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A redistributive GSA scheme to cope with socio-economic mortality differentials

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Abstract
Longevity risk is threatening the sustainability of traditional pension systems. To deal with this issue, decumulation strategies alternative to annuities have been proposed in the literature. However, heterogeneity in mortality experiences in the pool of policyholders due to socio-economic classes generates inequity, because of implicit wealth transfers from the more disadvantaged to the wealthier classes. We address this issue in a Group Self-Annuitization (GSA) scheme in the presence of stochastic mortality by proposing a redistributive GSA scheme where benefits are optimally shared across classes. The expected present values of the benefits in a standard GSA scheme show relevant gaps across socio-economic groups, which are reduced in the redistributive GSA scheme. We explore sensitivity to pool size, interest rates and mortality assumptions.

1. Introduction

In light of the increasing pressure imposed on traditional pension systems by longevity risk, both the actuarial academic literature and practice have started exploring alternative schemes, especially in the decumulation phase. Self-insurance schemes, such as tontines and Group Self-Annuitization (GSA) schemes – also called pooled annuity funds (Winter and Planchet, 2022) – are currently the most debated ones, because they can provide feasible and improved risk sharing. Tontines and GSA schemes are very similar, for they share the same structure: a group of retirees (possibly heterogeneous) enters the scheme paying an initial contribution (possibly different) to the fund. The fund evolves over time due to the investment returns, and the pool size evolves over time too, due to the members' death. At specific time points (time can be modelled in a discrete or continuous framework), benefits are paid out to the surviving members of the pool, with the budget constraint that the wealth is not exhausted. The baseline benefits received by the survivors are increased by the mortality credits coming from the wealth of the deceased members. According to Milevsky and Salisbury (2015), in pooled annuity funds or GSA schemes, the payout rate can be found as a function of time and the number of survivors in the pool, i.e. it is path-dependent, see Piggott *et al*. (2005) and Stamos (2008). Instead, in tontines the optimal payout rate can be found at time 0 in a deterministic way: "the defining feature of a tontine" is to have a deterministic total payout, see Milevsky and Salisbury (2016). In a very special case, that is in the absence of interest rate risk, the GSA scheme proposed by Piggott *et al*. (2005) turns out to provide benefits that are identical to those of the natural tontine of Milevsky and Salisbury (2015) for the logarithmic utility function, see also Section 2.1.

Self-insurance schemes have several advantages, which have been identified and studied in the literature. Their main one vis-à-vis traditional pension schemes lies in their ability to pool and mitigate longevity risk, as pointed out by a vast literature, since the seminal contribution by Piggott *et al*. (2005).

While standard retirement products expose the issuer to potential losses and solvency concerns because of longevity risk, i.e. unexpected changes from the expected mortality of the policyholders, in self-insurance schemes longevity risk is shared among participants. In some cases, an intermediary may intervene, but its cost in terms of capital requirements is lower than with traditional pension solutions (Chen *et al*. 2019). Another advantage is that it is easy to cope with moral hazard issues within the pool, by adopting a pre-determined withdrawal structure (Fullmer and Sabin 2018). Finally, self-insurance schemes can indeed constitute a valuable alternative for policyholders (see Denuit *et al*. 2022). It has been shown that in several instances the tontines may be the preferred option for a policyholder (see for instance Chen *et al*. 2021), especially if optimally designed in combination with other products, such as annuities (see Chen *et al*. 2019; Chen *et al*. 2020), or long-term care benefits (Hieber and Lucas 2022).

However, there are some aspects of self-insurance schemes that need further scrutiny. In particular, it has been pointed out that the longevity risk sharing mechanism may lead to unfairness if the policyholders are heterogeneous in their survival probabilities. Indeed, while previous works typically assume homogeneity in the mortality dynamics of policyholders, some authors (see, for instance Donnelly 2015) have raised the issue that pooling together individuals with different characteristics (age, contributions to the fund, and health status) cannot be performed in a standard GSA scheme without discriminating at least some individuals.

Inequality among policyholders arising from heterogeneous pooling has always been a common problem in insurance. Self-insurance products are clearly not exempt from this issue. This problem has even become more urgent recently since the promotion of UN's 2030 Agenda for sustainable development goals. Among these, goal 10.4 stresses the importance of implementing policies aimed at reducing inequalities.

Heterogeneity in self-insurance schemes has been considered in a few previous works. When mortality is deterministic, a GSA-type scheme can be designed to be actuarially fair (Donnelly *et al*. 2014), even when its members are heterogeneous in terms of wealth and a-priori mortality rates. However, this is in general not true when mortality is stochastic, unless one allows for different contributions or benefit calculation rules for the different groups that are pooled together. In the context of tontines, Milevsky and Salisbury (2016) introduced the notion of equitability to capture the idea that a self-annuitizing pool of heterogeneous cohorts should be designed so that they would all be equally (un)-happy. Building on this, Chen and Rach (2023) find that, by allowing the participation rate (i.e. the price for an individual to participate in the scheme) to differ among cohorts, the scheme can be both individually (to each policyholder) and collectively (in aggregate, to the whole scheme) fair.

Inspired by the importance of goal 10.4 of UN's 2030 Agenda, we contribute to this stream of literature as well, addressing the issue of reducing inequalities among different socio-economic classes in a GSA scheme with stochastic mortality. We propose a GSA scheme with heterogeneous policyholders whose mortalities are described by different stochastic processes in which, while contributions are equal across individuals, benefits differ, as if they were set by a planner (as in Dhaene and Milevsky 2024). We call this scheme a *redistributive GSA scheme*. We set the benefits' path of each socio-economic class by differentiating the initial benefit according to the optimal weights of an optimization problem that has the objective of minimizing the squared distance between the actuarially fair expected value of the benefits for the reference population (the amount paid by all individuals) and that of each individual in the group (similar to Bernard *et al*. 2024). The main contribution of this paper consists in implementing a redistributive mechanism in a GSA scheme in the presence of stochastic mortalities that are heterogeneous due to the members' socio-economic status. Our work is close to Qiao and Sherris (2013), who studied a GSA scheme with stochastic mortality. While they consider an open fund where new cohorts enter, we consider a closed fund and tackle the issue of redistribution among different socio-economic classes.

In detail, our analysis unfolds as follows. We start by considering a traditional GSA scheme where (baseline) benefits are set equal among policyholders and are actuarially fair for a reference group.

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We assume that the scheme pools together groups of individuals with (stochastic) mortalities different than that of the reference group, and that each group belongs to a different socio-economic class. Our analysis is motivated by the presence of relevant mortality differentials across socio-economic classes. These differentials are beyond the cohort-based ones considered by Milevsky and Salisbury (2016) and Chen and Rach (2023) and may prevent the viability of self-insurance schemes. Since Antonovsky (1967), many papers (Chetty *et al*. 2016; Wen *et al*. 2021; Cairns *et al*. 2022, among others), stress how the socio-economic class heavily affects the mortality experience. Accordingly, there will be expected gains/losses for different groups who enroll in the same scheme. We use a British dataset on historical mortality by socio-economic classes previously used in the actuarial literature (see Haberman *et al*. 2014; Wen *et al*. 2021; Cairns *et al*. 2022) and calibrate a stochastic mortality model for three different socio-economic classes. Then, we adopt a simulative approach and analyze the distribution of the expected present value (EPV) of the benefits obtained from the GSA scheme for the different groups of individuals. Our results show that large differences appear across policyholders belonging to different socio-economic groups when they are pooled together in the same traditional GSA scheme. Indeed, as Donnelly (2015) points out, unfairness arises since those who on average die earlier (the poorest) subsidize those who on average die later (the wealthiest). This is a socially undesirable outcome as pointed out also by UN's 2030 Agenda. We quantify such transfer to be in the order of 30%. This happens because the actuarial fairness principle, which would imply differentials in pricing the GSA product to different sub-groups, is violated. We then evaluate to what extent our proposed redistributive GSA scheme is able to restore fairness, comparing the distribution of the EPV of the different sub-groups when the benefits are optimally redistributed. The scheme achieves the objective of improving equity across sub-groups because the simulated distributions of their EPVs become more similar. Finally, we study how our results depend on three key factors: the size of the pool, which matters because idiosyncratic mortality can be perfectly diversified only in large samples, the level of interest rates and mortality assumptions. While the the size of the pool affects the dispersion of EPVs within each group, the level of interest rates is inversely linked to inequity. Indeed, when interest rates are lower, longevity differences matter more and thus inequity is more pronounced. Mortality heterogeneity is also inversely linked to inequity. More heterogeneous pools show more distant EPV levels across sub-groups, while inequity decreases if the selected cohorts are closer in mortality to the reference one. In any case, the redistributive GSA scheme we propose is able to restore fairness across sub-groups.

The paper is organized as follows: Section 2 describes the decumulation products, Section 2.2 introduces our proposed redistributive GSA scheme, Section 3 describes the mortality modelling approach, Section 4 provides the empirical application, Section 5 discusses sensitivity of the results and finally Section 6 concludes.

2. The decumulation products

2.1. GSA scheme

We assume that a community of l_x workers aged x retire at time $t = 0$ and are free to decide the decumulation strategy to follow. They evaluate the competing strategies according to their expected present values at time $t = 0$. As a benchmark decumulation strategy, we consider an immediate annuity product with annual benefit b_A paid at the beginning of each year. In the following and in the remainder of the paper, we will be assuming that there is a *reference population* whose survival probabilities are used by the insurance company to price a lifetime annuity. Assuming that

$$\{{}_tp_x^r\}_{t=0,\ldots,\omega-x-1}, \tag{2.1}$$

is the vector that collects all the survival probabilities for a head aged x over t years for the reference population, and ignoring commission expenses and safety loadings, the single premium of the unitary immediate lifetime annuity paid in advance sold to a policyholder aged x is

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$$\ddot{a}_x = \sum_{t=0}^{\omega-x-1} {}_tp_x^r v^t,$$

where $v^t = (1+i)^{-t}$ is the t-years financial discount factor and ω denotes the limiting age (i.e. $p_\omega = 0$). Then, the expected present value at time $t = 0$ of the benefits paid by the annuity with periodic payment b_A for a policyholder aged x whose survival probabilities are $\{{}_tp_x\}_{t=0,\ldots,\omega-x-1}$ is

$$EPV_A(0) = \sum_{t=0}^{\omega-x-1} {}_tp_x v^t b_A.$$

The alternative we focus on in this paper is a collective self-insurance scheme, namely the group self-annuitization scheme proposed by Piggott *et al*. (2005). The scheme works as follows. When it is set up, the scheme pools together into a fund the resources collected by the l_x policyholders. To make the comparison with the annuity fair, we assume that each individual contributes $b_A\ddot{a}_x$ to the fund. Hence, the total fund at time $t = 0$ is

$$F(0) = l_x b_A \ddot{a}_x.$$

At time 0, each individual receives a benefit b_A, equal to the annuity benefit, therefore

$$b_{GSA}(0) = b_A. \tag{2.2}$$

Assuming investment at the risk-free interest rate level i, at time $t = 1$ the fund is valued

$$F(1) = (F(0) - l_x b_A)(1+i) = l_x b_A(\ddot{a}_x - 1)(1+i),$$

where the annuity price $\ddot{a}_x$ (as well as all the other annuity prices $\ddot{a}_{x+t}$ for all $t \geq 1$) is computed using the survival probabilities (2.1) of the reference population. Given the actual number of survivors at age $x+1$, l^*_{x+1}, the benefit at time $t = 1$ received by each survivor in the scheme, $b_{GSA}(1)$, obtained sharing among the survivors the annuitized value of the fund at time 1, is

$$b_{GSA}(1) = \frac{1}{l^*_{x+1}}\left(\frac{F(1)}{\ddot{a}_{x+1}}\right) = \frac{1}{l^*_{x+1}}\left(\frac{l_x b_A(\ddot{a}_x - 1)(1+i)}{\ddot{a}_{x+1}}\right).$$

Considering that (i) the recursive relationship for the annuity prices is

$$\ddot{a}_{x+1} = (\ddot{a}_x - 1)(1+i)/p^r_x,$$

and that (ii) the number of survivors at time 1, l^*_{x+1}, is given by $l_x p^*_x$, p^*_x being the realized 1-year survival probability at age x, $b_{GSA}(1)$ can be rewritten as

$$b_{GSA}(1) = \frac{1}{l^*_{x+1}}\left(\frac{l_x b_A(\ddot{a}_x - 1)(1+i)}{(\ddot{a}_x - 1)(1+i)/p^r_x}\right) = b_A\left(\frac{p^r_x}{p^*_x}\right).$$

Observing that at a generic time $t \geq 1$

$$l^*_{x+t} = l^*_{x+t-1} p^*_{x+t-1}, \tag{2.3}$$

the benefit of the GSA scheme at time t is

$$\begin{aligned} b_{GSA}(t) &= \frac{F(t)}{l^*_{x+t}\ddot{a}_{x+t}} = \frac{l^*_{x+t-1} b_{GSA}(t-1)\,(\ddot{a}_{x+t-1} - 1)\,(1+i)}{l^*_{x+t}\ddot{a}_{x+t}} = \\ &= b_{GSA}(t-1)\frac{l^*_{x+t-1}}{l^*_{x+t}}\frac{(\ddot{a}_{x+t-1} - 1)\,(1+i)}{(\ddot{a}_{x+t-1} - 1)\,(1+i)\,/p^r_{x+t-1}} = \\ &= b_{GSA}(t-1)\left(\frac{p^r_{x+t-1}}{p^*_{x+t-1}}\right). \end{aligned}$$

Hence,

$$b_{GSA}(t) = b_{GSA}(t-1)\cdot MEA_t, \tag{2.4}$$

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where

$$MEA_t = \frac{p^r_{x+t-1}}{p^*_{x+t-1}}. \tag{2.5}$$

The factor MEA_t, $t = 1, 2, ..., \omega - x - 1$ is the mortality experience adjustment, that is the ratio of the expected to the actual survival rates in year $[t-1, t]$. Notice that in Piggott *et al*. (2005) the interest rate risk is considered too, and the benefit at time t turns out to be equal to the benefit at time $t-1$ times the adjustment for the mortality risk (MEA_t) and the adjustment for the interest rate risk (IRA_t). In this paper, we ignore interest rate risk by assuming a constant interest rate i over time.[1]

The realized survival probabilities p^*_{x+t}, $t = 0, 1, ...$ are found via simulation of the number of deaths in the pool, and therefore simulation of the realized number of survivors l^*_{x+t}. Piggott *et al*. (2005) assume the randomness in mortality to increase linearly with age. Differently from Piggott *et al*. (2005), we model the number of survivors in each time period by simulating the evolution of the stochastic mortality intensity process $\lambda(t)$ and the time of death of each individual in the pool as the first jump time of a *doubly stochastic process* with intensity λ (see Section 3).

The time-0 expected present value for an individual enrolled in the GSA scheme aged x and whose survival probabilities are $\{{}_tp_x\}_{t=0,\ldots,\omega-x-1}$ is

$$EPV_{GSA}(0) = \sum_{t=0}^{\omega-x-1} v^t\, {}_tp_x b_{GSA}(t),$$

where the GSA benefits $b_{GSA}(t)$ are as in (2.4).

2.2. The redistributive GSA scheme

In the description of the GSA scheme, we have neglected the fact that the scheme can pool together individuals belonging to different sub-populations, which can display different mortality patterns. A simplified description of this situation is to consider three different categories of individuals: high-risk (HR), medium-risk (MR) and low-risk (LR) individuals, with survival probabilities $\{{}_tp^{HR}_x\}_{t=0,\ldots,\omega-x-1}$, $\{{}_tp^{MR}_x\}_{t=0,\ldots,\omega-x-1}$ and $\{{}_tp^{LR}_x\}_{t=0,\ldots,\omega-x-1}$, respectively. In this context, by "high (low) risk individual" we refer to an individual with lower (higher) survival probabilities with respect to the medium ones at every time horizon:

$${}_tp^{HR}_x < {}_tp^{MR}_x < {}_tp^{LR}_x \qquad \text{for all } t = 0, \ldots, \omega - x - 1.$$

Remark 1. This classification of individuals is simplified because it assumes implicitly that each individual will always remain in the same risk class. However, the risk profile of an individual can change over time, due to, for example, the outbreak of new illnesses. Changes in the risk profiles of a number of members can have a non-negligible impact on the scheme's benefits and fairness. In fact, suppose that a group of LR (MR) members at a certain point in time $t_0 > 0$ moves to the MR (HR) class. After time t_0, the mortality experience of those individuals should be described by a different mortality process. Ignoring this transition in the simulation of deaths produces a lower estimation of the number of deaths, leading to lower estimated benefits to the survivors. And vice versa: ignoring a significant move from HR (MR) to MR (LR) produces over-estimation of the benefits. In turn, the mis-estimation of the benefits would produce mismatching in the redistributive mechanism. In this paper, we do not address the challenging issue of changes in the risk classification over time.

[1] Notice that in the assumption of absence of interest rate risk the benefit provided by the GSA is identical to the benefit provided by the natural tontine proposed by Milevsky and Salisbury (2015), in a framework of maximization of expected utility with a logarithmic utility function. This coincidence no longer holds if the interest rate risk were to be considered.

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If we calculate the expected present values at $t=0$ of the GSA benefits for the three categories, $EPV^{j}_{GSA}(0)$, $j=HR, MR, LR$, using their own respective survival probabilities, we obtain that

$$\sum_{t=0}^{\omega-x-1} {}_tp_x^{HR} v^t b_{GSA}(t) < \sum_{t=0}^{\omega-x-1} {}_tp_x^{MR} v^t b_{GSA}(t) < \sum_{t=0}^{\omega-x-1} {}_tp_x^{LR} v^t b_{GSA}(t),$$

that is

$$EPV^{HR}_{GSA}(0) < EPV^{MR}_{GSA}(0) < EPV^{LR}_{GSA}(0). \tag{2.6}$$

It is clear that, due to the different survival probabilities of each sub-population, distributing the same benefit to each sub-group leads to a solidarity transfer across individuals, in particular from high-risk to low-risk individuals. If, as usual, high-risk individuals belong to lower socio-economic classes, such transfer exhacerbates inequality. It is thus interesting to study redistribution mechanisms within the scheme to reduce those inequalities.

We consider a redistribution mechanism obtained with a simulative approach. In particular, we simulate 10,000 scenarios of mortality patterns for each sub-population and implement an optimal redistribution policy by minimizing for each simulation k ($k=1, \ldots, 10,000$) the squared distance between a re-scaling of $EPV^{j,k}_{GSA}(0)$ ($j=HR, MR, LR$) through the use of redistributive shares α^j and the baseline $EPV_{GSA}(0)$ level of the medium policyholder in the absence of other sub-populations in the scheme.

The redistributive scheme works as follows. We assume that each retiree pays $b_A\ddot{a}_x = 1000$ in the pooled fund at time $t=0$. The unique pool has value $N_0 b_A \ddot{a}_x = 1000N_0$, where N_0 is the total number of retirees aged x given by the sum of the number N_0^j of retirees in each sub-population $j \in \{HR, MR, LR\}$:

$$N_0 = N_0^{HR} + N_0^{MR} + N_0^{LR}.$$

The optimal redistributive shares, α^j_*, are found via a simulative approach by solving the following problem:

$$\min_{\{\alpha^j\}_{j\in\{LR,MR,HR\}}} \left\{ \frac{1}{10,000} \sum_{k=1}^{10,000} \frac{1}{3} \sum_{j\in\{LR,MR,HR\}} \left(1000 - \alpha^j EPV^{j,k}_{GSA}(0)\right)^2 \right\} \tag{2.7}$$

$$s.t. \sum_{j\in\{LR,MR,HR\}} \alpha^j = 3. \tag{2.8}$$

where $EPV^{j,k}_{GSA}(0)$ is the expected present value of GSA benefits for sub-population j in simulation k. At time $t=0$, the optimal redistributive share α^j_* defines the benefits paid to sub-group j in the redistributive scheme:

$$b^j_{RE}(0) = \alpha^j_* b^j_{GSA}(0) = \alpha^j_* b_A. \tag{2.9}$$

Due to the recursive mechanism (2.4), for each $t > 0$, it holds

$$b^j_{RE}(t) = b^j_{RE}(t-1) \cdot MEA_t, \tag{2.10}$$

with MEA_t defined as in Equation (2.5).

The approach results to be financially sustainable since the constraint in (2.8) ensures that the funds awarded to each individual are a fraction of the total funds available at each time t.

The effect of the redistribution can be measured evaluating the EPV of the benefits obtained by the groups after the redistribution:

$$EPV^j_{RE}(0) = \sum_{t=0}^{\omega-x-1} {}_tp^j_x v^t b^j_{RE}(t) = \alpha^j_* EPV^j_{GSA}(0). \tag{2.11}$$

The goal after the redistribution is to obtain a reduced gap among the $EPV_{RE}(0)$ of the different sub-populations.

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3. Mortality modelling

3.1. The theoretical framework

We consider a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $\{\mathcal{G}_t : t \geq 0\}$ of sub-σ-algebras of $\mathcal{F}$. We introduce a nonexplosive counting process M_t with intensity $\lambda(t)$ and another filtration $\{\mathcal{F}_t : t \geq 0\}$ such that $\mathcal{F}_t \subset \mathcal{G}_t$.

The process M_t is said to be *doubly stochastic* driven by $\{\mathcal{F}_t : t \geq 0\}$, if $\lambda(t)$ is $(\mathcal{F}_t)$-predictable and for all t,s with $t < s$, conditional on the σ-algebra $\mathcal{G}_t \vee \mathcal{F}_s$ generated by $\mathcal{G}_t \cup \mathcal{F}_s$, the process $M_s - M_t$ is Poisson distributed with parameter

$$\int_t^s \lambda(u)du.$$

Conditional on the knowledge of a particular trajectory $t \to \lambda(t, \widetilde{\omega}) = \widetilde{\lambda}(t)$ for fixed $\widetilde{\omega} \in \Omega$, the counting process M becomes a Poisson process with (conditionally deterministic) time varying intensity $\widetilde{\lambda}(t)$.

A stopping time τ is said to be *doubly stochastic with intensity* λ if the underlying counting process whose first jump time is τ is doubly stochastic with intensity λ. The specification of the sub-filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ is meant to indicate that the first jump time of M is a stopping time with respect to $(\mathcal{G}_t)_{t \geq 0}$, but outside the span of $(\mathcal{F}_t)_{t \geq 0}$, which carries sufficient information to reveal the intensity $\widetilde{\lambda}(t)$ (i.e. the likelihood that the jump will happen) but not enough to predict the occurrence of the jump. Doubly stochastic processes with a stopping time are typically exploited to model the survival process of individuals. In particular, the time of death is typically modelled as a doubly stochastic stopping time with intensity given by the mortality intensity λ that is a stochastic force of mortality. The mortality intensity is typically modelled as an affine process in order to exploit well-known analytical results for the computation of the survival function (see Duffie *et al*. 2000). For the specification of the affine stochastic mortality intensity, we consider non-mean reverting processes among those introduced by Luciano and Vigna (2008). In particular, we model the mortality intensity $\lambda_x(t)$ of an individual aged $x + t$ of a specific cohort with initial age x with a Feller process with dynamics given by

$$d\lambda_x(t) = a\lambda_x(t)dt + \sigma\sqrt{\lambda_x(t)}dW_x(t), \tag{3.1}$$

where $a > 0$ and $\sigma \geq 0$ represent the drift and diffusions parameters associated to the processes, $W_x(t)$ being a Brownian motion. Following Duffie *et al*. (2000), given an affine mortality intensity $\lambda_x(t)$, the survival probability $S_x(t)$ describing the probability of an individual aged x to survive t years is:

$$S_x(t) = {}_tp_x = \mathbb{E}[e^{-\int_0^t \lambda_x(u)du} | \mathcal{G}_0] = e^{\alpha(t) + \beta(t)\lambda_x(0)}, \tag{3.2}$$

where $\alpha(\cdot)$ and $\beta(\cdot)$ solve appropriate ODEs and the initial observed intensity $\lambda_x(0)$ is taken to be equal to $-ln(\hat{p}_x)$, where $\hat{p}_x$ is the observed survival rate at age x.

In the case of the *Feller process,* we have

$$\begin{cases} \alpha(t) = 0 \\ \beta(t) = \frac{1 - e^{bt}}{c + de^{bt}} \end{cases} \tag{3.3}$$

with $b = -\sqrt{a^2 + 2\sigma^2}$, $c = \frac{b+a}{2}$, $d = \frac{b-a}{2}$.

The inequality constraint

$$e^{bt}\left(\sigma^2 + 2d^2\right) > \sigma^2 - 2dc \tag{3.4}$$

must hold in order for the survival process $S_x(t)$ to be a decreasing function of t.

3.2. Methodology for the simulation of the number of deaths in one simulated scenario

Let τ be the time of death of the individual modelled as a doubly stochastic stopping time with intensity λ. For the death time of each individual in the group self-annuitization pool it holds

$$\mathbb{P}(\tau > t) = \mathbb{P}(M_t = 0) = \mathbb{E}[e^{-\int_0^t \widetilde{\lambda}(u)du} | \mathcal{F}_0], \tag{3.5}$$

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where the knowledge of the parameter $\int_0^t \widetilde{\lambda}(u)du$ is conditional to the filtration $\mathcal{F}_t$, M_t being the nonexplosive counting process with intensity $\lambda(t)$ described in Section 3.1. It can also be proven that, conditional on the particular trajectory of the intensity process $\widetilde{\lambda}(t)$, τ satisfies

$$\tau = \inf_{t \geq 0} \left\{ t{:}E_1 \leq \int_0^t \widetilde{\lambda}(u)du \right\}, \tag{3.6}$$

where $E_1 \sim Exp(1)$.

In order to obtain the number of deaths in the GSA pool and the p^*_{x+t-1} probabilities in one simulated scenario, we first simulate for each sub-population $j \in \{HR, MR, LR\}$ one trajectory of the intensity process $\tilde{\lambda}(t)$ for $t \in [0, \omega - x]$; then, assuming the future remaining lifetime of pool members to be independent, we simulate, for $j \in \{HR, MR, LR\}$, $N_0^j = 1000$ independent realizations from an Exp(1) random variable; finally, thanks to (3.6) every single extraction leads to the death time of the different $N_0^j = 1000$ individuals. Thus, for a given j, the exponential random vector

$$\{E_n^j\} = \{E_1^j, E_2^j, ..., E_{1000}^j\},$$

for $n = 1, ..., N_0^j = 1000$, leads through Equatrion (3.6) to the death time random vector for the simulated scenario

$$\{\tau_n^j\} = \{\tau_1^j, \tau_2^j, ..., \tau_{1000}^j\}.$$

It follows that the number of individuals of sub-population j dying between ages $x + t - 1$ and $x + t$ in the simulated scenario is given by

$$d^j_{x+t} = \mathbb{E}\left[\sum_{n=1}^{N_0^j} \mathbb{1}_{\left\{t-1<\tau_n^j\leq t\right\}}\right]. \tag{3.7}$$

This leads to the computation of the survivors l^{*j}_{x+t} at age $x + t$ belonging to sub-population j in the GSA pool as

$$l^{*j}_{x+t} = l^{*j}_{x+t-1} - d^j_{x+t}, \tag{3.8}$$

and for each t the total number of survivors in the GSA pool is given by

$$l^*_{x+t} = \sum_{j\in\{HR,MR,LR\}} l^{*j}_{x+t}. \tag{3.9}$$

From the sequence $\{l^*_{x+t}\}$ provided by Equation (3.9), one can compute the realized survival probabilities $\{_tp^*_x\}$ as in (2.3) in the simulated scenario. Finally, using (2.4), it is possible to compute in the simulated scenario the benefits $\{b_{GSA}(t)\}_{t=0,1,...,\omega-x-1}$ for all survivors in the GSA scheme.

4. Numerical application and calibration

We intend to apply the redistributive GSA scheme using the methodology illustrated in Sections 2–3 to a specific dataset of different socio-economic classes. Section 4.1 illustrates the dataset.

4.1. The dataset

Individuals belonging to distant socio-economic classes bear a different longevity risk. More disadvantaged social classes are more exposed to riskier environments, riskier habits and to access barriers to healthcare. According to Ardito *et al*. (2019), the longevity gap is an increasing function of the degree of deprivation, and it is more pronounced for male individuals than for females. In addition, the gap shrinks with the policyholder's age.

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To model the mortality differences in socio-economic classes, we use a dataset which groups the English population in ten deciles, according to the English Indices of Deprivation (version of 2015).[2] The deciles are the result of the subdivision of the 32,844 lower-layer super output areas in ten equal-sized groups such that the first decile corresponds to the most deprived areas and the tenth decile corresponds to the least deprived ones. The ranking is based on income deprivation, employment deprivation, education, skills and training deprivation, health deprivation and disability, crime, barriers to housing and services, living environment deprivation.

The dataset has been used in other papers on mortality differentials by socio-economic classes such as Cairns *et al*. (2022), Wen *et al*. (2021), Haberman *et al*. (2014). As stressed by Wen *et al*. (2021), "the health deprivation and disability" factor does not perfectly suit the purposes of a research analysis based on socio-economic mortality differentials. It concerns the quality of the health status rather than a mere economic wealth based discriminant. Still, considering the correlation among the health status and all the remaining factors, the English Indices of Deprivation dataset is a useful and proficient tool to spot differences in mortality experience according to the wealth level.

The dataset provides figures for the number of deaths and risk exposures (mid-year estimates) in each population decile area by gender and single year of age for calendar years 2001–2017. Therefore, our calibration relies on seventeen observations for each sub-population. We select three sub-populations out of the ten deciles of the dataset: we consider as low socio-economic status representatives the first decile sub-population individuals, as middle class representatives the fifth decile sub-population individuals and as high socio-economic status representatives the tenth decile sub-population individuals. Therefore, given the differences in the survival probability levels, we consider three categories of policyholders by assuming first decile retirees as high-risk individuals ($j = HR$), fifth decile retirees as medium-risk individuals ($j = MR$) and tenth decile retirees as low risk individuals ($j = LR$). Notice that, as stressed in Remark 1, in making this risk classification, we are ignoring the fact that the risk profile of each individual can change over time. Considering the construction of this dataset, a change of risk class can occur not only due to a sudden change in the health status, but also due to the members' mobility or the re-classification of the geographical districts used to construct the dataset.

Although policyholders contribute identically to the scheme funding, high socio-economic status individuals (i.e. LR individuals) will receive on average benefits for a longer period with respect to lower socio-economic groups retirees (i.e. HR individuals). Thus, lower socio-economic classes will bear the longevity risk associated to the richest.

4.2. Calibration

For the calibration procedure, we focus on the cohort of male individuals born in 1936, entering the scheme in 2001 at the age of $x = 65$[3] and, as mentioned in Section 4.1, we consider the three sub-populations of HR, MR and LR individuals belonging to the first, fifth and tenth percentile, respectively. The mortality intensity of each sub-population follows the Feller process in (3.1):

$$d\lambda^j_{65}(t) = a^j \lambda^j_{65}(t)dt + \sigma^j \sqrt{\lambda^j_{65}(t)}dW^j_{65}(t) \qquad j = HR, MR, LR. \tag{4.1}$$

We calibrate the processes (4.1) on seventeen observations, from 2001 to 2017, over the age span 65 to 81 and we obtain closed-form expressions for the remaining survival probabilities up to the limiting age $\omega = 110$.

In detail, we first derive from the dataset described in Section 4.1 three vectors of observed empirical survival probabilities $\{{}_t\hat{p}^j_{65}\}$ for $j \in \{HR, MR, LR\}$ relative to males born in 1936 and observed yearly from 2001 to 2017. Then, we calibrate the parameters of the processes $\{\lambda^{HR}_{65}, \lambda^{MR}_{65}, \lambda^{LR}_{65}\}$ by minimizing

[2]The dataset is available on the Office for National Statistics website: https://www.ons.gov.uk/peoplepopulationandcommunity/birthsdeathsandmarriages/deaths/adhocs/009299numberofdeathsandpopulationsindeprivationdecileareasbysexandsingleyearofageenglandandwalesregisteredyears2001to2017.

[3]In the remainder of the paper, we will sometimes use x, sometimes use 65 as the initial age.

Table 1. *Feller model's calibration.*

	a^j	σ^j	$\lambda^j_{65}(0)$	Error	e^j_{65}
HR	0.074664	0.000254	0.024836	0.004898	15.0129
MR	0.073965	0.00025	0.015865	0.004491	19.0285
LR	0.073313	0.000478	0.011364	0.004024	22.3383

the mean squared error (MSE), that is the average squared difference between the model implied and the observed survival probabilities:

$$\min_{a^j,\sigma^j} \sum_{t=1}^{17} \frac{1}{17} \left({}_t\hat{p}^j_{65} - {}_tp^j_{65} \right)^2 , \tag{4.2}$$

for $j \in \{HR, MR, LR\}$, where ${}_tp^j_{65}$ is defined as in (3.2)–(3.3) and applied to the j-th sub-population. From the calibration procedure we get the minimizing parameters a^j, σ^j that are reported in Table 1 together with $\lambda^j_{65}(0) = -\log {}_{}\hat{p}^j_{65}$ and the calibration error. The observed mortality intensity at age 65, $\lambda^j_{65}(0)$, ranges from 0.011 to 0.025. The value of the long term mean parameter a^j ranges between 0.073 and 0.075 while the volatility σ^j ranges from 0.00025 to 0.00048. The low volatility levels are associated to a low calibration error ranging from 0.004 to 0.0049. The Supplementary Material reports for each $j \in \{HR, MR, LR\}$ the calibrated model implied survival function ${}_tp^j_{65}$ vs. the observed survival curve and shows that the two overlap quite well, their differences being below 1% in absolute value for all ages and risk classes.

Table 1 reports also the fitted life expectancies at age 65, e^j_{65}, which can be computed from the model-implied survival curves. We see that $e^{HR}_{65} = 15.0129$ for the high-risk population, $e^{MR}_{65} = 19.0285$ for the medium-risk population, $e^{LR}_{65} = 22.3383$ for the low risk population. These figures are informative, because they highlight a remarkable difference in life expectancy among the different sub-populations.

4.3. Simulation of the benefits under the GSA scheme

We have run 10,000 Monte Carlo simulations of the processes (4.1). In order to compute the GSA and the annuity benefits, we need a reference population (see Section 2). We, therefore, set the reference population r to be the fifth decile male policyholders, that is $r = MR$, that implies

$$\{{}_tp^r_x\}_{t=0,\ldots,\omega-x-1} = \{{}_tp^{MR}_x\}_{t=0,\ldots,\omega-x-1}. \tag{4.3}$$

Accordingly, given that the premium paid by the policyholders is 1000, the annuity level benefit is

$$b_A = \frac{1000}{\ddot{a}^r_x} = \frac{1000}{\ddot{a}^{MR}_x}. \tag{4.4}$$

In the calculation of the annuity, we set the interest rate $i = 2\%$. In the case of the GSA scheme, the starting point for the GSA benefits as in Equation (2.2) is given by

$$b_{GSA}(0) = \frac{1000}{\ddot{a}^{MR}_x} = b_A = 62.565.$$

Considering a cohort of three equally sized sub-populations of $N^j_0 = 1000$ males, $j \in \{HR, MR, LR\}$ (so that $N_0 = 3000$) coming from the first, the fifth and the tenth deciles of the English population, the simulation process is based on the following algorithm:

- For each simulated scenario, we simulate the trajectory of $\lambda^j_{65}(t)$ for $t = 1, \ldots, 45$ for each sub-population j. We run 10,000 simulated scenarios.

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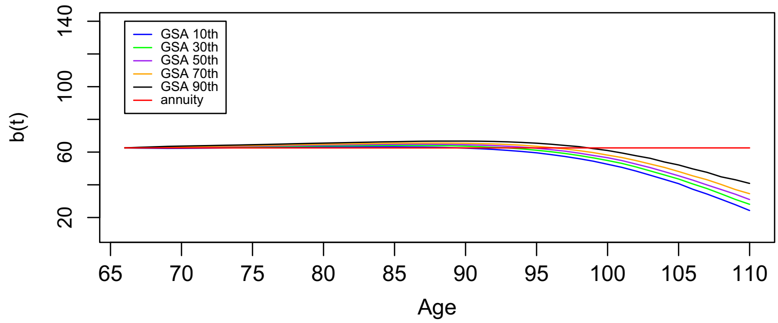

Figure 1. *Cash flow streams: GSA versus annuity.*

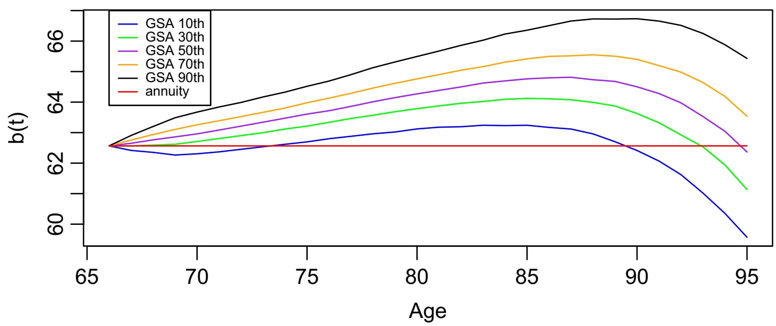

Figure 2. *Cash flow streams: GSA versus annuity. Detail, age 65–95.*

- For each simulated scenario we use the procedure illustrated in Section 3.2 to simulate the number of deaths and obtain the realized survivors $\{l^*_{x+t}\}$ and the realized survival probabilities ${}_tp^*_x$ for that scenario.
- For each of the 10,000 scenarios we compute the benefits for the GSA scheme $b_{GSA}(t)$ for $t = 1, \ldots, 45$ using the $\{{}_tp^*_x\}_{t=1,\ldots,45}$ as illustrated in Section 2.
- Therefore, we get the distribution of the 10,000 paths for the GSA benefits $b_{GSA}(t)$ for $t = 1, \ldots, 45$.

Figure 1 shows the level annuity benefit b_A and the 10th, 30th, 50th, 70th and 90th percentiles of the distribution of benefits $b_{GSA}(t)$ over ages 65–110 obtained from the 10,000 simulations. The GSA and annuity benefits are unique across individuals; therefore, Figure 1 holds indiscriminately for each sub-population $j = \{HR, MR, LR\}$.

Figure 2 reports the scheme benefits over ages 65–95, while Figure 3 reports the expectation and the standard deviation of the GSA benefits for all ages 65–110.

At time $t = 0$, we clearly have $b_{GSA}(0) = b_A$. In most cases, the $b_{GSA}(t)$ distribution outperforms the level annuity benefit until about the age $x = 90$, when it bends towards a minimum of approximately 20 (see Figure 1). In line with this behaviour, we see from Figure 3 that the expectation of the GSA benefit $\mathbb{E}(b_{GSA}(t))$ in the first years slightly increases with age and is decreasing after age 90; the standard deviation of the GSA benefit $\sigma(b_{GSA}(t))$ is an increasing function of age. From Figures 1 and 2, we observe a systematic rise and decline of GSA benefits before and after an approximate age of 90. In the comparison with the annuity, which provides flat benefits, we see that the GSA benefit is larger than the annuity one from age 65 to age 95 (the age when the GSA benefit equals the annuity one depends on the scenario, with the median value being 95 years). This rise–decline pattern can be due to the coexistence of different sub-populations with different mortality in the same pool. This creates a number of deaths larger in

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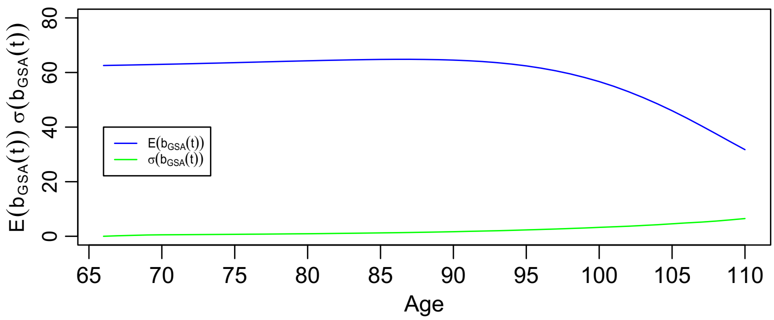

Figure 3. $\mathbb{E}(b_{GSA}(t))$, $\sigma(b_{GSA}(t))$.

the first years and lower in later years with respect to the reference population, leading to an immediate rise and a subsequent fall in the benefits. Sensitivity analyses (see the Supplementary Material) support this intuition: by considering three sub-populations that are more similar to the reference one we obtain a benefit behaviour that is more similar to that of the annuity.

Regardless of the causes, this pattern could be considered a negative feature for retirees. There is indeed evidence that for elderly people liquidity needs increase with age, for persons older than 85 being six times higher than for persons below 65 (see Weinert and Gründl 2021). Also, Weinert and Gründl (2021) talk about a retirement smile: high consumption levels in the early retirement years followed by a consumption decline as people become more and more home-bound, before a consumption peak at very old ages due to costly nursing care.

However, many considerations are in order. First, most of the members of the GSA scheme would never experience such decline in benefits because they die before age 95 (the probability of dying between ages 65 and 95 from the dataset is ${}_{30}q_{65}^{HR} = 0.9387$ for HR, ${}_{30}q_{65}^{MR} = 0.8277$ for MR, ${}_{30}q_{65}^{LR} = 0.7115$ for LR). Thus, this problem would be suffered only by a minority of individuals, while most of them would in any case enjoy the extra benefit of GSA in the first years after retirement.

Second, it is true that the annuity guarantees a level benefit that does not decrease with age. However, the annuity benefits reported here do not consider commission expenses (that are higher for annuities than for self-insurance schemes): if expenses were taken into account, the annuity benefit would be reduced and the positive gap between GSA and annuity benefits in the first years would be larger. Therefore, a possible way to address the decreasing GSA benefits in old age could be to invest in the first decades the extra money exceeding the annuity benefits in financial instruments and use the accumulated money after age 95. A natural way to do it and transfer longevity risk, would be to buy a deferred lifetime annuity that starts to pay periodic benefits at age 95: given the low probability of reaching age 95, such deferred annuity would likely be affordable with the extra benefit from the GSA scheme.

Last but not least, the retirees actually surviving to age 95 and over would mainly belong to the less deprived socio-economic class (LR), which would probably be less affected by the decline of benefits in elderly age.

4.4. Calculation of EPV(0) for each socio-economic class

The benefit from the GSA scheme is the same for all individuals. However, different socio-economic classes experience different lifetime durations. Therefore, the value of the benefit's stream will be different. A natural way to measure such value is the expected present value at time $t = 0$ of the benefits using the class-specific survival function.

Therefore, for each of the 10,000 simulated scenarios of $b_{GSA}(t)$ we calculate $EPV_{GSA}^{j}(0)$, where

$$EPV_{GSA}^{j}(0) = \sum_{t=0}^{\omega-x-1} v^t \, {}_t p_x^j \, b_{GSA}(t),$$

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Table 2. *$EPV_A^j(0)$ and some percentiles of the distribution of $EPV_{GSA}^j(0)$, $j \in \{HR, MR, LR\}$.*

	$EPV^{HR}(0)$	$EPV^{MR}(0)$	$EPV^{LR}(0)$
Annuity	831.1444	1000	1130.149
GSA 90th	852.5442	1026.2595	1157.8222
GSA 70th	846.7466	1018.3843	1147.8504
GSA 50th	842.7923	1012.941	1141.095
GSA 30th	838.8706	1007.7905	1134.8674
GSA 10th	833.5499	1000.481	1125.7446

where in ${}_tp_x^j$ we plug the calibrated parameters a^j, σ^j ($j \in \{HR, MR, LR\}$) of Tab. 1. Thus, for each socio-economic class $j \in \{HR, MR, LR\}$, we obtain a distribution of the 10,000 $EPV_{GSA}^j(0)$.

Similarly, we calculate the expected present value of the benefits for the annuity, $EPV_A^j(0)$, where

$$EPV_A^j(0) = \sum_{t=0}^{\omega-x-1} v^t {}_tp_x^j b_A.$$

Because of (4.3) and (4.4) when $j = MR$ we have $EPV_A^{MR}(0) = 1000$.

Table 2 collects $EPV_A^j(0)$ and some percentiles of the distribution of $EPV_{GSA}^j(0)$.

Table 2 shows the effect of ignoring mortality differentials across socio-economic classes: if the same benefit is awarded to all individuals regardless of their heterogeneous contribution to the overall risk, the expected present value of cash flow streams at time $t = 0$ is considerably different across sub-groups when the actualization process is weighted by the specific mortality experience of each sub-population individually. Without redistribution, the more deprived socio-economic individuals (HR) enjoy a value about 15–17% lower than the reference individuals (MR) who, in turn, enjoy a value about 12–13% lower than the one of the least deprived individuals (LR). More importantly, on average, the transfer from HR to LR individuals is about 30% of the benefit enjoyed by the reference individuals. This strong solidarity from the high-risk retiree to the low-risk retiree can be dealt with by means of the redistributive scheme illustrated in Section 2.2.

4.5. Simulation of redistributive GSA benefits

We implement the redistributive mechanism as described in Section 2.2. In particular, using the simulated $EPV_{GSA}^{j,k}(0)$ for every j and every k, obtained in Sections 4.3–4.4, we find numerically the optimal shares α_*^j, $j \in \{HR, MR, LR\}$ that minimize the performance criterion in (2.7) satisfying restriction (2.8), using the library "*Rsolnp*" of the R software environment.

Table 3 reports the optimal redistributive shares α_*^j, $j \in \{HR, MR, LR\}$.

Table 3. *α_*^j, $b_{RE}^j(0)$, b_A for $j \in \{HR, MR, LR\}$.*

	HR	MR	LR
α_*^j	1.1643	0.9717	0.864
$b_{RE}^j(0)$	72.8451	60.7946	54.0554
b_A	62.565	62.565	62.565

From Table 3, we see that the redistributive mechanism ensures that retirees enjoying a more favourable mortality experience are penalized by a lower initial benefit:

$$b_{RE}^{HR}(0) > b_A > b_{RE}^{MR}(0) > b_{RE}^{LR}(0). \quad (4.5)$$

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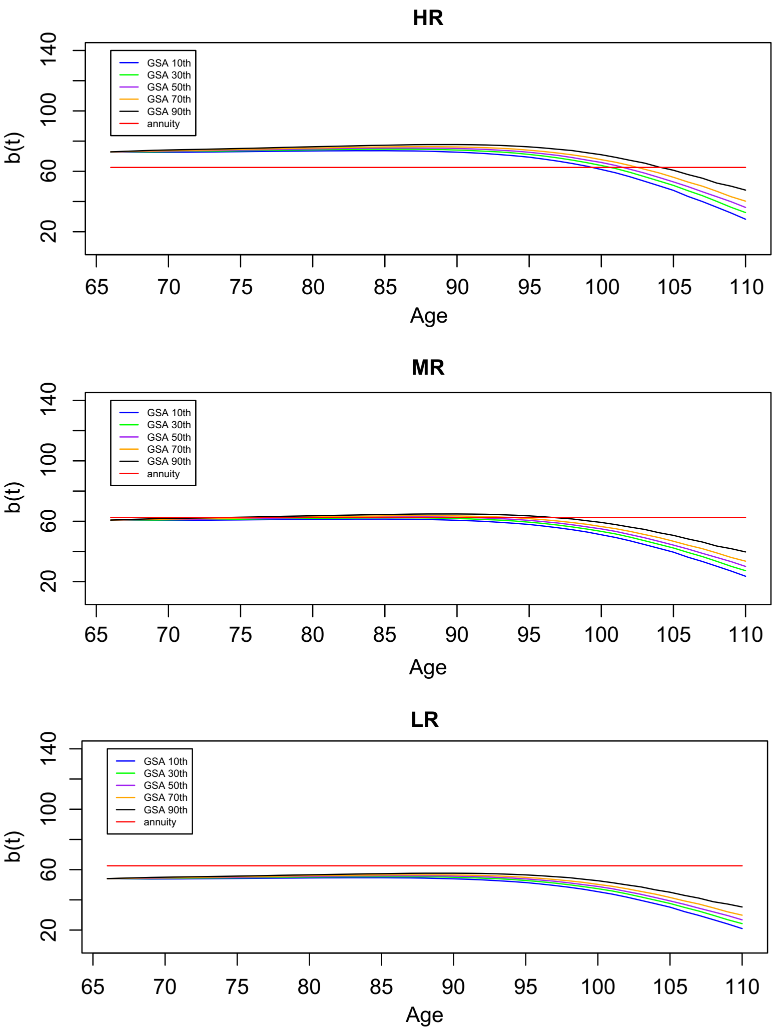

Figure 4. *Annuity benefits and relevant percentiles of the GSA scheme benefits distribution. Top panel: HR individuals; Medium panel: MR individuals; Bottom panel: LR individuals.*

Figure 4, top, medium and bottom panels report the benefits $b_{RE}(t)$ paid by the redistributive scheme to HR, MR and LR members, respectively. Figures 5 and 6 report, respectively, the expectation and the standard deviation of $b^j_{RE}(t)$ for all $j \in \{HR, MR, LR\}$ together with the expectation and standard deviation of the GSA benefit. Table 4 reports the optimal shares and some percentiles of the distribution of $EPV^j_{RE}(0)$.

We observe the following:

- Remarkably and as expected, the effect of redistribution becomes evident in Table 4. Thanks to the redistributive mechanism, the EPV(0) gaps across socio-economic groups have been significantly shrunk. The $EPV_{RE}(0)$ of HR individuals is still lower than that of LR individuals, but now the transfer from HR to LR individuals is on average less than 0.5–1%.
- For high-risk policyholders, redistribution triggers a wider dispersion of the $b_{RE}(t)$ distribution at each time t with respect to the case with no redistribution, as it emerges by observing Figure 6.
- When analysed over time, the dispersion is an increasing function of the policyholder's riskiness. We compute two types of ranges: the minimum and the maximum values reached by

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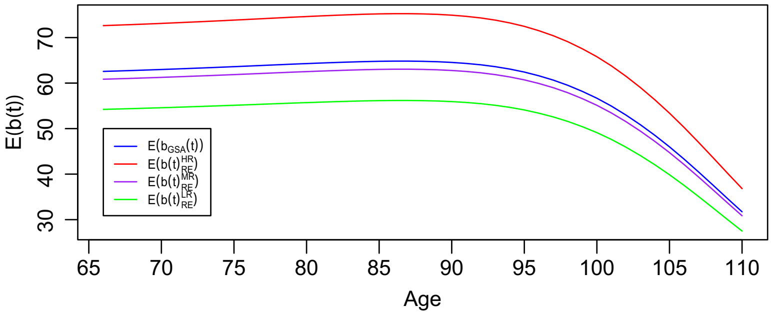

Figure 5. $\mathbb{E}(b_{GSA}(t))$ *and* $\mathbb{E}(b^j_{RE}(t)), j \in \{HR, MR, LR\}$.

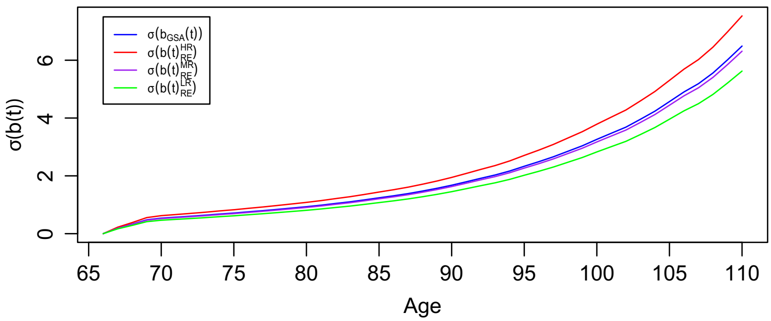

Figure 6. $\sigma(b_{GSA}(t))$ *and* $\sigma(b^j_{RE}(t)), j \in \{HR, MR, LR\}$.

$b_{GSA}(t)$ over time and over paths, $[\min_t b_{GSA}(t), \max_t b_{GSA}(t)]$, and the minimum value reached by $b_{GSA}(t)$ over time over the 10th percentile and the maximum value reached by $b_{GSA}(t)$ over time over the 90th percentile, $[\min_t b_{GSA}(t)10\text{th}, \max_t b_{GSA}(t)90\text{th}]$. We conclude the following:

1. Before redistribution, $[\min_t b_{GSA}(t), \max_t b_{GSA}(t)]$ ranges in [16.96, 81.70]. After redistribution, for high-risk individuals $b^{HR}_{RE}(t)$ ranges in [19.74, 95.12], for medium-risk individuals it ranges in [16.48, 79.39], finally for low-risk individuals it ranges in [14.65, 70.59].
2. Before redistribution $[\min_t b_{GSA}(t)10\text{th}, \max_t b_{GSA}(t)90\text{th}]$ ranges in [24.29, 66.73], while after redistribution for high-risk individuals $b_{RE}(t)$ ranges in [28.28, 77.70], for medium-risk individuals it ranges in [23.60, 64.84], while for low-risk individuals it ranges in [20.99, 57.66].

- For all the sub-populations $j \in \{HR, MR, LR\}$ the expectation of the redistributive GSA benefit $\mathbb{E}(b^j_{RE}(t))$ is slightly increasing for approximately 25 years and then it is decreasing with age, while the standard deviation of the redistributive GSA benefit $\sigma(b^j_{RE}(t))$ is an increasing function of age (Figures 5 and 6).

5. Sensitivity analysis

This section explores sensitivity of the GSA and our proposed redistributive GSA scheme to the relevant scheme features. We have performed three analyses. The first considers changes in the sample size, the second in the risk-free interest rate and the third in mortality assumptions. In the following, we discuss the main results and refer the reader to the Supplementary Material for a detailed analysis.

Size

When the sample size is small, idiosyncratic mortality risk is not well diversified and, fixing any time horizon, the volatility in the distribution of benefits is larger. This happens in both the GSA and

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Table 4. *$EPV_A^j(0)$ and some percentiles of the distribution of $EPV_{RE}^j(0)$, $j \in \{HR, MR, LR\}$ for the redistributive GSA scheme.*

	$EPV^{HR}(0)$	$EPV^{MR}(0)$	$EPV^{LR}(0)$
α_*^j	1.1643	0.9717	0.864
Annuity	831.1444	1000	1130.149
GSA 90th	992.6261	997.219	1000.3433
GSA 70th	985.8759	989.5666	991.7278
GSA 50th	981.2719	984.2774	985.8912
GSA 30th	976.7058	979.2726	980.5106
GSA 10th	970.5108	972.1699	972.6287

the redistributive GSA scheme, because the mortality credits are in both cases shared among a small number of survivors. Consequently, the expected present value of the benefits is more dispersed across policyholders. Vice versa, a larger sample size leads to a less volatile benefit distribution and to more certainty for the scheme members. As a consequence, the distribution of the expected present values is more compressed. The pool size has a small effect on the redistributive share, with a slightly larger redistribution when the sample size is small.

Interest rates

Interestingly, changing the interest rate produces different effects on the GSA scheme with or without redistribution. In the traditional GSA scheme when the interest rate is lower the distribution of expected present values shifts to the left for the HR members and to the right for the LR ones. In other words, inequity among different socio-economic classes increases with low interest rates. This is in line with intuition: with a low interest rate the future cash flows become more valuable to the benefit of those who live longer (LR individuals) and to the detriment of those who live shorter (HR individuals). And vice versa: when instead the interest rate is higher, the EPV distribution shifts to the right for HR members and to the left for LR ones because the different life expectancy is less important, so the inequity among socio-economic classes decreases. On the other hand, in the redistributive GSA scheme, with a lower interest rate both distributions of EPVs of HR and LR individuals shift to the left and the redistributive shares are more distant from one another. Vice versa, when the interest rate is higher both distributions of EPVs are shifted to the right and the optimal redistributive shares are closer. Thanks to the redistributive mechanism, in the presence of lower interest rates the penalization of HR members is spread among all participants to the pool. This seems to suggest that the redistributive mechanism mitigates the inequity among socio-economic classes, especially in times of low interest rates.

Mortality

We perform two different sensitivity analyses with respect to pool members' mortality assumptions.

First, we consider different deciles of our dataset for the HR (3rd instead of 1st) and LR (7th instead of 10th) members. Indeed, we consider a pool of members whose mortality experiences are less different among classes than in the basecase. The results align with expectations. Indeed, the benefits of the GSA scheme depart less from those of the annuity. Interestingly, they do not decline at later ages for all the quantiles of the distribution, but only for lower ones (below the 30th percentile) and at a later age relative to the basecase. Consistently, the redistributive GSA scheme shows both redistributive shares α_j^* and benefit distributions closer to one another across classes relative to our baseline case.

Second, we analyze the impact of considering different cohorts. We thus repeat our analysis for individuals who were 60 and 70, respectively, in 2001. The cash flow benefit distributions in the GSA scheme are very little affected. The redistributive shares in the redistributive schemes are similar in the two cases and slightly closer to one another relative to the basecase. This is due to the fact that the mortality differences between the high-risk and low-risk individuals in these two cohorts are slightly smaller than the baseline ones.

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6. Conclusions

In this paper, we propose a simple way of coping with heterogeneity in closed self-insurance pools. Our redistributive GSA scheme optimally sets different benefits to each sub-group within the pool by minimizing the distance of the expected present values of policyholders relative to a benchmark, which is the expected present value of a policyholder belonging to the reference group. While heterogeneity in self-insurance schemes has been analyzed by previous works, we contribute by taking a simulative approach and measuring inequity by studying the distributions of EPVs across sub-groups. Up to our knowledge, we are the first who address the problem of reducing inequality among socio-economic classes in a GSA scheme in the presence of stochastic mortality. We study sensitivity of our results to different pool sizes, risk-free interest rates and mortality assumptions, highlighting in particular that a lower risk-free rate worsens inequity and that the redistributive mechanism mitigates this increased inequity.

Further research is envisaged, with the aim of extending our framework to go beyond the assumption of risk-neutrality and introduce in the valuation of the benefits different risk attitudes for different policyholders. Another interesting avenue of future research is the investigation of how the variabilities of the stochastic mortality intensities affect the inequalities among policyholders of different socio-economic classes. Finally, also the study of how the correlation structure among different mortality intensities impacts the extent of inequalities would be worth exploring. Indeed, the doubly stochastic setup for the mortality intensity makes this possible.

Supplementary material. The supplementary material for this article can be found at https://doi.org/10.1017/asb.2025.16.

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Conflicts of interest. The authors have no relevant financial or non-financial interests to disclose.

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