is a totally ordered division ring;*
(4) *Q is a division ring.*

Moreover, when these conditions apply, $Q = Q_c$.

The fact that an f-ring with zero l-radical is a subdirect sum of totally ordered integral domains suggests the possibility that such an f-ring R is a qf-ring if and only if it is an Ore ring, and moreover that when R is a qf-ring, $Q = Q_c$. The following examples discount these.

5.3. *Example.* Let R be the sub-f-ring of $\mathbf{Q}^{\mathbf{Z}}$, the f-ring of all rational valued functions on the integers, consisting of those functions that are constant off finite sets. Then R is clearly a commutative f-ring with zero singular ideal. Moreover, in this case $Q = \mathbf{Q}^{\mathbf{Z}}$; cf. **(18**, (2.1)**)**. Also, it is clear that each non zero-divisor of R is invertible, so $Q_c = R$. Thus a qf-ring which is an Ore ring need not have $Q = Q_c$.

5.4. *Example.* It is known that the group ring $\mathbf{Q}[F]$ over the rationals on the free group F with two generators a and b admits a total order, that it is not an Ore ring, but that it can be embedded in a totally ordered division ring K; cf. **(15)**. Let R be the sub-f-ring of $K^{\mathbf{Z}}$ consisting of those functions that assume only finitely many values outside of $\mathbf{Q}[F]$. Then R is easily seen to be a qf-ring with zero singular ideal (and $Q = K^{\mathbf{Z}}$). However, R is not an Ore ring since $\mathbf{Q}[F]$ is not. Thus, a semi-prime qf-ring need not be an Ore ring.

Finally, we remark that we have been unable to determine whether or not in general every semi-prime Ore f-ring is a qf-ring. In the next section, however, we shall find one class of semi-prime f-rings for which the two properties are equivalent.

6. Semi-prime f-rings with a maximum condition. An l-ideal I (left, right, or two-sided) of an f-ring R is *closed* in case $I = I^{\perp\perp}$. For a semi-prime f-ring an l-ideal I is closed if and only if it is a right annihilator ideal; moreover, each closed l-ideal is two-sided; cf. (**4**, p. 63, Corollary 2).

6.1. THEOREM. *Let R be a semi-prime* f-*ring satisfying the maximum condition for closed* l-*ideals. Then R is a* qf-*ring if and only if it is an Ore ring. Moreover, if R is a* qf-*ring, then $Q = Q_c$.*

Proof. By (**1**, Lemmas 1 and 4) R has a finite set of maximal closed l-ideals $\{M_1, \ldots, M_n\}$, each $I_i = M_i^{\perp}$ is a totally ordered closed l-ideal of R, the sum $D = I_1 + \ldots + I_n$ is direct, and $D^{\perp} = 0$. For each $i = 1, \ldots, n$, let $x_i \in I_i$ be non-zero. As each I_i is totally ordered, it is clear that $x_i^{\perp} = I^{\perp} = M_i$, and as the sum of the I_i's is direct, it is clear that if $x = x_1 + \ldots + x_n$, then $x^{\perp} = D^{\perp} = 0$. Thus, since R is semi-prime, x is a non zero-divisor. If $a \in R$, then as each I_i is a two-sided ideal, $ax_i \in I_i$. Therefore, we have that $ax \in D$, and since $x^{\perp} = 0$, that $(ax)^{\perp} = a^{\perp}$.

Now suppose that R is a qf-ring and that $a, d \in R$ with d a non zero-divisor. By Theorem 4.3 and the fact that each I_i is totally ordered, there exist $s_i, t_i \in R$ such that

$$ax_i s_i = dx_i t_i \qquad (i = 1, \ldots, n),$$

and such that $ax_i s_i = 0$ only if $ax_i = 0$. As each I_i has no non-zero zero-divisors, we may assume that $x_i s_i \neq 0$ for all $i = 1, \ldots, n$. Thus

$$(x_1 s_1 + \ldots + x_n s_n)^{\perp} = D^{\perp} = 0,$$

whence $x_1 s_1 + \ldots + x_n s_n$ is not a zero-divisor. Moreover, since

$$a(x_1 s_1 + \ldots + x_n s_n) = d(x_1 t_1 + \ldots + x_n t_n)$$

R is an Ore ring.

The same basic approach, namely reducing to D by multiplying everything in sight by x, is used in proving that if R is an Ore ring, then it is a qf-ring. We omit the details.

Finally, if R is a qf-ring, then Theorem 4.3 implies that each I_i has no proper essential submodule or supermodule in R; thus R is finite dimensional in the sense of (**11**) and so, by (**11**, Theorem 4.4), $Q = Q_c$.

Now it is an easy matter to prove an f-ring version of Goldie's theorem (**8**, Theorem 4.4) for semi-prime rings with a maximum condition.

6.2. THEOREM. *Let R be a* qf-*ring. Then R is a semi-prime* f-*ring with the maximum condition for closed* l-*ideals if and only if Q is a direct sum of totally ordered division rings.*

Proof. For the necessity we first observe that by the previous theorem R is an Ore ring. Next it follows from Theorem 4.3 that any independent sequence of right ideals of the ring R is pairwise orthogonal. Therefore, R has no infinite independent set of non-zero right ideals. So R satisfies Goldie's r.q. conditions (**8**, p. 206), and hence by (**8**, Theorem 4.4), $Q = Q_c$ is a semi-simple artinian ring. But since R is a qf-ring, Q is an f-ring; hence Q must have the asserted structure. In our case the converse is absolutely trivial since R is a sub-f-ring of Q, and R inherits both the maximum condition for closed l-ideals and its l-radical from Q.

It is known (**8**, Theorem 4.1) that a noetherian semi-prime ring, indeed any ring with Goldie's r.q. conditions, is an Ore ring. Thus, by Theorem 6.1, every semi-prime f-ring with the maximum condition for right (ring) ideals is a qf-ring. The following example shows, however, that the maximum condition for right l-ideals does not force a semi-prime f-ring to be a qf-ring.

6.3. *Example.* The free group F on two generators a, b admits a total order (**14**), whence the set

$$S = \{x \in F; x \geqslant e\}$$

is a sub-semigroup, where e is the identity of F. We may assume that $a, b \in S$. The semigroup ring R of S over the rational field can be totally ordered lexicographically in such a way that the natural mapping $S \to R$ preserves the order of S. Then R is a totally ordered integral domain, and every non-zero right (or left) l-ideal contains 1. In particular, R satisfies the maximum condition for right l-ideals. However, R is not a qf-ring since $aR \cap bR = 0$.

7. Self-injective f-rings. A ring R is *right self-injective* in case the right R-module R_R is injective. If R has a zero singular ideal, then Q is right self-injective (and regular). Utumi (**18**, Theorem 4) has proved that a regular ring R is right self-injective if and only if every family $\{a_\alpha + e_\alpha R\}_{\alpha \in \Omega}$ of cosets of principal right ideals which has the finite intersection property has non-void intersection. The main purpose of this section is to characterize self-injective f-rings by means of a type of order completeness.

7.1. LEMMA. *In a regular* f-*ring R every idempotent is central and every one-sided ideal is two-sided.*

Proof. A totally ordered regular ring is a division ring; thus a regular f-ring is strongly regular. The result now follows from (**2**, Theorem 3.4).

7.2. COROLLARY. *A semi-prime* f-*ring is right self-injective if and only if it is left self-injective.*

Proof. By Lemmas 4.1 and 7.1 together with the right- and left-hand versions of Utumi's characterization of self-injectivity.

In view of this corollary we shall dispense, in what follows, with the qualification "right" when speaking of self-injective f-rings.

If R is a regular ring and if $a \in R$, then there is an $x \in R$ such that $axa = a$; we denote by e_a the idempotent ax. By Lemma 7.1 if R is an f-ring, then also $e_a = xa$.

7.3. THEOREM. *Let R be a regular* f-*ring. Then the following conditions are equivalent:*

(1) *R is self-injective;*

(2) *for every pairwise orthogonal set S in R there is an $x \in R$ such that $xe_a = a$ for all $a \in S$;*

(3) *every pairwise orthogonal set of positive elements of R has a supremum in R.*

Proof. (1) implies (2). Let $S \subseteq R$ be pairwise orthogonal. Then $SR = \oplus \sum_{a \in S} e_a R$, so there is a $\phi \in \mathrm{Hom}_R(SR, R)$ such that $\phi(e_a) = a$ for all $a \in S$. As R is self-injective, it follows from (**5**, Theorem I.3.2) that there is an $x \in R$ satisfying the desired condition: $xe_a = a$ for all $a \in S$.

(2) implies (3). Let S be a pairwise orthogonal set of positive elements. Then there exists a set S' of pairwise orthogonal idempotents maximal with respect to $S \perp S'$. Set

$$T = S' \cup \{a + e_a; a \in S\}.$$

Then T is a maximal orthogonal set in R^+, and by (2) there is an $x \in R$ such that $xe_t = t$ for all $t \in T$. So for each $a \in S$ and each $j \in S'$ we have

$$xe_a = a + e_a \text{ and } xe_j = e_j = j.$$

We claim that $x - 1$ is the supremum of S. For first let $a \in S$. Then for all $b \in S$, and for all $j \in S'$,

$$(a - x + 1)^+ e_b = (ae_b - b - e_b + e_b)^+ = 0,$$
$$(a - x + 1)^+ e_j = (ae_j - e_j + e_j)^+ = 0.$$

So $(a - x + 1)^+ \in T^\perp$, whence by the maximality of T, $(a - x + 1)^+ = 0$ or $x - 1 \geqslant a$. Thus $x - 1$ is an upper bound for S. Next suppose $y \geqslant a$ for all $a \in S$. Then $y \geqslant 0$ and for each $a \in S$, $ye_a \geqslant ae_a = a$. So for each $a \in S$ and each $j \in S'$,

$$(x - 1 - y)^+ e_a = (xe_a - e_a - ye_a)^+ = (a + e_a - e_a - ye_a)^+ = 0,$$
$$(x - 1 - y)^+ e_j = (e_j - e_j - ye_j)^+ = 0.$$

Thus $(x - 1 - y)^+ \in T^\perp$ and, as before, $y \geqslant x - 1$. Hence $x - 1$ is the supremum of S.

(3) implies (1). By (**5**, Theorem I.3.2) it will suffice to show that if I is a (right) ideal of R and if $\phi \in \mathrm{Hom}_R(I, R)$, then there is an $x \in R$ such that

$$\phi(a) = xa \qquad (a \in I).$$

Let $\{e_\alpha\}_{\alpha\in\Omega}$ be a maximal set of orthogonal idempotents in I and for each $a \in \Omega$ set

$$x_\alpha = \phi(e_\alpha).$$

As $x_\alpha e_\alpha = x_\alpha$ and $x_\alpha e_\beta = 0$ for all $\alpha \neq \beta$ in Ω we infer that $\{x_\alpha^+\}_{\alpha\in\Omega}$ and $\{x_\alpha^-\}_{\alpha\in\Omega}$ are pairwise orthogonal sets in R^+. Let their suprema be s and t, respectively, and set $x = s - t$. Since $x_\alpha^+ \perp x_\beta^-$ for all $\alpha, \beta \in \Omega$, we have $s \wedge t = 0$ (**3**, p. 231), whence $x^+ = s$ and $x^- = t$. To complete the proof it will clearly suffice to show that for each idempotent $e \in I$,

$$\phi(e) = xe.$$

If $f \in R$ is an idempotent orthogonal to all the e_α, then $f \in I^\perp$, whence

$$(\phi(e) - xe)f = \phi(ef) - xef = 0.$$

Thus $(\phi(e) - xe)I^\perp = 0$. Now for each $\alpha \in \Omega$ we have (**13**, Theorem 25.1)

$$xe_\alpha = x^+e_\alpha - x^-e_\alpha = x_\alpha^+ - x_\alpha^- = x_\alpha.$$

Therefore, for each $\alpha \in \Omega$,

$$(\phi(e) - xe)e_\alpha = \phi(e_\alpha)e - xe_\alpha e = x_\alpha e - x_\alpha e = 0.$$

Then by the maximality of $\{e_\alpha\}_{\alpha\in\Omega}$ we have $(\phi(e) - xe)I = 0$. As R is regular, it follows from Theorem 4.3 that R is a qf-ring. So from statement (1) in the proof of that theorem and the fact that $I + I^\perp$ annihilates $\phi(e) - xe$, we conclude that $\phi(e) = xe$, and the proof is complete.

8. Left qf-rings. The maximal left ring of quotients L of a ring R is defined, in the obvious way, as the opposite ring of the right ring of quotients of the opposite ring of R. In general, L and Q are not isomorphic, and, in fact, a right self-injective ring need not be left self-injective (**18**, §5).

In Corollary 7.2 we saw that for regular f-rings, left and right self-injectivity are equivalent. This suggests that for f-rings we may be able to find even stronger results relating L and Q. As yet, however, our information is skimpy. We do not know, for example, whether every right qf-ring is a left qf-ring, or for R both a right and left qf-ring whether $L = Q$. With respect to this last problem we do have one positive result.

8.1. THEOREM. *If R is a semi-prime right and left* qf-*ring, then its maximal ring of right quotients is also a maximal ring of left quotients.*

Proof. Let Q be the f-ring of right quotients of R. Since Q is left self-injective (Corollary 7.2), it will suffice to show that ${}_RQ$ is an essential extension of ${}_RR$. So let $q \in Q$ be non-zero. As Q is regular, there is a $q' \in Q$ such that $qq'q = q$. By Lemma 7.1 $qq' = q'q$ is a central idempotent. Now R_R is essential in Q_R; so there exist $a, d \in R$ such that $qd = a \neq 0$. Since Q is an f-ring, $|a| \wedge |d| \neq 0$;

so since R is a left qf-ring, the left-hand version of Theorem 4.3 implies that $hd = ka \neq 0$ for some $h, k \in R$. Let $d' \in Q$ such that $dd'd = d$; then

$$dkq = dd'dkq = dkqd'd = dkqdd' = dkad' = dhdd' = dd'dh = dh \neq 0.$$

Thus, $Rq \cap R \neq 0$ and ${}_RR$ is essential in ${}_RQ$.

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