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ON SHEAF REPRESENTATION OF A BIREGULAR NEAR-RING

BY

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ABSTRACT. It is shown that R is a biregular near-ring if and only if it is isomorphic with the near-ring of sections of a sheaf of reduced near-rings over a Boolean space. Also, some ideal properties of a biregular near-ring are proved. These are considered as generalizations of some works of R. Pierce on biregular rings.

1. Introduction. R. Arens and I. Kaplansky [1] called a ring with identity biregular if every two sided principal ideal is generated by a central idempotent, and they gave a topological representation of such a ring. Then A. Grothendieck and J. Dieudonne [7] proved that a commutative ring is isomorphic with the ring of sections of a sheaf of local rings. More sheaf representations of rings, modules and algebras were found by J. Dauns and K. Hofmann [4], R. Pierce [11], J. Lambek [9], K. Koh [8], and B. Davey [5]. A lot of applications of the representation theory were also obtained by R. Pierce [11], O. Villamayor and D. Zelinsky [13], G. Bergman [3], A. Magid [10], F. DeMeyer [6] and G. Szeto [12]. The purpose of the present paper is to generalize the Pierce representation for a biregular ring [11] to the case of near-rings. We shall show that R is a biregular near-ring if and only if it is isomorphic with the near-ring of sections of a sheaf of reduced near-rings over a Boolean space, where a near-ring T is called reduced if $TaT = T$ or 0 for each a in T and T is biregular. Also, some ideal properties of a biregular near-ring are proved.

The author would like to thank Professor B. Banaschewski for his many valuable comments and suggestions. In particular, he has called my attention to the related work of B. Davey [5] (see Remark 2), and suggested the theorem that the central idempotents of a biregular near-ring form a Boolean algebra (See Lemma 3.1 and Theorem 3.2).

2. Preliminaries. A near-ring R is an algebraic structure $(R, +, \cdot)$ such that the following axioms are satisfied: (1) $(R, +)$ is a group (not necessarily commutative) with identity 0, (2) $(R, \cdot)$ is a semigroup, and (3) the multiplication $\cdot$ is left distributive over the addition $+$. A left R-subgroup G is a

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subgroup of $(R, +)$ such that $RG \subset G$. It is easy to see that $RrR = \{\sum r_i r t_i$ in R for r_i, t_i in R and r a fixed element in $R\}$ is a left R-subgroup in R. A biregular near-ring R is a near-ring with a multiplicative identity such that $RaR = Re$ for some central idempotent e in R. Every near-ring R with 1 in which every element is a power of itself and $0R = 0$ is biregular ([2]). More examples can be constructed by using our representation theorem of this paper (see Example 4.2). An ideal I of a near-ring R is a subset of R such that (1) $(I, +)$ is a normal subgroup of $(R, +)$, (2) $RI \subset I$, and (3) $((x+a)y - xy)$ is in I for all x, y in R and a in I. Throughout, let R be a near-ring with an identity 1, $B(R)$ the set of all central idempotents of R. *Then*, $B(R)$ is a Boolean algebra under the joint $e \vee f = e + f - ef$, the meet $e \wedge f = ef$, and the complement $1 - e$ of e for all e, f in $B(R)$, when R is a distributively generated near-ring or a near-ring with no non-zero nilpotent elements such that $0R = 0$. Here, we are particularly interested in the above fact that $B(R)$ is a Boolean algebra for a biregular near-ring R. As usual, when $B(R)$ is a Boolean algebra, we denote the set of all maximal ideals of $B(R)$ by Spec $B(R)$ with the hull-kernel topology. It is known that Spec $B(R)$ is compact, Hausdorff and totally disconnected with a system of basic open sets $\Gamma(e) = \{x$ in Spec $B(R)$ such that $e \notin x\}$ for e in $B(R)$.

3. Ideals of a biregular near-ring. In this section, we shall give several properties of the ideals of a biregular near-ring R, which provide us with all necessary materials for the main representation theorem. We begin with a proof of the fact due to Professor B. Banaschewski that $B(R)$ is a Boolean algebra for a biregular near-ring R.

LEMMA 3.1. *Let R be a biregular near-ring. Then the following statements hold*: (1) $0R = 0$. (2) *If e is in $B(R)$ then $(1-e)$ is in $B(R)$*. (3) *For any e, f in $B(R)$, $e - ef + f$ is in $B(R)$*. (4) *For any e, f in $B(R)$, ef $(= e - ef + f)$ is the least upper bound of e and f where $u \leq v$ means that $u = uv$ for u, v in $B(R)$*.

Proof. (1) Since $0r = R0R = Re$ for some e in $B(R)$, $0 = ae$ and $e = 0b$ for some a, b in R. Then $0 = e0 = 0(0b) = 0b = e$, which makes 0 central, and therefore $0R = 0$.

(2) Since $R(1-e)R = Rf$ for some f in $B(R)$, one has $1 - e = af$ and $f = \sum b_i(1-e)c_i$ with suitable a, b_i, c_i in R. Then $fe = ef = \sum b_i e(1-e)c_i = \sum 0c_i = 0$, further, $1 - e = f(1-e) = f - fe = f$, so that $1 - e = f$ which is in $B(R)$.

(3) If $R(e - ef + f)R = Ru$ for some u in $B(R)$ then $e - ef + f = au$ and $u = \sum b_i(e - ef + f)c_i$ with suitable a, b_i, c_i in R and therefore,

$$\begin{aligned} e - ef + f = (e - ef + f)u &= \sum (e - ef + f)b_i(e - ef + f)c_i \\ &= \sum ((e - ef + f)b_i e - (e - ef + f)b_i ef + (e - ef + f)b_i f)c_i \\ &= \sum (eb_i - efb_i + fb_i)c_i = \sum b_i(e - ef + f)c_i = u, \end{aligned}$$

which proves the assertion.

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(4) We note that $\vee$ understood for the partial order of the central idempotents given by $u \le v$ if and only if $u = uv$, for which meet is then given by the product. Now since $e(e-ef+f)=e$ and $f(e-ef+f)=f$, one has $e,f \le e-ef+f$. On the other hand, if $e,f \le u$ for any u in $B(R)$ then $u(e-ef+f)= ue-uef+uf=e-ef+f$, and hence $e-ef+f \le u$. This proves that $e \vee f$ is the least upper bound of e and f.

THEOREM 3.2. *Let R be a biregular near-ring. Then $B(R)$ is a Boolean algebra under the join $e \vee f = e+f-ef$, the meet $e \wedge f = ef$, and the complement $1-e$ of e for all e in $B(R)$.*

Proof. We want to show that $B(R)$ is a Boolean algebra under the consideration of Lemma 3.1-(4) such that $1-e$ is a complement of e. In fact, for any e, f, u in $B(R)$, $u \wedge (e \vee f) = u(e-ef+f) = ue-uef+uf = ue-ueuf+uf = (u \wedge e) \vee (u \wedge f)$. Thus $B(R)$ is a distributive lattice. Moreover, since $e(1-e)=0$ and $(1-e) \vee e = 1-e-(1-e)e+e = 1-e+0-e = 1$, $1-e$ is a complement of e for any e in $B(R)$. Then, the proof is complete once we show that $e-ef+f= e+f-ef$. Since $1 = e \vee (1-e) = e+1-e$, $1-e = -e+1$, for any e in $B(R)$, and hence $f-ef = -ef+f$ for any e, f in $B(R)$. Thus $e-ef+f = e+f-ef = e \vee f$.

Next are several properties of the ideals of a biregular near-ring R.

LEMMA 3.3 *Every left R-subgroup RaR for an a in R is an ideal.*

Proof. Since R is biregular, $RaR = Re$ for a central idempotent e. For re, $r'e$ in Re, $re-r'e = re+e(-r') = e(r-r') = (r-r')e$ which is in Re. For r, t in R,

$$(1-e)(t+re-t) = (1-e)t+0+(1-e)(-t) = (1-e)(t-t) = 0,$$

so $(t+re-t)$ is in Re. Hence $(RaR,+)$ is a normal subgroup of $(R,+)$. Clearly, $R(Re) \subset Re$. Now, for t, t', r in R, we have

$$(1-e)((t+re)t'-tt') = ((1-e)t+0)t'-(1-e)tt' = (1-e)(tt'-tt') = 0,$$

so $((t+re)t'-tt)$ is in Re. Thus RaR is an ideal.

THEOREM 3.4. *Every two sided R-subgroup in R is an ideal. where a two sided R-subgroup G in R is a left R-subgroup such that $GR \subset G$.*

Proof. Let G be a two sided R-subgroup. Then $(G,+)$ is a subgroup of $(R,+)$ and $RGR \subset G$ by definition. Now, for g in G, $RgR = Re$ which is contained in G for some e in $B(R)$, so RgR is an ideal by the lemma. But then $(r+g-r)$ is in G and $((t+g)t'-tt')$ is in G for all r, t and t' in R and g in G. Thus G is an ideal.

Following R. Pierce ([11], P. 45), we call an ideal I regular if $I= (I \cap B(R))R$.

THEOREM 3.5. *Every ideal of R is regular.*

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Proof. Let I be an ideal. Clearly, $(I \cap B(R))R \subset I$. Conversely, for any a in I, $aR \subset I$ since $((0+a)r-0r)$ is in I, and hence $RaR = Re$ is contained in I for some e in $B(R)$. So, $a = re = er$ for some r in R. Thus a is in $(I \cap B(R))R$.

Since $I \cap B(R)$ is an ideal of $B(R)$, Theorem 3.5 implies that the mapping $F: I \to I \cap B(R)$ is one-to-one from the set of ideals of R to the set of ideals of $B(R)$. Now we claim that F is onto.

THEOREM 3.6. *If J is an ideal of $B(R)$, then JR is an ideal of R.*

Proof. Clearly, $((JR), +)$ is a subgroup of $(R, +)$ and $RJR \subset JR$. For any r in R, $(r+\sum e_i r_i - r)$ is in JR for e_i in J and r_i in R because $R(\sum e_i r_i)R = Re$ for some e in J, and so $(1-e)(r+\sum e_i r_i - r) = 0$. Similarly, we can show that $(1-e)((r+\sum e_i r_i)t - rt) = 0$ for all r, t in R, and hence $(r+\sum e_i r_i)t - rt$ is in $R(\sum e_i r_i)R \subset JR$. Thus JR is an ideal.

We remark that the above theorem holds even without the requirement of the biregularity of R, since taking $e = \vee e_i$ gives the same proof. Now noting that $J = JR \cap B(R)$ for the ideal J of $B(R)$ and that every ideal of R is regular, we have:

THEOREM 3.7. *The mapping $F: I \to (I \cap B(R))$ is bijective from the set of ideals of R to the set of ideals of $B(R)$.*

An ideal P is called a prime ideal of R if $RaRbR \subset P$ implies either RaR or $RbR \subset P$.

COROLLARY 3.8. *Every prime ideal of R is also maximal.*

4. A sheaf representation. Let R be a biregular near-ring. From Section 2, we have a Boolean spectrum $\operatorname{Spec} B(R)$ of R, and xR is a maximal ideal of R for each $x \in \operatorname{Spec} B(R)$ by Theorem 3.7. Let T be a disjoint union of the quotient near-rings R/xR for all x in $\operatorname{Spec} B(R)$. We shall show that T can be topologized so that T is a sheaf with stalks R/xR over $\operatorname{Spec} B(R)$, and R is isomorphic with the near-ring of sections of T.

For each $\hat{r}$ in R, let r be a function from $\operatorname{Spec} B(R)$ to T defined by $\hat{r}(x) = \bar{r}$ in R/xR. We can show that the set $\{\hat{r}(\Gamma(e))$ for all r in R, e in $B(R)\}$ forms a system of basic open sets for a topology imposed on T. In fact, what one needs is that for any a in $\hat{r}(\Gamma(e)) \cap \hat{s}(\Gamma(f))$ there exists another such set containing a and contained in the given intersection. To see this, let $a = \hat{r}(x) = \hat{s}(x)$ where x is in $\Gamma(e) \cap \Gamma(f)$. Then $(r-s)$ is in xR, that is, $(r-s) = \sum u_i t_i$ with suitable u_i in x and t_i in R, so that $(r-s) = (r-s)u$ for $u = \vee u_i$ in x. Then, for $v = (1-u)ef$, $x \in \Gamma(v) \subset \Gamma(e) \cap \Gamma(f)$, and $y \in \Gamma(v)$ implies $u \in y$ so that $(r-s) \in yR$, that is, r and s coincide on $\Gamma(v)$. Hence $a \in \hat{r}(\Gamma(v)) \subset \hat{r}(\Gamma(e)) \cap \hat{s}(\Gamma(f))$. Also, let x be a point in $\hat{r}^{-1}(\hat{r}'(\Gamma(e))$. Then $\hat{r}(x) = \hat{r}'(x)$ where $x \in \Gamma(e)$. Thus there is an e_0 in $B(R)$ such that $\hat{r} = \hat{r}'$ on $\Gamma(e_0)$. This implies $\hat{r}(\Gamma(e_0 e)) \subset \hat{r}'(\Gamma(e))$, and hence $\hat{r}$ is a continuous function from Spec $B(R)$ to T. By summarizing the above results, the natural projection $\pi: T \to \text{Spec } B(R)$, and the operations as given in the fibres R/xR, is a sheaf of near-rings such that $\hat{r}$, $r \in R$, are continuous sections.

We call a near-ring R reduced if RrR is either R or O for each r in R and R is biregular.

THEOREM 4.1. *A near-ring R is biregular if and only if it is isomorphic to the near-ring of sections of a sheaf of reduced near-rings with* Spec $B(R)$ *as base space.*

Proof. Let F be the mapping: $r \to \hat{r}$ for each r in R. From the above remark, $\hat{r}$ is a section of T. Suppose $\hat{r} = 0$. Then $\hat{r}(x) = \bar{r} = 0$ in R/xR for each x in Spec $B(R)$. Hence r is in $\cap(xR)$ for all x in Spec $B(R)$. This follows that $r = 0$ because $\bigcap(xR) = 0$ (this is a good exercise). Thus F is one-to-one. Now let f be a section of T. Then $f(x) = \bar{r} = \hat{r}(x)$ for an r in R. By a standard property in sheaf theory, $f = \hat{r}$ on a basic open set $\Gamma(e)$. Vary x over Spec $B(R)$, so Spec $B(R)$ is covered by such $\Gamma(e)$'s. Hence the usual partition property of Spec $B(R)$ ([11], P. 12) gives a finite set of orthogonal idempotents e_i with $i = 1, 2, \ldots, k$ for some integer k summing to 1 and r_i in R such that $f = \hat{r}_i$ on $\Gamma(e_i)$ for each i. Thus $r = \sum r_i e_i$ is the element in R such that $f = \hat{r}$. This proves that F is onto. Clearly, F is a near-ring homomorphism, and R/xR is a reduced near-ring for each x. By using the argument of ([11], P. 44) the converse is immediate.

We conclude the paper with an example which shows that the generalization from a biregular ring to a biregular near-ring is non-vacuous.

EXAMPLE 4.2. Let T be the subset $\{1, 1/2, \ldots, 1/n \ldots\}$ with the usual relative topology of real numbers, F a near-field with discrete topology. Let R be the set of all continuous functions from T to F. Then R is a near-ring with the componentwise addition and multiplication. Moreover, $B(R) = \{e/e(u) = 1 \text{ or } 0\}$ for each u in T, Spec $B(R) = \{x_i/e \in x_i$ with $e(1/i) = 0$ for each positive integer $i\} \cup \{x_0/e \in x_0$ with $e(0) = 0\}$. Then Spec $B(R) \cong T$ such that $R/xR \cong F$ for each x. Thus R is a biregular near-ring by Theorem 4.1, but not a ring since F is not a ring.

REMARKS 1. The near-ring of sections of a sheaf of reduced near-rings might not be biregular even for rings if the sheaf is not over a Boolean space: Any product is a sheaf (with discrete base space) but the product $\pi M_n(K)$ of all n by n matrix rings over a field K is not biregular, where n runs through the set of positive integers.

2. The representation obtained here may also be viewed as a particular instance of certain universal algebraic results ([5], Theorem 4.5). For biregular near-rings, the ideals Re determine a sublattice of the congruence lattice of R

(the associated congruences are $\{(a, b)/a-b \in Re\}$) which is isomorphic to $B(R)$ and satisfies the conditions under which Theorem 4.5 holds. The detailed verification of this, however, together with an adequate translation of the general considerations into the present setting, would not be any shorter than our approach.

References

1. R. Arens and I. Kaplansky, *Topological representations of algebras*, Trans. Amer. Math. Soc. **63** (1949), 457–481.

2. H. Bell, *Near-rings in which each element is a power of itself*, Bull. Austral. Math. Soc. **2** (1970), 363–368.

3. G. Bergman, *Hereditary commutative rings and centres of hereditary rings*, Proc. London Math. Soc. **23** (1971), 214–236.

4. J. Dauns and K. Hofmann, *The representation of biregular rings by sheaves*, Math. Zeitschr. **91** (1966), 103–123.

5. B. Davey, *Sheaf space and sheaves of universal algebras*, Math. Zeitschr. **134** (1973), 275–290.

6. F. DeMeyer, *Separable polynomials over a commutative rings*, Roc. Mtn. J. Math. **2** (1972), 299–310.

7. A. Grothendieck and J. Dieudonne, *Elements de Géometrie Algébrique*, I.H.E.S. Publications Math. Paris, 1960.

8. K. Koh, *On functional representations of a ring without nilpotent elements*, Canad. Math. Bull. **14** (1971), 359–368.

9. J. Lambek, *On the representation of modules by sheaves of factor modules*, Canad. Math. Bull. **14** (1971), 359–368.

10. A. Magid, *Pierce's representation and separable algebras*, Illinois J. Math. **15** (1971), 114–121.

11. R. Pierce, *Modules over commutative regular rings*, Mem, Amer. Math. Soc. **70** (1967).

12. G. Szeto, *On the Wedderburn theorem*, Canad. Math. J. **3** (1973), 525–530.

13. O. Villamayor and D. Zelinsky, *Galois theory for rings with infinitely many idempotents*, Nagoya J. Math. **35** (1969) 83–98.

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