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# EXPLICIT FORMULAS FOR THE COEFFICIENTS OF $\alpha$-CONVEX FUNCTIONS, $\alpha \geqq 0$

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**1. Introduction.** Let the function

$$(1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

be analytic in the unit disk $\Delta = \{z| \ |z| < 1\}$, with

$$f(z)f'(z)/z \neq 0$$

there, and let $\alpha$ be a real number. Then $f(z)$ is said to be $\alpha$-convex in $\Delta$ if and only if the inequality

$$\mathrm{Re}\left[ (1 - \alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] > 0$$

holds in $\Delta$. The class $\mathscr{M}(\alpha)$ of $\alpha$-convex functions was introduced in [**8**] and was studied in detail in the series [**5**]-[**10**], where in particular it is shown that $\alpha$-convex functions are univalent and starlike for all $\alpha$ $(-\infty \leqq \alpha \leqq +\infty)$, that is, the inequality

$$\mathrm{Re}\frac{zf'(z)}{f(z)} > 0$$

holds in $\Delta$. Moreover, in [**8**] for $0 < \alpha \leqq 1$, and in [**6**] for $\alpha > 0$, it is proved that each function $f(z)$ $\alpha$-convex in $\Delta$ is a univalent Bazilevič function of the form

$$(2) \quad f(z) = \left[\frac{1}{\alpha}\int_0^z [\sigma(\zeta)]^{1/\alpha}\frac{d\zeta}{\zeta}\right]^{\alpha},$$

where $\sigma(z) \in S^* \equiv \mathscr{M}(0)$ [**1**]. In [**3**], [**4**], and [**5**] it is noted that the best upper bounds on the coefficients of $f(z) \in \mathscr{M}(\alpha)$, $\alpha > 0$, might be given by the coefficients of the "$\alpha$-convex Koebe function"

$$(3) \quad K(z) = \left[\frac{1}{\alpha}\int_0^z \zeta^{(1/\alpha)-1}(1 - \zeta)^{-2/\alpha}d\zeta\right]^{\alpha}.$$

In [**3**] and [**4**] explicit cumbersome formulas for the coefficients of the extremal function (3) are given; these are based on a technique used in [**2**].

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(We remark that the formulas (19-20) in [**4**] are incorrect unless $n$ is replaced by $k$; and formula (21) in [**4**] must be studied for $k < n$, but not for $k \leqq n$, thus perpetuating an oversight in formula (22) in [**2**] ). If we use the technique used in [**14**], we can obtain what is possibly the simplest combinatorial form yet for the Taylor coefficients in problems of this kind. We do just that to obtain formulas for the coefficients of $\alpha$-convex functions, $\alpha > 0$. Hence we are able to obtain sharp upper bounds of the moduli $|a_{n+1}|$ of the coefficients of the functions (1) for $n = 1, \ldots, [\alpha] + 1$, if $\alpha > 0$ is not a positive integer, ( $[\alpha]$ denotes the greatest integer less than $\alpha$) and for all $n = 1, 2, \ldots$, if $\alpha$ is a positive integer. Thus we verify "the coefficient conjecture" for some $n$. En route we obtain additional results that include earlier ones due to Pinchuk [**11**], Robertson [**12**] and Schild [**13**].

All our results are sharp with all extremal functions explicitly given.

**2. Explicit formulas for the coefficients of the powers of the "nucleus" of the Bazilevič functions.** If $f(z) \in \mathcal{M}(\alpha)$ and if we put

$$(4) \qquad (1 - \alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) = p(z),$$

then $p(z) \in C$, where $C$ is the class of Carathéodory analytic functions

$$(5) \qquad p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$$

which satisfy the inequality $\operatorname{Re} p(z) > 0$ in $\Delta$. From (4) we obtain

$$(6) \qquad \frac{f(z)}{z}\left(\frac{zf'(z)}{f(z)}\right)^{\alpha} = P(z),$$

where

$$(7) \qquad P(z) = \exp\left\{\int_0^z \frac{p(t) - 1}{t} dt\right\}.$$

By using the identity

$$(8) \qquad \frac{zf'(z)}{f(z)} = 1 + z\frac{d}{dz} \ln \frac{f(z)}{z},$$

and the substitution

$$(9) \qquad \frac{f(z)}{z} = [u(z)]^{\alpha},$$

in (6), we obtain for $\alpha \neq 0$,

$$(10)\quad \frac{du}{dz} = -\frac{u}{\alpha z} + \frac{[P(z)]^{1/\alpha}}{\alpha z}.$$

From (9) and (10) we obtain, for $\alpha > 0$,

$$(11)\quad f(z) = \left[\frac{1}{\alpha}\int_0^z \zeta^{(1/\alpha)-1}[P(\zeta)]^{1/\alpha}d\zeta\right]^\alpha = z + \ldots .$$

If we set

$$(12)\quad \sigma(z) = zP(z)$$

then (6) yields the relation

$$(13)\quad \frac{z\sigma'(z)}{\sigma(z)} = p(z), \quad p(z) \in C,$$

which expresses the starlikeness of $\sigma(z)$ in $\Delta$. Hence it follows that the representations (2) and (11) are equivalent.

*Definition.* We call the function $P(z) \equiv \sigma(z)/z$ the nucleus of the corresponding Bazilevič function defined by (2) or (11).

For $\alpha \neq 0$, the function

$$(14)\quad [P(z)]^{1/\alpha} = 1 + \sum_{n=1}^{\infty} P_n(\alpha)z^n,$$

is analytic near $z = 0$. Hence, in order to find the coefficients of the function (11) we must first find the coefficients of (14) in terms of the coefficients of $p(z)$ in (5).

THEOREM 1. *The coefficients* $P_n(\alpha)$, $n = 1, 2, \ldots, (\alpha \neq 0)$ *of the function* (14) *have the explicit representation*

$$(15)\quad P_n(\alpha) = \sum_{k=1}^{n} \frac{1}{\alpha^k} C_{nk}\left(\frac{p_1}{1}, \ldots, \frac{p_{n-k+1}}{n-k+1}\right),$$

*where*

$$(16)\quad C_{nk}\left(\frac{p_1}{1}, \ldots, \frac{p_{n-k+1}}{n-k+1}\right) = \sum \prod_{s=1}^{n-k+1} \frac{1}{\nu_s!}\left(\frac{p_s}{s}\right)^{\nu_s}$$

*where the sum is taken over all solutions in non-negative integers* $\nu_1, \ldots, \nu_{n-k+1}$ *of the system*

$$(17)\quad \begin{array}{l} \nu_1 + \nu_2 + \ldots + \nu_{n-k+1} = k, \\ \nu_1 + 2\nu_2 + \ldots + (n-k+1)\nu_{n-k+1} = n. \end{array}$$

*Proof.* It follows from (5) and (7) that the function (15) can be represented by the composite function

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$$[P(z)]^{1/\alpha} = e^u \circ \sum_{n=1}^{\infty} \frac{p_n}{\alpha n} z^n,$$

where $\circ$ denotes the substitution

$$u = \sum_{n=1}^{\infty} (p_n/\alpha n) z^n.$$

By a more precise version of a Faà di Bruno formula for derivatives of composite functions [**14**, Theorem 1], applied to the $n^{\text{th}}$ derivative of the composite function above, at the point $z = 0$, we obtain the representation

$$(18) \quad D^n_{z=0}[P(z)]^{1/\alpha} = n! \sum_{k=1}^{n} \sum \prod_{s=1}^{n-k+1} \frac{1}{\nu_s!}\left(\frac{p_s}{\alpha s}\right)^{\nu_s},$$

where the interior sum is taken over all solutions in non-negative integers $\nu_1, \dots, \nu_{n-k+1}$ of the system (17). From (18) we obtain the representation (15), (16) for $P_n(\alpha)$.

This completes the proof of Theorem 1.

For an arbitrary $x$, let $\langle x \rangle_k$ denote the factorial polynomial

$$\langle x \rangle_k = x(x+1)\dots(x+k-1) \quad (k = 1, 2, \dots; \langle x \rangle_0 = 1).$$

From Theorem 1 we obtain the following result.

THEOREM 2. *For* $\alpha > 0$, *the coefficients* $P_n(\alpha)$, $n = 1, 2, \dots$, *of the function* (14) *satisfy the sharp inequalities*

$$(19) \quad |P_n(\alpha)| \leqq \frac{\left\langle \frac{2}{\alpha} \right\rangle_n}{n!},$$

*where equality holds only for the function*

$$(20) \quad [P_*(z)]^{1/\alpha} = (1 - \epsilon z)^{-2/\alpha} = 1 + \sum_{n=1}^{\infty} \frac{\left\langle \frac{2}{\alpha} \right\rangle_n}{n!} \epsilon^n z^n, \quad (|\epsilon| = 1).$$

*Proof.* From (15) and (16) we obtain the estimate

$$(21) \quad |P_n(\alpha)| \leqq \sum_{k=1}^{n} \frac{1}{\alpha^k} C_{nk}\left(\frac{|p_1|}{1}, \dots, \frac{|p_{n-k+1}|}{n-k+1}\right).$$

Since $p(z) \in C$, we can use the known inequalities $|p_s| \leqq 2$, $s = 1, \dots, n$, in (21) to obtain

$$(22)\quad |P_n(\alpha)| \leqq \sum_{k=1}^{n} \frac{1}{\alpha^k} C_{nk}\left(\frac{2}{1}, \dots, \frac{2}{n-k+1}\right),$$

where by (15)-(17) equality holds in (22), if and only if $p_s = 2\epsilon^s$, $|\epsilon| = 1$, $s = 1, \dots, n$. But if $p_1 = 2\epsilon$ holds for the Carathéodory function (5), then

$$p_s = 2\epsilon^s \quad \text{for } s = 1, 2, \dots .$$

Therefore equality holds in (22) only if the function (5) has the form

$$(23)\quad p_*(z) = \frac{1+\epsilon z}{1-\epsilon z} = 1 + 2\sum_{n=1}^{\infty} \epsilon^n 2^n \quad (|\epsilon| = 1),$$

or, by (7), only if the function (14) has the form (20). If we use the combinatorial identity

$$\sum_{k=1}^{n} \frac{1}{\alpha^k} C_{nk}\left(\frac{2}{1}, \dots, \frac{2}{n-k+1}\right) = \frac{\left\langle \frac{2}{\alpha} \right\rangle_n}{n!}, \quad (n = 1, 2, \dots; \alpha > 0),$$

then the inequality (22) becomes (19).

This completes the proof of Theorem 2.

We note two important consequences of Theorems 1 and 2.

i) For finite $\alpha$, $\alpha \geqq 1$, if we set

$$\beta = 1 - \frac{1}{\alpha} \quad (0 \leqq \beta < 1)$$

in (14) and if we use (12), we conclude that

$$(24)\quad \sigma_\beta(z) \equiv z[P(z)]^{1-\beta} = z + \sum_{n=1}^{\infty} P_n\left(\frac{1}{1-\beta}\right) z^{n+1}$$

is starlike of order $\beta$, that is, it satisfies the inequality

$$(25)\quad \mathrm{Re}\frac{z\sigma'_\beta(z)}{\sigma_\beta(z)} > \beta$$

in $\Delta$. This also follows from the relation

$$(26)\quad \frac{z\sigma'_\beta(z)}{\sigma_\beta(z)} = (1-\beta)p(z) + \beta, \quad p(z) \in C,$$

which is easily obtained from (7) and (24). Hence, for

$$\alpha = 1/(1-\beta), \quad 0 \leqq \beta < 1,$$

Theorem 1 yields the coefficients of the function (24) in terms of the coefficients of the associated function $p(z)$ in (26), and Theorem 2 gives us the known estimates [**11**]-[**13**]

$$(27)\quad \left|P_n\left(\frac{1}{1-\beta}\right)\right| \leqq \frac{\langle 2-2\beta\rangle_n}{n!} \quad (n = 1, 2, \dots)$$

for the class $S^*_\beta$ of functions $\sigma_\beta(z)$ starlike of order $\beta$; here equality holds only for the Koebe functions

$$\sigma_{\beta^*}(z) = \frac{z}{(1-\epsilon z)^{2(1-\beta)}} = z + \sum_{n=1}^{\infty} \frac{\langle 2-2\beta\rangle_n}{n!}\epsilon^n z^{n+1} \quad (|\epsilon| = 1).$$

For $\beta = 0$ the results reduce to the results for the class $S^* \equiv S^*_0$ of starlike univalent functions.

ii) For $0 \leqq \beta < 1$, it follows from (24) and (25) that the function

$$(28)\quad f_\beta(z) \equiv \int_0^z [P(\zeta)]^{1-\beta} d\zeta = z + \sum_{n=1}^{\infty} P_n\left(\frac{1}{1-\beta}\right)\frac{z^{n+1}}{n+1}$$

is convex of order $\beta$, that is, it satisfies the inequality

$$\mathrm{Re}\left(1 + \frac{zf''_\beta(z)}{f'_\beta(z)}\right) > \beta$$

in $\Delta$. This also follows from the relation

$$(29)\quad 1 + \frac{zf''_\beta(z)}{f'_\beta(z)} = (1-\beta)p(z) + \beta, \quad p(z) \in C,$$

which can be obtained from (24), (26), and (28). Hence, the coefficients of $f_\beta(z)$ can be expressed in terms of the coefficients of the associated function $p(z)$, using (29), and Theorem 1 for $\alpha = 1/(1-\beta)$, $0 \leqq \beta < 1$. From (27) and (28) we obtain the known inequalities [**11**, **12**, **13**]

$$\left|\frac{1}{n+1}P_n\left(\frac{1}{1-\beta}\right)\right| \leqq \frac{\langle 2-2\beta\rangle_n}{(n+1)!} \quad (n = 1, 2, \dots),$$

for the class $C^0_\beta$ of functions $f_\beta(z)$ convex of order $\beta$; here equality holds only for the functions.

$$f_{\beta^*}(z) = \frac{1}{\epsilon(1-2\beta)}\left[\frac{1}{(1-\epsilon z)^{1-2\beta}} - 1\right]$$

$$= z + \sum_{n=1}^{\infty} \frac{\langle 2-2\beta\rangle_n}{(n+1)!}\epsilon^n z^{n+1} \quad (|\epsilon| = 1)$$

for $\beta \neq 1/2$, and only for the function

$$f_{\beta^*}(z) = -\frac{1}{\epsilon}\ln(1 - \epsilon z) = z + \sum_{n=1}^{\infty} \frac{\epsilon^n}{n+1} z^{n+1} \quad (|\epsilon| = 1)$$

for $\beta = 1/2$. For $\beta = 0$, the results reduce to those for the class $C^0 \equiv C_0^0$ of convex univalent functions.

**3. Explicit formulas for the coefficients of Bazilevič functions.** Let $(x)_k$, for an arbitrary number $x$, denote the factorial polynomial

$$(x)_k = x(x-1)\dots(x-k+1), \quad (k = 1, 2, \dots; (x)_0 = 1),$$

and let the function in (11) have the expansion

$$(30) \quad f(z) = z + \sum_{n=1}^{\infty} a_{n+1}(\alpha) z^{n+1} \quad (\alpha > 0)$$

in $\Delta$. Now we shall determine the coefficients in (30) in terms of the coefficients of the function given in (14).

THEOREM 3. *The Taylor coefficients $a_{n+1}(\alpha)$, $n = 1, 2, \dots$ of the function* (11) *for an arbitrary $\alpha > 0$ have the explicit representation*

$$(31) \quad a_{n+1}(\alpha) = \sum_{k=1}^{n} (\alpha)_k C_{nk}\left(\frac{P_1(\alpha)}{\alpha+1}, \dots, \frac{P_{n-k+1}(\alpha)}{(n-k+1)\alpha+1}\right),$$

*where*

$$(32) \quad C_{nk}\left(\frac{P_1(\alpha)}{\alpha+1}, \dots, \frac{P_{n-k+1}(\alpha)}{(n-k+1)\alpha+1}\right) = \sum \prod_{s=1}^{n-k+1} \frac{1}{\nu_s!}\left(\frac{P_s(\alpha)}{(s\alpha+1)}\right)^{\nu_s},$$

*where the sum is taken over all solutions in non-negative integers $\nu_1, \dots, \nu_{n-k+1}$ of the system* (17). *If $\alpha$ is a positive integer* ($\alpha = 1, 2, \dots$), *then we also have the explicit representation*

$$(33) \quad a_{n+1}(\alpha) = \alpha! C_{n+\alpha,\alpha}\left(P_0(\alpha), \frac{P_1(\alpha)}{\alpha+1}, \dots, \frac{P_n(\alpha)}{n\alpha+1}\right), \; P_0(\alpha) = 1,$$

*for $n = 1, 2, \dots$, where*

$$(34) \quad C_{n+\alpha,\alpha}\left(P_0(\alpha), \frac{P_1(\alpha)}{\alpha+1}, \dots, \frac{P_n(\alpha)}{n\alpha+1}\right) = \sum \prod_{s=0}^{n} \frac{1}{\nu_{s+1}!}\left(\frac{P_s(\alpha)}{s\alpha+1}\right)^{\nu_{s+1}}$$

*where the sum is taken over all non-negative integers* $\nu_1, \ldots, \nu_{n+1}$ *satisfying*

$$(35)\quad \nu_1 + \nu_2 + \ldots + \nu_{n+1} = \alpha,$$
$$\nu_1 + 2\nu_2 + \ldots + (n + 1)\nu_{n+1} = n + \alpha.$$

*Remark.* For a positive integer $\alpha$ ($\alpha = 1, 2, \ldots$), (31) and (33) are identical.

*Proof.* From (11), (14), and (30), we obtain

$$(36)\quad f(z) = z[F(z)]^{\alpha} \equiv z + \sum_{n=1}^{\infty} a_{n+1}(\alpha)z^{n+1}, \quad \alpha > 0,$$

where

$$(37)\quad F(z) = 1 + \sum_{n=1}^{\infty} \frac{P_n(\alpha)}{\alpha n + 1} z^n, \quad z \in \Delta, \quad \alpha > 0.$$

Again the (more precise) Faà di Bruno formula for the $n^{\text{th}}$ derivative of composite functions [**14**, Theorem 1], applied to the $n^{\text{th}}$ derivative of the composite function

$$[F(z)]^{\alpha} \equiv t^{\alpha} \circ F(z)$$

at the point $z = 0$ (see the proof of Theorem 2 in [**14**]), yields the representation (31) immediately.

If $\alpha$ is a positive integer ($\alpha = 1, 2, \ldots$), then from (36) and (37) we obtain

$$(38)\quad f(z) = z^{1-\alpha}[zF(z)]^{\alpha} \equiv z + \sum_{n=1}^{\infty} a_{n+1}(\alpha)z^{n+1}, \quad z \in \Delta.$$

The same Faà di Bruno formula for the $n$th derivative of composite functions (used above), now applied to the composite function

$$[zF(z)]^{\alpha} \equiv t^{\alpha} \circ [zF(z)]$$

at the point $z = 0$, yields

$$(39)\quad [zF(z)]^{\alpha} = \sum_{n=\alpha}^{\infty} \alpha! C_{n\alpha}\left(P_0(\alpha), \frac{P_1(\alpha)}{\alpha + 1}, \ldots, \frac{P_{n-\alpha}(\alpha)}{(n - \alpha)\alpha + 1}\right) z^n,$$
$$P_0(\alpha) = 1,$$

in $\Delta$, where

$$(40)\quad C_{n\alpha}\left(P_0(\alpha), \frac{P_1(\alpha)}{\alpha + 1}, \ldots, \frac{P_{n-\alpha}(\alpha)}{(n - \alpha)\alpha + 1}\right)$$

$$= \sum \prod_{s=0}^{n-\alpha} \frac{1}{\nu_{s+1}!}\left(\frac{P_s(\alpha)}{s\alpha + 1}\right)^{\nu_{s+1}}$$

where the sum is taken over all non-negative integers $\nu_1, \ldots, \nu_{n-\alpha+1}$ satisfying

(41) $\nu_1 + \nu_2 + \ldots + \nu_{n-\alpha+1} = \alpha$

$\nu_1 + 2\nu_2 + \ldots + (n - \alpha + 1)\nu_{n-\alpha+1} = n.$

Now from (38)-(41) we obtain (33)-(35).

This completes the proof of Theorem 3.

From Theorem 3 and (20) we obtain the coefficients of the most general "α-convex Koebe function"

(42) $$K(z; \epsilon; \alpha) \equiv \left[\frac{1}{\alpha}\int_0^z \zeta^{(1/\alpha)-1}(1 - \epsilon\zeta)^{-2/\alpha}d\zeta\right]^\alpha \quad (|\epsilon| = 1; \alpha > 0),$$

for arbitrary $\alpha > 0$. We find

(43) $$K(z; \epsilon; \alpha) = z[F(z; \epsilon; \alpha)]^\alpha \equiv z + \sum_{n=1}^{\infty} K_{n+1}(\epsilon; \alpha)z^{n+1}$$

in $\Delta$, where

(44) $$F(z; \epsilon; \alpha) = 1 + \sum_{n=1}^{\infty} \epsilon^n c_n(\alpha)z^n$$

with

(45) $$c_n(\alpha) = \frac{\left\langle \frac{2}{\alpha} \right\rangle_n}{n!(n\alpha + 1)}, \quad (n = 1, 2, \ldots).$$

It is clear that the substitution $\zeta = zt$ in (42) yields the identity

(46) $$K(z; \epsilon; \alpha) = z\left[\mathscr{F}\left(\frac{2}{\alpha}, \frac{1}{\alpha}, 1 + \frac{1}{\alpha}; \epsilon z\right)\right]^\alpha \quad (|\epsilon| = 1; \alpha > 0),$$

where $\mathscr{F}$ is the Gauss hypergeometric function

$$\mathscr{F}\left(\frac{2}{\alpha}, \frac{1}{\alpha}, 1 + \frac{1}{\alpha}; \epsilon z\right) = \frac{1}{\alpha}\int_0^1 t^{(1/\alpha)-1}(1 - \epsilon zt)^{-2/\alpha}dt$$

that is, the analytic continuation of the series (44) into the $z$-plane cut along the ray

$$z = \rho e^{-i\arg\epsilon}, \quad \rho \geqq 1.$$

In particular, if $\alpha$ is a positive integer ($\alpha = 1, 2, \ldots$), then from (43) we obtain

$$(47)\quad K(z;\epsilon;\alpha) = z^{1-\alpha}[zF(z;\epsilon;\alpha)]^{\alpha}$$
$$\equiv z + \sum_{n=1}^{\infty} K_{n+1}(\epsilon;\alpha)z^{n+1}, \quad z \in \Delta.$$

As a corollary to the preceding, we have the following result concerning the coefficients of the function (42) (or (46)) and the function (47), respectively.

THEOREM 4. *The coefficients* $K_{n+1}(\epsilon;\alpha)$ *of the general "α-convex Koebe functions"* (42) *for an arbitrary* $\alpha > 0$ *have the explicit representation*

$$(48)\quad K_{n+1}(\epsilon;\alpha) = \epsilon^n K_{n+1}(\alpha) \quad (n = 1, 2, \dots),$$

*where*

$$(49)\quad K_{n+1}(\alpha) = \sum_{k=1}^{n} (\alpha)_k C_{nk}(c_1(\alpha), \dots, c_{n-k+1}(\alpha)),$$

*where*

$$(50)\quad C_{nk}(c_1(\alpha), \dots, c_{n-k+1}(\alpha)) = \sum \prod_{s=1}^{n-k+1} \frac{(c_s(\alpha))^{\nu_s}}{\nu_s!},$$

*where the* $c_1(\alpha), \dots, c_{n-k+1}(\alpha)$ *are defined in* (45) *and where the sum is taken over all solutions in non-negative integers* $\nu_1, \dots, \nu_{n-k+1}$ *of the system* (17).

*If* α *is a positive integer* ($\alpha = 1, 2, \dots$), *then we also have the representation* (48) *but with the simpler expression*

$$(51)\quad K_{n+1}(\alpha) = \alpha! C_{n+\alpha,\alpha}(c_0(\alpha), c_1(\alpha), \dots, c_n(\alpha)), \ c_0(\alpha) = 1$$

*for* $n = 1, 2, \dots$, *where*

$$(52)\quad C_{n+\alpha,\alpha}(c_0(\alpha), c_1(\alpha), \dots, c_n(\alpha)) = \sum \prod_{s=0}^{n} \frac{(c_s(\alpha))^{\nu_{s+1}}}{\nu_{s+1}!}$$

*and the sum is taken over all solutions in non-negative integers* $\nu_1, \dots, \nu_{n+1}$ *of the system* (35).

*Remark*. For a positive integer $\alpha$ ($\alpha = 1, 2, \dots$), (49) reduces to (51).

In [**3**], [**4**], and [**5**], as we have noted in the introduction, it is indicated that the coefficients of the function (3) might yield the sharp upper bounds on the modulus of the coefficients of the α-convex functions for $\alpha > 0$. Theorem 4 yields the simplest combinatorial form yet of that conjecture.

CONJECTURE. *The coefficients* $a_{n+1}(\alpha)$ *of the* α*-convex functions* (30) *satisfy the sharp inequalities*

(53) $|a_{n+1}(\alpha)| \leqq K_{n+1}(\alpha), \quad (n = 1, 2, \dots)$

*where the* $K_{n+1}(\alpha)$ *are given by* (49) *for arbitrary* $\alpha > 0$, *and by* (51) *for a positive integer* $\alpha$ ($\alpha = 1, 2, \dots$), *respectively, with equality only for the "$\alpha$-convex Koebe functions"* (42).

Our Theorems 2, 3 and 4 verify that conjecture in the following cases.

THEOREM 5. *The coefficients* $a_{n+1}(\alpha)$ *of the* $\alpha$*-convex functions* (30) *satisfy the sharp inequalities* (53) *for* $n = 1, \dots, [\alpha] + 1$ *if* $\alpha > 0$ *is not a positive integer* ( $[\alpha]$ *denotes the greatest integer less than* $\alpha$), *and for all* $n = 1, 2, \dots$ *if* $\alpha$ *is a positive integer* ($\alpha = 1, 2, \dots$) *where* $K_{n+1}(\alpha)$ *are given by* (49) *and* (51), *respectively, with equality only for the "$\alpha$-convex Koebe functions"* (42).

*Proof.* If $\alpha > 0$ is not a positive integer, then from (31) and (32) we obtain, for $n = 1, \dots, [\alpha] + 1$, the estimates

$$(54) \quad |a_{n+1}(\alpha)| \leqq \sum_{k=1}^{n} (\alpha)_k C_{nk}\left(\frac{|P_1(\alpha)|}{\alpha + 1}, \dots, \frac{|P_{n-k+1}(\alpha)|}{(n - k + 1)\alpha + 1}\right).$$

Now we can use the inequalities (19) in (54) to obtain

$$(55) \quad |a_{n+1}(\alpha)| \leqq \sum_{k=1}^{n} (\alpha)_k C_{nk}(c_1(\alpha), \dots, c_{n-k+1}(\alpha)) = K_{n+1}(\alpha),$$

with (45) and (49) in mind. Equality holds in (55), according to Theorems 2 and 3, only if the function (11) has the form (42).

If $\alpha$ is a positive integer ($\alpha = 1, 2, \dots$), the above proof is valid for all $n = 1, 2, \dots$. Indeed, then $(\alpha)_k = 0$ for $k > \alpha$, and hence, the summation in (54) and (55) is taken over $k = 1, \dots, n$ if $1 \leqq n \leqq \alpha$, and over $k = 1, \dots, \alpha$ if $n > \alpha$.

If $\alpha$ is a positive integer ($\alpha = 1, 2, \dots$), another proof can be obtained from (33) and (34). Indeed, from (33) and (34) for $n = 1, 2, \dots$ we obtain

$$(56) \quad |a_{n+1}(\alpha)| \leqq \alpha! C_{n+\alpha,\alpha}\left(|P_0(\alpha)|, \frac{|P_1(\alpha)|}{\alpha + 1}, \dots, \frac{|P_n(\alpha)|}{n\alpha + 1}\right).$$

Again we use (19), (45), (51) and (56) to obtain

$$|a_{n+1}(\alpha)| \leqq \alpha! C_{n+\alpha,\alpha}(c_0(\alpha), c_1(\alpha), \dots, c_n(\alpha)) = K_{n+1}(\alpha)$$

where equality holds, according to Theorems 2 and 3 only if the function (11) has the form (42).

This completes the proof of Theorem 5.

**4. Explicit formulas for the coefficients of the Carathéodory functions in terms of the coefficients of their associated $\alpha$-convex functions.** A solution to this "inverse problem" follows from the following general result.

THEOREM 6. *If the function*

$$(57)\quad f(z) = \sum_{n=1}^{\infty} a_n z^n \quad (a_1 = 1)$$

*is analytic in* $\Delta$, *then the following statements are valid.*

(i) *If*

$$(58)\quad \frac{zf'(z)}{f(z)} \equiv 1 + \sum_{n=1}^{\infty} p_n z^n,$$

*then* $p_n = ng_n$ $(n = 1, 2, \dots)$,

*where*

$$(59)\quad g_n = \sum_{k=1}^{n} (-1)^{k-1}(k-1)!C_{nk}(a_2, \dots, a_{n-k+2}),$$

*and*

$$(60)\quad C_{nk}(a_2, \dots, a_{n-k+2}) = \sum \prod_{s=1}^{n-k+1} \frac{(a_{s+1})^{\nu_s}}{\nu_s!},$$

*where the summation is taken over all solutions in non-negative* $\nu_1, \dots, \nu_{n-k+1}$ *of the system* (17). *The series* (58) *is valid for* $|z| < r_1$ *where* $r_1$ *is the distance from* $z = 0$ *to the nearest zero of the function* $f(z)/z$ *in* $\Delta$; *if* $f(z)/z \neq 0$ *in* $\Delta$, *then the series* (58) *is valid in all of* $\Delta$.

(ii) *If*

$$(61)\quad \frac{zf''(z)}{f'(z)} = \sum_{n=1}^{\infty} p_n z^n,$$

*then,* $p_n = nh_n$ $(n = 1, 2, \dots)$, *where*

$$h_n = \sum_{k=1}^{n} (-1)^{k-1}(k-1)!C_{nk}(2a_2, \dots, (n-k+2)a_{n-k+2}),$$

*and*

$$(62)\quad C_{nk}(2a_2, \dots, (n-k+2)a_{n-k+2}) = \sum \prod_{s=1}^{n-k+1} \frac{[(s+1)a_{s+1}]^{\nu_s}}{\nu_s!},$$

*where the summation is that for* (60). *The series* (61) *is valid for* $|z| < r_2$ *where* $r_2$ *is the distance from* $z = 0$ *to the nearest zero of the derivative* $f'(z)$ *in* $\Delta$; *if* $f'(z) \neq 0$ *in* $\Delta$, *then the expansion* (61) *is valid in* $\Delta$.

(iii) *For every number* $\alpha$, *we have*

$$(63)\quad (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) = 1 + \sum_{n=1}^{\infty} p_n z^n,$$

*where*

$$p_n = n[(1-\alpha)g_n + \alpha h_n] \quad (n = 1, 2, \dots),$$

*where the* $g_n$ *and* $h_n$ *are given in* (i) *and* (ii) *above. Moreover, the series* (63) *converges for* $|z| < r_3$ *where*

$$r_3 = \min(r_1, r_2);$$

*if* $f(z)f'(z)/z \neq 0$ *in* $\Delta$, *then the series in* (63) *is convergent there.*

*Proof.* (i) From the identity (8) it follows that it is sufficient to obtain an expansion for $\ln(f(z)/z)$. If we use the (improved) Faà di Bruno formula for the $n^{\text{th}}$ derivative of composite functions [**14**, Theorem 1], applied to the $n^{\text{th}}$ derivative of the composite function

$$\ln(f(z)/z) \equiv \ln t \circ (f(z)/z)$$

at the point $z = 0$, we obtain the formula

$$D^n_{z=0} \ln \frac{f(z)}{z} = n! g_n \quad (n = 1, 2, \dots),$$

where the $g_n$ are given by (59). Hence

$$\ln\left(\frac{f(z)}{z}\right) = \sum_{n=1}^{\infty} g_n z^n$$

holds; and this with (8) yields (58). The remarks concerning convergence follow at once.

(ii) From the identity

$$\frac{zf''(z)}{f'(z)} \equiv z\frac{d}{dz} \ln f'(z),$$

with

$$\ln f'(z) = \sum_{n=1}^{\infty} h_n z^n,$$

we again use the (improved) Faà di Bruno formula to prove (ii). Again the remarks concerning convergence are immediate.

(iii) This follows from (i) and (ii).

This completes the proof of Theorem 6.

In particular, when the function (57) is (i) starlike, (ii) convex or (iii) $\alpha$-convex, ($\alpha$ real), yield the result indicated in the title of this section.

**5. Conclusion.** If in (50) we put $c_s \equiv c_s(\alpha)$, $s = 1, \ldots, n - k + 1$, and we consider $c_1, \ldots, c_{n-k+1}$ as arbitrary variables, then the expressions (50) are isobaric homogeneous polynomials, of weight $n$ and of degree $k$, in those variables [**14**]. Those polynomials may be obtained by using the recursion formula introduced in [**14**]:

$$C_{nk} = \frac{1}{k} \sum_{\mu=1}^{n-k+1} c_\mu C_{n-\mu,k-1}$$

$$(1 \leqq k \leqq n;\ n \geqq 1;\ C_{n0} = 0;\ C_{00} = 1),$$

where $C_{nk} = C_{nk}(c_1, \ldots, c_{n-k+1})$. The first and last polynomials are

$$C_{n1} = c_n, \quad C_{nn} = \frac{1}{n!} c_1^n \quad (n \geqq 1).$$

The short table of the polynomials $C_{nk}$ for $1 \leqq n \leqq 5$, given below, is taken from [**14**].

$$C_{11} = c_1;\ C_{21} = c_2,\ C_{22} = \frac{1}{2} c_1^2;$$

$$C_{31} = c_3,\ C_{32} = c_1 c_2,\ C_{33} = \frac{1}{6} c_1^3;$$

$$C_{41} = c_4,\ C_{42} = c_1 c_3 + \frac{1}{2} c_2^2,$$

$$C_{43} = \frac{1}{2} c_1^2 c_2,\ C_{44} = \frac{1}{24} c_1^4;\ C_{51} = c_5,$$

$$C_{52} = c_1 c_4 + c_2 c_3,\ C_{53} = \frac{1}{2} c_1^2 c_3 + \frac{1}{2} c_1 c_2^2,$$

$$C_{54} = \frac{1}{6} c_1^3 c_2,\ C_{55} = \frac{1}{120} c_1^5.$$

It is clear that sums (16), (32), (34), (40), (50), (52), (60) and (62) are corresponding values of the polynomials $C_{nk}$.

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