substituting $l - m\cos C - n\cos B$, etc., for x, y, z in (T). The result is $(a, b, c, f, g, h)(l - m\cos C - n\cos B,\ m - n\cos A - l\cos C,\ n - l\cos B - m\cos C)^2 \times \Sigma = \Delta(l^2 + m^2 + n^2 - 2mn\cos A - 2nl\cos B - 2lm\cos C)^2$, the condition sought for.

The triangle and its escribed parabolas.

By A. J. PRESSLAND, M.A.

§ 1. The problem "to inflect a straight line between two sides of a triangle so that the intercepted portion is equal to the segments cut off" has been discussed in the third volume of the *Proceedings.*

If we discuss the same analytically; taking CB and CA as axes of x and y (Fig. 1) and calling each segment k, the equation of the line considered is

$$x/(a-k) + y/(b-k) = 1, \quad \ldots \quad \ldots \quad (\alpha)$$

where $$k^2 = (a-k)^2 + (b-k)^2 - 2(a-k)(b-k)\cos C \quad \ldots \quad (\beta)$$

The envelope of (α) considering k unrestricted by (β) is

$$(x+y)^2 - 2(a-b)(x-y) + (a-b)^2 = 0 \quad \ldots \quad (\gamma)$$

a parabola touching the axis of x at $(a-b, 0)$
and the axis of y at $(0, b-a)$

and which can be shown to touch AB

at the point $\left(\dfrac{a^2}{a-b},\ -\dfrac{b^2}{a-b}\right)$.

Its axis is $\quad x + y = 0$

and tangent at vertex $x - y = \dfrac{a-b}{2}$.

§ 2. If we consider $x/(a-k) + y/(b+k) = 1$
which cuts off equal portions from BC and CA produced, the envelope is

$$(x-y)^2 - 2(a+b)(x+y) - (a+b)^2 = 0.$$

which touches CB at $(a+b,\ 0)$ the point l,
CA at $(0,\ a+b)$ the point k,
AB at $\left(\dfrac{a^2}{a+b}, \dfrac{b^2}{a+b}\right)$ the point t,

the axis being $\quad x - y = 0$

and tangent at vertex $\quad x + y = \dfrac{a+b}{2}$.

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§ 3. The foci of these parabolas are

$$x = -y = (a-b)/4\sin^2\tfrac{1}{2}C$$
$$x = \;\; y = (a+b)/4\cos^2\tfrac{1}{2}C$$

§ 4. Hence if ABC be the triangle and we bisect the angles we get the orthic system F, D, E, O.

Let α, β, γ be mid points of sides,

a, b, c be mid points of OD, OE, OF;

then A, B, C, α, β, γ, a, b, c are on the nine point circle of DEF.

We have shown that α, β, γ, a, b, c are foci of parabolas touching all the sides of ABC, so that any tangent cuts off equal portions from two sides, viz. :—

γ and c parabolas from CB and CA,
β and b parabolas from BC and BA,
α and a parabolas from AB and AC,

the Greek letter corresponding to direct section,
the Italian letter corresponding to transverse section.

§ 5. The points of contact of the parabolas are :—

(i.) on lines through C, $(a \pm b)$ distant from C.

(ii.) on AB dividing AB internally and externally in ratio $a : b$, oppositely to the bisectors of the angle C.

Whence the four lines CT, CA, Ct, CB form an harmonic pencil;

T is $\left(\dfrac{a^2}{a-b}, -\dfrac{b^2}{a-b}\right)$ and therefore lies on kl,

t is $\left(\dfrac{a^2}{a+b}, \dfrac{b^2}{a+b}\right)$ and therefore lies on fg.

§ 6. Taking any tangent so that the intercepted portion is equal to m times the segment cut off we have the equation

$$k^2(m^2 - 2 + 2\cos C) + 2k(a+b)(1-\cos C) - c^2 = 0. \quad \ldots \quad (\delta)$$

This shows that we can get two $(1 : m : 1)$ lines.

Let these be

$$x/(a-k_1) + y/(b-k_1) = 1$$
$$x/(a-k_2) + y/(b-k_2) = 1.$$

Their point of intersection is

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$$(a-b)x=(a-k_1)(a-k_2)=\frac{a^2(m^2-4\sin^2\tfrac{1}{2}C)+4a(a+b)\sin^2\tfrac{1}{2}C-c}{m^2-4\sin^2\tfrac{1}{2}C}$$

$$(b-a)y=(b-k_1)(b-k_2)=\frac{b^2(m^2-4\sin^2\tfrac{1}{2}C)+4b(a+b)\sin^2\tfrac{1}{2}C-c^2}{m^2-4\sin^2\tfrac{1}{2}C}$$

Eliminating m we get as the locus of such points

$$\frac{(a-b)x-a^2}{(b-a)y-b^2}=\frac{4a(a+b)\sin^2\tfrac{1}{2}C-c^2}{4b(a+b)\sin^2\tfrac{1}{2}C-c^2}=\frac{a(a+b)(a-b)^2/c^2-a^2}{b(a+b)(a-b)^2/c^2-b^2}$$

$$=\frac{4a^2\sin^2\tfrac{1}{2}C-(a-b)^2}{4b^2\sin^2\tfrac{1}{2}C-(a-b)^2}\qquad\ldots\qquad\ldots\qquad\ldots\qquad(\epsilon)$$

which show that the line passes through T, through the focus γ, and through $\left(\frac{a(a^2-b^2)}{c^2},-\frac{b(a^2-b^2)}{c^2}\right)$.

The last is a special point K; which lies not only on $bx+ay=0$, the line through the origin parallel to AB; but also on the circumscribing circle.

§ 7. The point determined by the intersection of the $\left\{\begin{matrix}1:1:1\\1:1:1\end{matrix}\right\}$ lines is

$$(a-b)(a^2+b^2-c^2-ab)x=(2a-b)ab^2$$
$$(b-a)(a^2+b^2-c^2-ab)y=(2b-a)a^2b\,;$$

it lies on

$$\frac{x}{(2a-b)b}+\frac{y}{(2b-a)a}=0$$

the line joining the origin to the intersection of

$$x/b-y/a=1$$

and

$$x/(2a-b)-y/(2b-a)=1.$$

The corresponding point for the c – parabola is

$$x(a-b)(a^2+b^2-c^2+ab)=(2a+b)(ab^2)$$
$$y(b-a)(a^2+b^2-c^2+ab)=(2b+a)(ba^2).$$

§ 8. As (ϵ) passes through T it follows that the two $(1:m:1)$ lines being tangents are equally inclined to AB.

A similar set of theorems are true of the c – parabola, and as c and γ are mid points of the arcs AcB and AγB it follows that th loci (ϵ) and (ϵ') are at right angles.

§ 9. If we consider the $(l:m:n)$ line whose equation is

$$x/(a-lk)+y/(b-nk)=1$$

we get a similar set of theorems; which can be verified by projection from those already found; and the loci (ϵ_1) (ϵ_1') will pass through the point K as before.

$$(\epsilon_1) \text{ is } \quad \frac{(an-bl)x-a^2n}{(bl-an)y-b^2l}=\frac{a(b^2-a^2)(an-bl)+c^2na^2}{-b(b^2-a^2)(bl-an)+c^2b^2l}$$

and on it will be situated two special points.

(i.) When $m=n$ the point is

$$x=\frac{1}{a-b\lambda}\cdot\frac{2ab-b^2\lambda}{2\cos C-\lambda}.$$

$$y=\frac{1}{b\lambda-a}\cdot\frac{b^2-(b\lambda-a)^2}{2\cos C-\lambda}$$

where $\lambda=l/n.$

Eliminating λ we get

$$(kx-b^2+by)\left(kx-l^2+\frac{ak}{b}y\right)=(bx+ay-ab)^2$$

where $k=2b\cos C-a$

an hyperbola having one asymptote parallel to AB and the other parallel to

$$(k^2-b^2)x+b(k-a)y=0.$$

(ii.) When $l=m$ we get as locus an hyperbola having one asymptote parallel to AB and the other parallel to

$$\frac{ax}{(a^2-c^2)^2-a^2b^2}+\frac{y}{b(a^2-b^2-c^2)}=0.$$

§ 10. It can be shown by transversals that Ag, Bf and CT meet in a point Q.

The locus of Q is the minimum ellipse* circumscribing the triangle ABC.

A geometrical proof of this without projections will be given in § 14.

The theorem may be stated thus:—If three parallel lines be drawn through the vertices of a triangle their isotoms intersect on the minimum ellipse.

* Steiner's *Gesammelte Werke*, Vol. I., p. 208.

§ 11. Different series of parabolas may be obtained by considering the envelopes of

$$\begin{aligned} x/k \quad &+ y/(b+k) = 1, \\ x/(a+k) &+ y/(k) \quad = 1, \\ x/k \quad &+ y/(b-k) = 1, \\ x/(a-k) &+ y/k \quad = 1. \end{aligned}$$

§ 12. A principle of triality would seem to hold; because any of these parabolas might be referred to A or B instead of C as origin. Such theorems can be transformed by means of the transversal property.

$$\frac{\mathrm{AE{\cdot}AF}}{\mathrm{CA{\cdot}AB}} + \frac{\mathrm{AF{\cdot}BD}}{\mathrm{AB{\cdot}BC}} + \frac{\mathrm{CD{\cdot}AE}}{\mathrm{BC{\cdot}CA}} = 0,$$

where DEF is a transversal cutting BC in D, CA in E, AB in F.

§ 13. Corresponding to K we get two other points by drawing through B and A parallels to the opposite sides. Calling these points K_c, K_b, K_a we have

$$\text{arc AB} = \text{arc } K_aC = \text{arc } K_bC.$$

Then $$\angle K_aK_cK_b = \pi - K_aBK_b = \pi - 2C.$$

Thus $K_aK_bK_c$ has angles $\pi - 2A$, $\pi - 2B$, $\pi - 2C$ and is similar to the pedal or orthic triangle.

Again $$\text{arc AB} = \text{arc } K_bC;$$

therefore $$\text{arc } AK_b = \text{arc CB},$$

and $$\angle AK_aK_b = \angle BAC = A.$$

Now $$\angle K_bK_aC = \pi - 2A;$$

therefore $$\angle CK_aR = A;$$

therefore $$K_a\alpha\gamma = A = K_a\gamma\alpha,$$

and B, A, β, α are concyclic.

The sides of $K_aK_bK_c$ are therefore anti-parallel to those of ABC, and in pairs equally inclined to the sides of ABC.

The circumscribing circles of the triangles formed by $K_a\beta\alpha$, $\beta\delta C$, $K_a\delta\gamma$, $\alpha\gamma C$ meet in a point Q, say.

Let the circumcircle of $\gamma\delta C$ cut that of ABC in F;

then $$\begin{aligned} \gamma\delta C &= \alpha\gamma\delta - \alpha C\delta \\ &= A - C; \end{aligned}$$

therefore $$\gamma FC = A - C.$$

Now $$K_aFC = \pi - C \text{ for } K_aC = AB;$$

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therefore $K_aF\gamma = \pi - A$,

or $K_aF\gamma + K_a\alpha\gamma = \pi$,

and therefore the circumcircle of $K_a\alpha\gamma$ passes through F, and hence Q and F coincide.

If the sides of the triangles ABC, $K_aK_bK_c$ be taken three and three, 20 triangles may be obtained whose circumcircles all pass through F. Two of these circumcircles obviously coincide.

F is Steiner's point for the triangle ABC, as may be proved by analysis.

It may also be shown that C is on the radical axis of $AK_a\beta$ and $BK_b\alpha$.

The triangle $K_aK_bK_c$ is twice the linear dimensions of the orthic triangle of ABC, and in position it is this same triangle turned through two right angles.

If instead of turning the triangle $K_aK_bK_c$ through two right angles, we turn ABC through two right angles we get another position of F diametrically opposite its former one. This new position is called Tarry's point.

It should be noticed that as $K_aK_bK_c$ is derived from ABC so can ABC be derived from the median triangle of DEF.

§ 14. Minimum Ellipse.

ABC is the triangle, K the intersection of the lines AD, BE, CF, D, E, F, points of contact of parabola.

Join AG and produce to H so that GH = GA; let AH cut BE in M, and join KH cutting AC in L.

We shall show the anharmonic ratio.

K(ABHC) is constant; whence K is on the minimum ellipse.

Let $\lambda = \frac{AF}{BA} = \frac{AE}{CE} = \frac{BC}{CD}$ [Apollonius *Conics*, III. 41.]

We have by transversals

$$EK \cdot MH \cdot AL = MK \cdot AH \cdot EL,$$

$$BK \cdot AF \cdot CE = EK \cdot BF \cdot AC,$$

and $$AE \cdot BC \cdot \alpha M = CE \cdot \alpha B \cdot AM$$

or $$2AE \cdot \alpha M = CE \cdot AM;$$

therefore $$\frac{BK}{EK} = \frac{BF \cdot AC}{AF \cdot CE} = \frac{(\lambda+1)^2}{\lambda};$$

therefore $$\frac{BE}{EK} = \frac{\lambda^2+\lambda+1}{\lambda}.$$

Again $$\frac{C\alpha}{B\alpha}.\frac{BM}{EM}.\frac{EA}{CA}=1,$$

or $$\frac{BM}{EM}=\frac{CA}{EA}=\frac{\lambda+1}{\lambda};$$

therefore $$\frac{BE}{EM}=\frac{2\lambda+1}{\lambda};$$

therefore $$\frac{EM}{EK}=\frac{\lambda}{2\lambda+1}.\frac{\lambda^2+\lambda+1}{\lambda},$$

$$=\frac{\lambda^2+\lambda+1}{2\lambda+1};$$

or $$\frac{MK}{EK}=\frac{\lambda^2+3\lambda+2}{2\lambda+1}.$$

Now $$\frac{AM}{\alpha M}=2\frac{AE}{CE}=2\lambda;$$

therefore $$\frac{A\alpha}{\alpha M}=\frac{2\lambda+1}{1};$$

therefore $$\alpha M=\frac{A\alpha}{2\lambda+1},$$

and $$HM=\frac{A\alpha}{3}+\frac{A\alpha}{2\lambda+1},$$

$$=\frac{2(\lambda+2)}{3(2\lambda+1)}A\alpha,$$

$$=\tfrac{1}{2}.\frac{\lambda+2}{2\lambda+1}.AH.$$

Hence $$\frac{HM}{AH}.\frac{EK}{MK}=\tfrac{1}{2}\frac{(\lambda+2)}{(2\lambda+1)}.\frac{2\lambda+1}{\lambda^2+3\lambda+2},$$

$$=\tfrac{1}{2}\frac{1}{\lambda+1};$$

therefore $$\frac{AL}{EL}=2\frac{AE+CE}{CE};$$

whence $$\frac{AE\cdot CL}{CE\cdot AL}=\tfrac{1}{2};$$ the required result.

Since $$\frac{AL}{CL}=2\frac{AE}{CE}=\frac{AM}{\alpha M},$$

ML is parallel to BC.

It may be noticed that GL passes through mid point of CD.

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