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An Explicit Computation of the Blanchfield Pairing for Arbitrary Links

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Abstract. Given a link L, the Blanchfield pairing $\mathrm{Bl}(L)$ is a pairing that is defined on the torsion submodule of the Alexander module of L. In some particular cases, namely if L is a boundary link or if the Alexander module of L is torsion, $\mathrm{Bl}(L)$ can be computed explicitly; however no formula is known in general. In this article, we compute the Blanchfield pairing of any link, generalizing the aforementioned results. As a corollary, we obtain a new proof that the Blanchfield pairing is Hermitian. Finally, we also obtain short proofs of several properties of $\mathrm{Bl}(L)$.

1 Introduction

The Blanchfield pairing of a knot K is a nonsingular Hermitian pairing $\mathrm{Bl}(K)$ on the Alexander module of K [2]. Despite early appearances in high dimensional knot theory [25, 26, 35], the Blanchfield pairing is nowadays mostly used in the classical dimension. For instance, applications of $\mathrm{Bl}(K)$ in knot concordance include a characterization of algebraic sliceness [27] and a crucial role in the obstruction theory underlying the solvable filtration of [15] (see also [7, 12, 22, 30, 32]). Furthermore, $\mathrm{Bl}(K)$ has also served to compute unknotting numbers [3–5] and in the study of finite type invariants [33]. Finally, the Blanchfield pairing can be computed using Seifert matrices [21, 27, 31], is known to determine the Levine–Tristram signatures [5] and more generally the S-equivalence class of the knot [37].

In the case of links, the Blanchfield pairing generalizes to a Hermitian pairing $\mathrm{Bl}(L)$ on the torsion submodule of the Alexander module of L. Although $\mathrm{Bl}(L)$ is still used to investigate concordance [8, 11, 13, 19, 28, 36], unlinking numbers, and splitting numbers [6], several questions remain: is there a natural definition of algebraic concordance for links and can it be expressed in terms of the Blanchfield pairing? Can one compute unlinking numbers and splitting numbers by generalizing the methods of [3–5]? Does the Blanchfield pairing determine the multivariable signature of [10]?

A common issue seems to lie at the root of these unanswered questions: there is no general formula to compute the Blanchfield pairing of a link. More precisely, $\mathrm{Bl}(L)$ can currently only be computed if L is a boundary link [14, 24] or if the Alexander module of L is torsion [16]. Note that these formulas generalize the one-component case in orthogonal directions: if L is a boundary link whose Alexander module is torsion, then L must be a knot. The aim of this paper is to provide a general formula for the Blanchfield pairing of any colored link, while generalizing the previous formulas.

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By a *μ-colored link*, we mean an oriented link L in S^3 whose components are partitioned into μ sublinks $L_1 \cup \cdots \cup L_\mu$. The exterior $S^3 \setminus \nu L$ of L will always be denoted by X_L. Moreover, we write $\Lambda_S := \mathbb{Z}[t_1^{\pm 1}, \ldots, t_\mu^{\pm 1}, (1-t_1)^{-1}, \ldots, (1-t_\mu)^{-1}]$ for the localization of the ring of Laurent polynomials, and we use $Q = \mathbb{Q}(t_1, \ldots, t_\mu)$ to denote the quotient field of Λ_S. Using these notations, the Blanchfield pairing of the colored link L is a Hermitian pairing $\mathrm{Bl}(L)\colon TH_1(X_L; \Lambda_S) \times TH_1(X_L; \Lambda_S) \to Q/\Lambda_S$, where $TH_1(X_L; \Lambda_S)$ denotes the torsion submodule of the Alexander module $H_1(X_L; \Lambda_S)$ of L (see Section 2.2 for details). There are two main reasons for which we use Λ_S coefficients instead of the more conventional $\Lambda := \mathbb{Z}[t_1^{\pm 1}, \ldots, t_\mu^{\pm 1}]$ coefficients. The first is to ensure that the Alexander module $H_1(X_L; \Lambda_S)$ admits a square presentation matrix: the corresponding statement is false over Λ [18, 23]. The second is to guarantee that the Blanchfield pairing is non-degenerate after quotienting $TH_1(X_L; \Lambda_S)$ by the so-called *maximal pseudonull submodule* [23]. Note that for knots, the Alexander module over Λ_S is the same as the Alexander module over Λ [31, Proposition 1.2].

As we mentioned above, $\mathrm{Bl}(L)$ can currently only be computed if L is a boundary link (using boundary Seifert surfaces) or if the Alexander module of L is torsion (using some generalized Seifert surfaces known as C-complexes). Let us briefly recall this latter result. A *C-complex* for a μ-colored link L consists in a collection of Seifert surfaces $F_1, \ldots, F_\mu$ for the sublinks $L_1, \ldots, L_\mu$ that intersect only pairwise along clasps. Given such a C-complex and a sequence $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_\mu)$ of ± 1, there are 2^μ generalized Seifert matrices A^ε that extend the usual Seifert matrix [9, 10, 17]. The associated *C-complex matrix* is the Λ-valued square matrix

$$H := \sum_{\varepsilon} \prod_{i=1}^{\mu} (1 - t_i^{\varepsilon_i}) A^\varepsilon,$$

where the sum is on all sequences $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_\mu)$ of ± 1. In [16, Theorem 1.1], together with Stefan Friedl and Enrico Toffoli, we showed that if $H_1(X_L; \Lambda_S)$ is Λ_S-torsion, then the Blanchfield pairing $\mathrm{Bl}(L)$ is isometric to the pairing

$$\Lambda_S^n / H^T \Lambda_S^n \times \Lambda_S^n / H^T \Lambda_S^n \longrightarrow Q/\Lambda_S \quad (a, b) \longmapsto -a^T H^{-1} \overline{b}, \tag{1.1}$$

where the size n C-complex matrix H for L was required to arise from a *totally connected* C-complex, *i.e.*, a C-complex F in which each F_i is connected and $F_i \cap F_j \neq \varnothing$ for all $i \neq j$. Note that (1.1) also shows that the Alexander module $H_1(X_L; \Lambda_S)$ admits a square presentation matrix. This fact was already known [10, Corollary 3.6], but as we mentioned above, it is false if we work over Λ [18, 23].

In general, the Blanchfield pairing is defined on the torsion submodule $TH_1(X_L; \Lambda_S)$ of $H_1(X_L; \Lambda_S)$. To the best of our knowledge, this Λ_S-module has no reason to admit a square presentation matrix, and thus a direct generalization of (1.1) seems out of reach. In order to circumvent this issue, we adapt the definition of the pairing described in (1.1) as follows. Let Δ denote the order of $\mathrm{Tor}_{\Lambda_S}(\Lambda_S^n / H^T \Lambda_S^n)$, the Λ_S-torsion submodule of $\Lambda_S^n / H^T \Lambda_S^n$. Note that for any class $[x]$ in $\mathrm{Tor}_{\Lambda_S}(\Lambda_S^n / H^T \Lambda_S^n)$, there exists an x_0 in Λ_S^n such that $\Delta x = H^T x_0$. As we will see in Proposition 4.2, the assignment $(v, w) \mapsto \frac{1}{\Delta^2} v_0^T H \overline{w_0}$ induces a well-defined pairing

$$\lambda_H\colon \mathrm{Tor}_{\Lambda_S}(\Lambda_S^n / H^T \Lambda_S^n) \times \mathrm{Tor}_{\Lambda_S}(\Lambda_S^n / H^T \Lambda_S^n) \to Q/\Lambda_S,$$

which recovers minus the pairing described in (1.1) when $\det(H) \neq 0$. Our main theorem reads as follows.

Theorem 1.1 *The Blanchfield pairing of a colored link L is isometric to the pairing $-\lambda_H$, where H is any C-complex matrix for L.*

Theorem 1.1 generalizes [16, Theorem 1.1] to links whose Alexander module $H_1(X_L;\Lambda_S)$ is not torsion and recovers it if $H_1(X_L;\Lambda_S)$ is torsion. Note also that [16, Theorem 1.1] required H to arise from a *totally connected* C-complex, whereas Theorem 1.1 removes this extraneous assumption. As we mentioned above, Theorem 1.1 also recovers the computation of $\mathrm{Bl}(L)$ when L is a boundary link (see Theorem 4.7). Note that to the best of our knowledge, Theorem 1.1 was not even known in the case of oriented links (*i.e.*, $\mu = 1$), and the result might be of independent interest.

While the Blanchfield pairing of a knot is known to be Hermitian and nonsingular, the corresponding statements for links require some more care. The Hermitian property of $\mathrm{Bl}(L)$ was sorted out by Powell [34], whereas Hillman [23] quotients $TH_1(X_L;\Lambda_S)$ by its *maximal pseudonull submodule* in order to turn $\mathrm{Bl}(L)$ into a non-degenerate pairing (see also [6, §2.5]). Even though we avoid discussing the non-degeneracy of the Blanchfield pairing, we observe that Theorem 1.1 provides a quick proof that $\mathrm{Bl}(L)$ is Hermitian: namely, using Δ_L^{tor} to denote the first non-vanishing Alexander polynomial of L over Λ_S, we obtain the following corollary.

Corollary 1.2 *The Blanchfield pairing of a link L is Hermitian and takes values in $\Delta_L^{\mathrm{tor}^{-1}}\Lambda_S/\Lambda_S$.*

Since the definition of the pairing λ_H is quite manageable, we also use Theorem 1.1 to obtain quick proofs regarding the behavior of $\mathrm{Bl}(L)$ under connected sums, disjoint unions, band claspings, mirror images, and orientation reversals (see Propositions 4.4, 4.5, and 4.6).

We conclude this introduction by remarking that Theorem 1.1 is not a trivial corollary of the work carried out in [16]. As we will see in Section 3, removing the torsion assumption on the Alexander module leads to several additional algebraic difficulties.

This paper is organized as follows. Section 2 briefly reviews twisted homology and the definition of the Blanchfield pairing. Section 3, which constitutes the core of this paper, deals with the proof of Theorem 1.1. Section 4 provides the applications of Theorem 1.1.

1.1 Notation and Conventions

We use $p \mapsto \overline{p}$ to denote the usual involution on $\mathbb{Q}(t_1,\dots,t_\mu)$ induced by $\overline{t_i} = t_i^{-1}$. Furthermore, given a subring R of $\mathbb{Q}(t_1,\dots,t_\mu)$ closed under the involution, and given an R-module M, we use $\overline{M}$ to denote the R-module that has the same underlying additive group as M, but for which the action by R on M is precomposed with the involution on R. Finally, given any ring R, we think of elements in R^n as column vectors.

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2 Preliminaries

This section is organized as follows. Section 2.1 briefly reviews the definition of twisted homology, while Section 2.2 gives a definition of the Blanchfield pairing. References include [20, Section 2] and [21, Section 2].

2.1 Twisted Homology

Let X be a CW complex, let $\varphi\colon \pi_1(X) \to \mathbb{Z}^\mu$ be an epimorphism, and let $p\colon \widetilde{X} \to X$ be the regular cover of X corresponding to the kernel of φ. Given a subspace $Y \subset X$, we will write $\widetilde{Y} := p^{-1}(Y)$, and view $C_*(\widetilde{X}, \widetilde{Y})$ as a chain-complex of free left modules over the ring $\Lambda = \mathbb{Z}[t_1^{\pm 1}, \dots, t_\mu^{\pm 1}]$. Given a commutative ring R and an (R, Λ)-bimodule M, consider the chain complexes

$$C_*(X, Y; M) := M \otimes_\Lambda C_*(\widetilde{X}, \widetilde{Y}),$$
$$C^*(X, Y; M) := \operatorname{Hom}_\Lambda(\overline{C_*(\widetilde{X}, \widetilde{Y})}, M)$$

of left R-modules and denote the corresponding homology R-modules by $H_*(X, Y; M)$ and $H^*(X, Y; M)$. Taking R to be Λ_S and M to be either Λ_S, Q or Q/Λ_S, we may send a cocycle f in $\operatorname{Hom}_\Lambda(\overline{C_*(\widetilde{X}, \widetilde{Y})}, M)$ to the Λ_S-linear map defined by $\sigma \otimes p \mapsto p \cdot \overline{f(\sigma)}$. This yields a well-defined isomorphism of left Λ_S-modules

$$H^i(X, Y; M) \longrightarrow H_i(\overline{\operatorname{Hom}_{\Lambda_S}(C_*(X, Y; \Lambda_S), M)}).$$

We also consider the evaluation homomorphism

$$H_i(\overline{\operatorname{Hom}_{\Lambda_S}(C_*(X, Y; \Lambda_S), M)}) \longrightarrow \overline{\operatorname{Hom}_{\Lambda_S}(H_i(C_*(X, Y; \Lambda_S)), M)}.$$

The composition of these two homomorphisms gives rise to the left Λ_S-linear map

$$\mathrm{ev}\colon H^i(X, Y; M) \longrightarrow \overline{\operatorname{Hom}_{\Lambda_S}(H_i(X, Y; \Lambda_S), M)}.$$

We will also use repeatedly that the short exact sequence $0 \to \Lambda_S \to Q \to Q/\Lambda_S \to 0$ of coefficients gives rise to the long exact sequence

(2.1)
$$\cdots \to H^k(X, Y; Q) \to H^k(X, Y; Q/\Lambda_S) \to H^{k+1}(X, Y; \Lambda_S) \to H^{k+1}(X, Y; Q) \to \cdots$$

in cohomology. The connecting homomorphism $H^k(X, Y; Q/\Lambda_S) \longrightarrow H^{k+1}(X, Y; \Lambda_S)$ is sometimes referred to as the *Bockstein homomorphism* and will be denoted by BS. Finally, if X is a compact connected oriented n-manifold, there are Poincaré duality isomorphisms $H_i(X, \partial X; M) \cong H^{n-i}(X; M)$ and $H_i(X; M) \cong H^{n-i}(X, \partial X; M)$.

2.2 The Blanchfield Pairing

Let $L = L_1 \cup \cdots \cup L_\mu$ be a colored link and denote its exterior by X_L. Identifying $\mathbb{Z}^\mu$ with the free abelian group on $t_1, \dots, t_\mu$, the epimorphism $\pi_1(X_L) \to \mathbb{Z}^\mu$ given by $\gamma \mapsto t_1^{\mathrm{lk}(\gamma, L_1)} \cdots t_\mu^{\mathrm{lk}(\gamma, L_\mu)}$ gives rise to the *Alexander module* $H_1(X_L; \Lambda_S)$ of L. Denote by Ω the composition

$$
\begin{aligned}
TH_1(X_L;\Lambda_S) &\xrightarrow{\text{(i)}} TH_1(X_L,\partial X_L;\Lambda_S)\\
&\xrightarrow{\text{(ii)}} \ker(H^2(X_L;\Lambda_S)\to H^2(X_L;Q))\\
&\xrightarrow{\text{(iii)}} \frac{H^1(X_L;Q/\Lambda_S)}{\ker(H^1(X_L;Q/\Lambda_S)\overset{\mathrm{BS}}{\to} H^2(X_L;\Lambda_S))}\\
&\xrightarrow{\text{(iv)}} \overline{\mathrm{Hom}_{\Lambda_S}(TH_1(X_L;\Lambda_S),Q/\Lambda_S)}
\end{aligned}
$$

of the four Λ_S-homomorphisms defined as follows. The inclusion induced map $H_1(X_L;\Lambda_S)\to H_1(X_L,\partial X_L;\Lambda_S)$ is an isomorphism [16, Lemma 2.2] and leads to (i). Since $H^2(X_L;Q)$ is a Q-vector space, torsion elements in $H^2(X_L;\Lambda_S)$ are mapped to zero in $H^2(X_L;Q)$, and thus Poincaré duality induces (ii). The long exact sequence displayed in (2.1) implies that the Bockstein homomorphism leads to the homomorphism labeled (iii). Indeed, by exactness $\ker(H^2(X_L;\Lambda_S)\to H^2(X_L;Q))$ is equal to $\mathrm{im}(\mathrm{BS})\cong\frac{H^1(X_L;Q/\Lambda_S)}{\ker(\mathrm{BS})}$. To deal with (iv), we must show that elements of $\ker(\mathrm{BS})$ evaluate to zero on elements of $TH_1(X_L;\Lambda_S)$. Since $\ker(\mathrm{BS})=\mathrm{im}(H^1(X_L;Q)\to H^1(X_L;Q/\Lambda_S))$, elements of $\ker(\mathrm{BS})$ are represented by cocycles which factor through Q-valued homomorphisms. Since Q is a field, these latter cocycles vanish on torsion elements, and thus so do the elements of $\ker(\mathrm{BS})$.

Definition 2.1 The *Blanchfield pairing* of a colored link L is the pairing

$$\mathrm{Bl}(L)\colon TH_1(X_L;\Lambda_S)\times TH_1(X_L;\Lambda_S)\longrightarrow Q/\Lambda_S$$

defined by $\mathrm{Bl}(L)(a,b)=\Omega(b)(a)$.

It follows from the definitions that the Blanchfield pairing is sesquilinear over Λ_S, in the sense that $\mathrm{Bl}(L)(pa,qb)=p\,\mathrm{Bl}(L)(a,b)\overline{q}$ for any a,b in $H_1(X_L;\Lambda_S)$ and any p,q in Λ_S.

3 Proof of Theorem 1.1

We start by fixing some notation. As we mentioned in the introduction, a C-complex for a μ-colored link L consists in a collection of Seifert surfaces $F_1,\dots,F_\mu$ for the sublinks $L_1,\dots,L_\mu$ that intersect only pairwise along clasps. Pushing a C-complex into the 4-ball D^4 leads to properly embedded surfaces that only intersect transversally in double points. Let W be the exterior of such a pushed-in C-complex in D^4, *i.e.*, W is the complement in D^4 of a tubular neighborhood of the pushed-in C-complex (see [16, §3] for details). We wish to study the cochain complexes of ∂W, W and $(W,\partial W)$ with coefficients in Λ_S, Q and Q/Λ_S. These nine cochain complexes fit in a commutative diagram whose columns and rows are exact.

Keeping this motivating example in mind, we make a short detour which will only involve homological algebra. More precisely, given a commutative ring R, we will consider the following commutative diagram of cochain complexes of R-modules whose

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columns and rows are assumed to be exact:

(3.1)
$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r, "h_B"] \arrow[d, "\nu_B"'] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & D \arrow[r, "h_D"] \arrow[d, "\nu_D"'] & E \arrow[r] \arrow[d, "\nu_E"'] & F \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H \arrow[r, "h_H"] \arrow[d] & J \arrow[r, "h_J"] \arrow[d] & K \arrow[r] \arrow[d] & 0 . \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

We will write $H^*(D) \to H^*(J)$ for the homomorphism induced by any composition of the cochain maps from D to J. Also $H^*(J) \to H^{*+1}(C)$ will denote the composition of the connecting map from $H^*(J)$ to $H^{*+1}(B)$ with the homomorphism induced by the cochain map from B to C. Alternatively, the latter map can also be described as the composition of the homomorphism induced by the cochain map from J to K with the connecting homomorphism $\delta_K^v : H^*(K) \to H^{*+1}(C)$. Furthermore, δ_K^h will denote the connecting homomorphism from $H^*(K)$ to $H^{*+1}(H)$. Note that these connecting maps are of degree +1 since we are working with *co*chain complexes. Finally, we will use the same notation for cochain maps as for the homomorphisms they induce on cohomology.

We now argue that there is a well-defined homomorphism from $\nu_D \ker(H^*(D) \to H^*(J))$ to $H^{*-1}(K)/\ker(\delta_K^h)$ that we will denote by $(\delta_K^h)^{-1}$. Indeed, if $[x]$ belongs to $\ker(H^*(D) \to H^*(J))$, the definition of the latter kernel implies that $(h_H \circ \nu_D)([x]) = 0$. Using the long exact sequence in cohomology induced by the bottom row of (3.1), there is a $[k]$ in $H^{*-1}(K)$ that satisfies $\delta_K^h([k]) = \nu_D([x])$. Define $(\delta_K^h)^{-1}(\nu_D([x]))$ as the class of $[k]$ in $H^{*-1}(K)/\ker(\delta_K^h)$. We now check that $(\delta_K^h)^{-1}$ is well defined. If $[k]$ and $[k']$ are elements of $H^{*-1}(K)$ satisfying $\delta_K^h([k]) = \nu_D([x]) = \delta_K^h([k'])$, then $[k] - [k']$ lies in $\ker(\delta_K^h)$. Consequently, the classes of $[k]$ and $[k']$ agree in the quotient $H^*(B)/\ker(\delta_K^h)$, as desired.

Similarly, we will argue that there is a well-defined homomorphism from $h_D \ker(H^*(D) \to H^*(J))$ to $\frac{H^*(B)}{\ker(\nu_B)}$, which we will denote by ν_B^{-1}. Indeed, if $[x]$ belongs to $\ker(H^*(D) \to H^*(J))$, the definition of the latter kernel implies that $(\nu_E \circ h_D)([x]) = 0$. Using the long exact sequence in cohomology induced by the middle column of (3.1), there is a $[b]$ in $H^*(B)$ that satisfies $\nu_B([b]) = h_D([x])$. Define $\nu_B^{-1}(h_D([x]))$ as the class of $[b]$ in $\frac{H^*(B)}{\ker(\nu_B)}$. We now check that ν_B^{-1} is well defined. If $[b]$ and $[b']$ are elements of $H^{*-1}(B)$ that satisfy $\nu_B([b]) = h_D([x]) = \nu_B([b'])$, then $[b] - [b']$ lies in $\ker(\nu_B)$. Consequently, the classes of $[b]$ and $[b']$ agree in the quotient $H^*(B)/\ker(\nu_B)$, as desired.

Finally, we claim that δ_K^v induces a well-defined map $\frac{H^{*-1}(K)}{\ker(\delta_K^h)} \to \frac{H^*(C)}{\operatorname{im}(H^{*-1}(J) \to H^*(C))}$. To see this, we must show that if $[k]$ lies in the kernel of δ_K^h, then $\delta_K^v([k])$ belongs to $\operatorname{im} := \operatorname{im}(H^{*-1}(J) \xrightarrow{h_J} H^{*-1}(K) \xrightarrow{\delta_K^v} H^*(C))$. By exactness of the bottom row of

(3.1), we have $\ker(\delta_K^h) = \operatorname{im}(h_J)$. Consequently $[k]$ lies in $\operatorname{im}(h_J)$ and thus $\delta_K^v([k])$ belongs to im, proving the claim.

We delay the proof of the following lemma to the appendix. Note that the statement of this lemma was inspired by [1, Lemma 4.4].

Lemma 3.1 *Given nine cochain complexes as in* (3.1)*, the diagram below anticommutes:*

$$\begin{tikzcd}
\ker(H^*(D) \to H^*(J)) \arrow[r, "\nu_D"] \arrow[d, "h_D"] & \nu_D \ker(H^*(D) \to H^*(J)) \arrow[d, "(\delta_K^h)^{-1}"] \\
h_D \ker(H^*(D) \to H^*(J)) \arrow[d, "\nu_B^{-1}"] & \dfrac{H^{*-1}(K)}{\ker(\delta_K^h)} \arrow[d, "\delta_K^v"] \\
\dfrac{H^*(B)}{\ker(\nu_B)} \arrow[r, "h_B"] & \dfrac{H^*(C)}{\operatorname{im}(H^{*-1}(J) \to H^*(C))}.
\end{tikzcd}$$

This concludes our algebraic detour, and we now return to topological matters: namely, to the nine cochain complexes that arose when we considered the exterior W of a pushed-in C-complex in the 4-ball.

Use $i^W_{\Lambda_S,Q}$ to denote the homomorphism from $H^2(W;\Lambda_S)$ to $H^2(W;Q)$ induced by the inclusion of Λ_S into Q. We also use $i^{W,\partial W}_{\Lambda_S}$ to denote the homomorphism from $H^2(W;\Lambda_S)$ to $H^2(\partial W;\Lambda_S)$. More generally, we will often implicitly follow this notational scheme, for instance $i^{(W,\partial W),W}_{Q/\Lambda_S}$ will denote the map from $H^2(W,\partial W;Q/\Lambda_S)$ to $H^2(W;Q/\Lambda_S)$.

Since BS plays the role of the boundary map δ_K^h in our algebraic detour, there is a well-defined map BS^{-1} from $i^{W,\partial W}_{\Lambda_S}\ker(H^2(W;\Lambda_S) \to H^2(\partial W;Q))$ to $\frac{H^1(\partial W;Q/\Lambda_S)}{\ker(H^1(\partial W;Q/\Lambda_S)\xrightarrow{\mathrm{BS}}H^2(\partial W;\Lambda_S))}$. Similarly, translating the role of ν_B into this setting, there is a well-defined map $(i_Q^{(W,\partial W),W})^{-1}$ from $i^W_{\Lambda_S,Q}\ker(H^2(W;\Lambda_S) \to H^2(\partial W;Q))$ to $\frac{H^2(W,\partial W;Q)}{\ker(H^2(W,\partial W;Q)\to H^2(W;Q))}$. Furthermore, we will denote by δ_{Q/Λ_S} the boundary map that arises in the long sequence of the pair $(W,\partial W)$ with Q/Λ_S coefficients.

Applying Lemma 3.1 to the cochain complexes of ∂W, W, and $(W,\partial W)$ with coefficients in Λ_S, Q, and Q/Λ_S immediately yields the following lemma.

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Lemma 3.2 *Let W be the exterior of a pushed-in C-complex in D^4. The following diagram anticommutes:*

$$\begin{tikzcd}
\ker(H^2(W;\Lambda_S) \to H^2(\partial W;Q)) \arrow[r, "i^{W,\partial W}_{\Lambda_S}"] \arrow[d, "i^W_{\Lambda_S,Q}"] & i^{W,\partial W}_{\Lambda_S} \ker(H^2(W;\Lambda_S) \to H^2(\partial W;Q)) \arrow[d, "BS^{-1}"] \\
i^W_{\Lambda_S,Q} \ker(H^2(W;\Lambda_S) \to H^2(\partial W;Q)) \arrow[d, "(i_Q^{(W,\partial W),W})^{-1}"] & \frac{H^1(\partial W;Q/\Lambda_S)}{\ker(H^1(\partial W;Q/\Lambda_S) \xrightarrow{BS} H^2(\partial W;\Lambda_S))} \arrow[d, "\delta_{Q/\Lambda_S}"] \\
\frac{H^2(W,\partial W;Q)}{\ker(H^2(W,\partial W;Q) \to H^2(W;Q))} \arrow[r, "i^{(W,\partial W)}_{Q,Q/\Lambda_S}"] & \frac{H^2(W,\partial W;Q/\Lambda_S)}{\operatorname{im}(H^1(\partial W;Q) \to H^2(W,\partial W;Q/\Lambda_S))}.
\end{tikzcd}$$

Recall from Section 2.1 that the Poincaré duality provides isomorphisms from $H_1(\partial W;\Lambda_S)$ to $H^2(\partial W;\Lambda_S)$ and from $H_2(W,\partial W;\Lambda_S)$ to $H^2(W;\Lambda_S)$. Both these maps will be denoted by PD. Furthermore, we use ∂ to denote the map from $H_2(W,\partial W;\Lambda_S)$ to $H_1(\partial W;\Lambda_S)$ that arises in the long exact sequence of the pair $(W,\partial W)$. We will abbreviate $TH_1(\partial W;\Lambda_S)$ by T. Finally, we recall that a C-complex $F = F_1 \cup \cdots \cup F_\mu$ is *totally connected* if each F_i is connected and $F_i \cap F_j \neq \varnothing$ for all $i \neq j$.

Lemma 3.3 *Let W be the exterior of a pushed-in C-complex in D^4.*

(i) *Poincaré duality restricts to a well-defined map $\partial^{-1}(T) \to \ker(H^2(W;\Lambda_S) \to H^2(\partial W;Q))$.*

(ii) *If the C-complex is totally connected, then Poincaré duality restricts to a well-defined map $T \to i^{W,\partial W}_{\Lambda_S} \ker(H^2(W;\Lambda_S) \to H^2(\partial W;Q))$.*

Proof In order to prove both statements, we will consider the following commutative diagram:

$$\begin{tikzcd}
H_2(W,\partial W;\Lambda_S) \arrow[r, "PD"] \arrow[d, "\partial"] & H^2(W;\Lambda_S) \arrow[r, "i^W_{\Lambda_S,Q}"] \arrow[d, "i^{W,\partial W}_{\Lambda_S}"] & H^2(W;Q) \arrow[d, "i^{W,\partial W}_Q"] \\
H_1(\partial W;\Lambda_S) \arrow[r, "PD"] & H^2(\partial W;\Lambda_S) \arrow[r, "i^{\partial W}_{\Lambda_S,Q}"] & H^2(\partial W;Q).
\end{tikzcd} \tag{3.2}$$

We start with the first assertion. Given x in $\partial^{-1}(T)$, the goal is to show that $PD(x)$ lies in $\ker(H^2(W;\Lambda_S) \to H^2(\partial W;Q))$ or in other words, we wish to show that $(i^{\partial W}_{\Lambda_S,Q} \circ i^{W,\partial W}_{\Lambda_S} \circ PD)(x)$ vanishes. Since $\partial(x)$ is a torsion element of $H_1(\partial W;\Lambda_S)$, there exists a non-zero λ in Λ_S for which $\lambda\partial(x) = 0$. The commutativity of (3.2) now implies that $\lambda(i^{\partial W}_{\Lambda_S,Q} \circ i^{W,\partial W}_{\Lambda_S} \circ PD)(x) = (i^{\partial W}_{\Lambda_S,Q} \circ PD)(\lambda\partial(x)) = 0$. Since $H^2(W;Q)$ is a vector space and λ is non-zero, the first claim is proved.

Next we deal with the second claim. Given a in T, we must find a d in $\ker(H^2(W;\Lambda_S) \to H^2(\partial W;Q))$ such that $i^{W,\partial W}_{\Lambda_S}(d) = PD(a)$. Since we now assume the C-complex to be totally connected, [16, Corollary 3.2] implies that $H_1(W;\Lambda_S) = 0$ and thus ∂ is surjective. Consequently, there exists an x in $H_2(W,\partial W;\Lambda_S)$ for which $\partial(x) = a$. Since a is torsion, x is actually in $\partial^{-1}(T)$ and so the first claim implies

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that $PD(x)$ lies in $\ker(H^2(W;\Lambda_S) \to H^2(\partial W;Q))$. Thus we set $d := PD(x)$ and observe that the commutativity of (3.2) implies $PD(a) = PD(\partial(x)) = i_{\Lambda_S}^{W,\partial W}(PD(x)) = i_{\Lambda_S}^{W,\partial W}(d)$, as desired. ∎

Next we deal with the evaluation maps which were described in Section 2.1, from $H^2(W,\partial W;Q)$ to $\overline{\operatorname{Hom}_{\Lambda_S}(H_2(W,\partial W;\Lambda_S),Q)}$ and from $H^2(W,\partial W;Q/\Lambda_S)$ to $\overline{\operatorname{Hom}_{\Lambda_S}(H_2(W,\partial W;\Lambda_S),Q/\Lambda_S)}$.

Lemma 3.4 *Let W be the exterior of a pushed-in C-complex in D^4.*

(i) *The evaluation map on $H^2(W,\partial W;Q)$ induces a well-defined map*

$$\mathrm{ev}\colon \frac{H^2(W,\partial W;Q)}{\ker(H^2(W,\partial W;Q)\to H^2(W;Q))} \longrightarrow \overline{\operatorname{Hom}_{\Lambda_S}(\partial^{-1}(T),Q)}.$$

(ii) *The evaluation map on $H^2(W,\partial W;Q/\Lambda_S)$ induces a well-defined map*

$$\mathrm{ev}\colon \frac{H^2(W,\partial W;Q/\Lambda_S)}{\operatorname{im}(H^1(\partial W;Q)\to H^2(W,\partial W;Q/\Lambda_S))} \longrightarrow \overline{\operatorname{Hom}_{\Lambda_S}(\partial^{-1}(T),Q/\Lambda_S)}.$$

Proof From now on, we will write $\langle\varphi,x\rangle$ instead of $(\mathrm{ev})(\varphi)(x)$. We start by proving the first assertion. By exactness we have

$$\ker(H^2(W,\partial W;Q)\longrightarrow H^2(W;Q)) = \operatorname{im}(H^1(\partial W;Q)\xrightarrow{\delta_Q} H^2(W,\partial W;Q)),$$

where δ_Q denotes the boundary map in the long exact sequence of the pair. Consequently, the goal is to show that for all φ in $H^1(\partial W;Q)$ and all x in $\partial^{-1}(T)$, one has $\langle\delta_Q\varphi,x\rangle = 0$. Consider the following commutative diagram:

$$\begin{tikzcd}
H^1(\partial W;Q) \arrow[r, "\mathrm{ev}"] \arrow[d, "\delta_Q"] & \overline{\operatorname{Hom}_{\Lambda_S}(H_1(\partial W;\Lambda_S),Q)} \arrow[d, "\partial^*"] \\
H^2(W,\partial W;Q) \arrow[r, "\mathrm{ev}"] & \overline{\operatorname{Hom}_{\Lambda_S}(H_2(W,\partial W;\Lambda_S),Q)}.
\end{tikzcd} \tag{3.3}$$

Since ∂x is torsion, there exists a non-zero λ in Λ_S for which $\lambda\partial(x)$ vanishes. The diagram in (3.3) now gives $\lambda\langle\delta_Q\varphi,x\rangle = \lambda\langle\varphi,\partial x\rangle = \langle\varphi,\lambda\partial(x)\rangle = 0$. Since this equation takes place in the field Q and λ is non-zero, we get $\langle\delta_Q\varphi,x\rangle = 0$, as desired.

To prove the second claim, start with φ in $H^1(\partial W;Q)$ and x in $\partial^{-1}(T)$. Consider the change of coefficient homomorphism $i^{\partial W}_{Q,Q/\Lambda_S}\colon H^1(\partial W;Q) \to H^1(\partial W;Q/\Lambda_S)$ and the connecting homomorphism $\delta_{Q/\Lambda_S}\colon H^1(\partial W;Q/\Lambda_S) \to H^2(W,\partial W;Q/\Lambda_S)$. In order to show that $\langle(\delta_{Q/\Lambda_S}\circ i^{\partial W}_{Q,Q/\Lambda_S})(\varphi),x\rangle = 0$, consider the same commutative diagram as displayed in (3.3) but with Q/Λ_S coefficients:

$$\begin{tikzcd}
H^1(\partial W;Q/\Lambda_S) \arrow[r, "\mathrm{ev}"] \arrow[d, "\delta_{Q/\Lambda_S}"] & \overline{\operatorname{Hom}_{\Lambda_S}(H_1(\partial W;\Lambda_S),Q/\Lambda_S)} \arrow[d, "\partial^*"] \\
H^2(W,\partial W;Q/\Lambda_S) \arrow[r, "\mathrm{ev}"] & \overline{\operatorname{Hom}_{\Lambda_S}(H_2(W,\partial W;\Lambda_S),Q/\Lambda_S)}.
\end{tikzcd} \tag{3.4}$$

Since φ is Q-valued and $\partial(x)$ is torsion, the result follows from the commutativity of (3.4). Indeed, $\langle(\delta_{Q/\Lambda_S} \circ i^{\partial W}_{Q,Q/\Lambda_S})(\varphi), x\rangle = \langle(i^{\partial W}_{Q,Q/\Lambda_S})(\varphi), \partial(x)\rangle$ and the latter term vanishes since cocycles which factor through Q vanish on torsion elements. ■

Recall that we use BS^{-1} to denote the map from $i^{W,\partial W}_{\Lambda_S} \ker(H^2(W;\Lambda_S) \to H^2(\partial W;Q))$ to $\frac{H^1(\partial W;Q/\Lambda_S)}{\ker(H^1(\partial W;Q/\Lambda_S)\to H^2(\partial W;\Lambda_S))}$ that appeared in Lemma 3.2. Combining the previous results, we obtain the following lemma.

Lemma 3.5 *Let L be a colored link and let W be the exterior of a pushed-in totally connected C-complex for L. The squares and triangle in the following diagram commute, while the top pentagon anticommutes. Furthermore, the map* $\Gamma := \mathrm{ev} \circ \mathrm{BS}^{-1} \circ PD$ *coincides with the adjoint of the Blanchfield pairing* $\mathrm{Bl}(L)$.

(3.5)

$$\begin{tikzcd}
\partial^{-1}(T) \arrow[r, "\partial"] \arrow[dd, "(i_Q^{(W,\partial W),W})^{-1}\circ i^W_{\Lambda_S,Q}\circ PD"] & T \arrow[d, "\mathrm{BS}^{-1}\circ PD"] \arrow[rdd, bend left, "\Gamma"] & \\
& \frac{H^1(\partial W;Q/\Lambda_S)}{\ker(\mathrm{BS})} \arrow[d, "\delta_{Q/\Lambda_S}"] \arrow[rd, "\mathrm{ev}"] & \\
\frac{H^2(W,\partial W;Q)}{\ker(H^2(W,\partial W;Q)\to H^2(W;Q))} \arrow[r, "i^{(W,\partial W)}_{Q,Q/\Lambda_S}"] \arrow[d, "\mathrm{ev}"] & \frac{H^2(W,\partial W;Q/\Lambda_S)}{\mathrm{im}(H^1(\partial W;Q)\to H^2(W,\partial W;Q/\Lambda_S))} \arrow[rd, "\mathrm{ev}"] & \overline{\mathrm{Hom}_{\Lambda_S}(T,Q/\Lambda_S)} \arrow[d, "\partial^*"] \\
\overline{\mathrm{Hom}_{\Lambda_S}(\partial^{-1}(T),Q)} \arrow[rr] & & \overline{\mathrm{Hom}_{\Lambda_S}(\partial^{-1}(T),Q/\Lambda_S)}.
\end{tikzcd}$$

Proof We start by arguing that the maps in (3.5) are well defined. For the upper right evaluation map, this follows from the same argument as the one that was used in Section 2.2, just before Definition 2.1. All the other maps are well defined thanks to Lemmas 3.2, 3.3, and 3.4. The top pentagon anticommutes thanks to Lemmas 3.3 and 3.2. The top triangle commutes by definition of Γ, the bottom square clearly commutes, while the commutativity of the rightmost square follows from (3.4). To prove the second assertion, we start by noting that [16, Lemma 5.2] implies that the inclusion induced map $H_1(X_L;\Lambda_S) \to H_1(\partial W;\Lambda_S)$ is an isomorphism. Using this fact, we observe that Γ is defined exactly as the adjoint Ω of the Blanchfield pairing was (see Section 2.2). ■

Looking at the leftmost column of (3.5), we wish to define a pairing on $\partial^{-1}(T)$. To do this, we start by considering the composition

$$\begin{aligned}
\Theta\colon \partial^{-1}(T) &\xrightarrow{PD} \ker(H^2(W;\Lambda_S) \to H^2(\partial W;Q)) \\
&\xrightarrow{i^W_{\Lambda_S,Q}} i^W_{\Lambda_S,Q} \ker(H^2(W;\Lambda_S) \to H^2(\partial W;Q)) \\
&\longrightarrow \frac{H^2(W,\partial W;Q)}{\ker(H^2(W,\partial W;Q) \to H^2(W;Q))} \\
&\xrightarrow{\mathrm{ev}} \overline{\mathrm{Hom}_{\Lambda_S}(\partial^{-1}(T),Q)}
\end{aligned}$$

of Λ_S-linear homomorphisms, where the third arrow denotes the homomorphism $(i_Q^{(W,\partial W),W})^{-1}$ which was described in the discussion leading up to Lemma 3.2. Note that the first map is well defined thanks to Lemma 3.3, the second map is obviously well defined, the discussion prior to Lemma 3.2 ensures that the third map is well defined, and the fourth map is well defined thanks to Lemma 3.4. We define the desired pairing on $\partial^{-1}(T)$ by

$$\theta(x,y) := \Theta(y)(x).$$

Recall from Lemma 3.5 and its proof that the pairing defined by Γ on $TH_1(\partial W;\Lambda_S)$ coincides with the Blanchfield pairing on $TH_1(X_L;\Lambda_S)$. Using these identifications, Lemma 3.5 implies the following proposition.

Proposition 3.6 *Let L be a colored link and let W be the exterior of a pushed-in totally connected C-complex for L. The following diagram commutes:*

$$\begin{tikzcd} \partial^{-1}(TH_1(\partial W;\Lambda_S)) \times \partial^{-1}(TH_1(\partial W;\Lambda_S)) \arrow[r, "-\theta"] \arrow[d, "\partial\times\partial"] & Q \arrow[d] \\ TH_1(\partial W;\Lambda_S) \times TH_1(\partial W;\Lambda_S) \arrow[r, "\mathrm{Bl}(L)"] & Q/\Lambda_S. \end{tikzcd} \tag{3.6}$$

As (3.6) suggests, the computation of the Blanchfield pairing now boils down to the computation of θ. The remainder of the proof is devoted to this task.

Henceforth, we will assume that W is the exterior of a pushed-in *totally connected* C-complex. The intersection form λ on W is defined as the adjoint of the composition (3.7)

$$\Phi\colon H_2(W;\Lambda_S) \xrightarrow{i} H_2(W,\partial W;\Lambda_S) \xrightarrow{PD} H^2(W;\Lambda_S) \xrightarrow{\text{ev}} \overline{\mathrm{Hom}_{\Lambda_S}(H_2(W;\Lambda_S),\Lambda_S)}.$$

In other words, $\lambda(x,y) := \Phi(y)(x)$ (see for instance [16, Section 2.3] for details). In particular, we notice that Φ vanishes on $\ker(i)$ and descends to a map on $H_2(W;\Lambda_S)/\ker(i)$ that we also denote by Φ.

Since we assumed that W is the exterior of a pushed-in totally connected C-complex, [16, Corollary 3.2] implies that $H_1(W;\Lambda_S)=0$. Thus, there is an exact sequence

$$H_2(W;\Lambda_S) \xrightarrow{i} H_2(W,\partial W;\Lambda_S) \xrightarrow{\partial} H_1(\partial W;\Lambda_S) \longrightarrow 0.$$

Consequently, we will henceforth identify $H_1(\partial W;\Lambda_S)$ with the cokernel of the map i. In particular, elements of $H_1(\partial W;\Lambda_S)$ will be denoted by $[x]$, where x lies in $H_2(W,\partial W;\Lambda_S)$. Furthermore, we will identify the boundary map ∂ with the quotient map of $H_2(W,\partial W;\Lambda_S)$ onto $\operatorname{coker}(i)$. In other words, we allow ourselves to write $\partial(x)$ and $[x]$ interchangeably.

Let Δ be the order of $TH_1(\partial W;\Lambda_S)$ and let x,y be in $\partial^{-1}(T)$. Since $[x]$ and $[y]$ are torsion, there exists x_0 and y_0 in $H_2(W;\Lambda_S)$ such that $\Delta x = i(x_0)$ and $\Delta y = i(y_0)$. Define a Q-valued pairing ψ on $\partial^{-1}(T)$ by setting

$$\psi(x,y) := \frac{1}{\Delta^2}\lambda(x_0,y_0).$$

Observe that ψ is well defined: if x_0 and x_0' both satisfy $i(x_0)=\Delta x = i(x_0')$, then $x_0 - x_0'$ lies in $\ker(i)$ and thus $\lambda(x_0-x_0',y)=0$, as we observed above. The same reasoning applies to the second variable. In particular, we could have very well taken x_0

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and y_0 in $H_2(W;\Lambda_S)/\ker(i)$. Summarizing, we have two Q-valued pairings defined on $\partial^{-1}(T)$ and we wish to show that they agree.

Proposition 3.7 *θ is equal to ψ.*

Before diving into the proof, let us set up some notation. First, we define a map $j\colon \partial^{-1}(T) \to \operatorname{im}(i)$ as follows. Given x in $\partial^{-1}(T)$, we set $j(x) := i(x_0)$, where x_0 is any element of $H_2(W;\Lambda_S)$ that satisfies $i(x_0) = \Delta x$. The map j is easily seen to be well defined. Next, we set

$$K := \ker(H^2(W;\Lambda_S) \xrightarrow{i^{W,\partial W}_{\Lambda_S}} H^2(\partial W;\Lambda_S) \xrightarrow{i^{\partial W}_{\Lambda_S,Q}} H^2(\partial W;Q)).$$

Note that K already appeared in Lemma 3.2, as well as in the definition of θ. The discussion leading up to Lemma 3.2 also provided a homomorphism $(i_Q^{(W,\partial W),W})^{-1}$ whose domain was $i^W_{\Lambda_S,Q}(K)$. For the moment however, we will rename it as

$$k^*\colon i^W_{\Lambda_S,Q}(K) \longrightarrow \frac{H^2(W,\partial W;Q)}{\ker(H^2(W,\partial W;Q) \to H^2(W;Q))}$$

and recall its definition. Given ϕ in K, the definition of K implies that $(i_Q^{\partial W,W} \circ i^W_{\Lambda_S,Q})(\phi)$ vanishes. Using the exactness of the long exact sequence of the pair $(W,\partial W)$ with Q coefficients, it follows that $i_Q^{(W,\partial W),W}(\xi) = i^W_{\Lambda_S,Q}(\phi)$ for some $\xi \in H^2(W,\partial W;Q)$. The map k^* is defined by $k^*(i^W_{\Lambda_S,Q}(\phi)) = [\xi]$.

Remark 3.8 Note that if $\phi = i_{\Lambda_S}^{(W,\partial W),W}(\varphi)$ for some φ in $H^2(W,\partial W;\Lambda_S)$, then the description of k^* becomes more concrete. The reason is that we can pick ξ to be $i^{(W,\partial W)}_{\Lambda_S,Q}(\varphi)$. Indeed, we have

$$i_Q^{(W,\partial W),W}(\xi) = (i_Q^{(W,\partial W)} \circ i^{(W,\partial W)}_{\Lambda_S,Q})(\varphi) = (i^W_{\Lambda_S,Q} \circ i_{\Lambda_S}^{(W,\partial W),W})(\varphi) = i^W_{\Lambda_S,Q}(\phi),$$

where the second equality follows from the diagram below:

$$\begin{tikzcd}
H^2(W,\partial W;\Lambda_S) \arrow[r, "i^{(W,\partial W)}_{\Lambda_S,Q}"] \arrow[d, "i^{(W,\partial W),W}_{\Lambda_S}"] & H^2(W,\partial W;Q) \arrow[d, "i^{(W,\partial W),W}_{Q}"] \\
H^2(W;\Lambda_S) \arrow[r, "i^W_{\Lambda_S,Q}"] & H^2(W;Q).
\end{tikzcd}$$

Summarizing, we have $(k^* \circ i^W_{\Lambda_S,Q} \circ i_{\Lambda_S}^{(W,\partial W),W})(\varphi) = i^{(W,\partial W)}_{\Lambda_S,Q}(\varphi)$.

Let us temporarily write V instead of $H_2(W;\Lambda_S)$. Proposition 3.7 will follow if we manage to show that all the maps in (3.8) are well defined and produce a commutative diagram. Indeed, in this diagram, there are several routes which lead from the upper right corner to the lower left corner. Taking the uppermost route produces the pairing

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ψ, while the lowermost route produces θ:

(3.8)

$$\begin{tikzcd}
\frac{V}{\ker(i)} \arrow[r, "i", "\cong"'] \arrow[d, "\Phi"] & \operatorname{im}(i) \arrow[d, "PD"] & \partial^{-1}(T) \arrow[l, "j"'] \arrow[d, "PD"] \\
\overline{\operatorname{Hom}_{\Lambda_S}(\frac{V}{\ker(i)}, \Lambda_S)} \arrow[d] & i_{\Lambda_S}^{(W,\partial W),W} \circ PD(V) \arrow[l, "\mathrm{ev}"'] \arrow[d, "i^W_{\Lambda_S,Q}"] & K \arrow[d, "i^W_{\Lambda_S,Q}"] \\
\overline{\operatorname{Hom}_{\Lambda_S}(\frac{V}{\ker(i)}, Q)} \arrow[d, "\frac{1}{\Delta^2} j^*(i^{-1})^*"] & i^W_{\Lambda_S,Q} \circ i_{\Lambda_S}^{(W,\partial W),W} \circ PD(V) \arrow[l, "\mathrm{ev}"'] \arrow[r, "\cdot\frac{1}{\Delta}"] \arrow[d, "k^*"] & i^W_{\Lambda_S,Q}(K) \arrow[d, "k^*"] \\
\overline{\operatorname{Hom}_{\Lambda_S}(\partial^{-1}(T), Q)} & k^* \circ i^W_{\Lambda_S,Q} \circ i_{\Lambda_S}^{(W,\partial W),W} \circ PD(V) \arrow[l, "\frac{1}{\Delta}\mathrm{ev}"'] \arrow[r, "\cdot\frac{1}{\Delta}"] & \frac{H^2(W,\partial W;Q)}{\ker(H^2(W,\partial W;Q) \to H^2(W;Q))} \arrow[ll, bend left, "\mathrm{ev}"]
\end{tikzcd}.$$

We begin by arguing that all the maps in (3.8) are well defined. We already checked that the rightmost vertical maps are well defined, see Lemma 3.2 and Lemma 3.3. The middle Poincaré duality map is well defined: this follows immediately from the equality $PD \circ i = i_{\Lambda_S}^{(\partial W,W),W} \circ PD$. Next, we deal with the two horizontal maps on the bottom right. First observe that $i_{\Lambda_S}^{(\partial W,W),W} \circ PD(V)$ is a subspace of K; indeed,

$$K = \ker(H^2(W;\Lambda_S) \xrightarrow{i_{\Lambda_S}^{W,\partial W}} H^2(\partial W;\Lambda_S) \xrightarrow{i_{\Lambda_S,Q}^{\partial W}} H^2(\partial W;Q)),$$

and $i_{\Lambda_S}^{W,\partial W} \circ i_{\Lambda_S}^{(W,\partial W),W} = 0$ by exactness. Consequently $i^W_{\Lambda_S,Q} \circ i_{\Lambda_S}^{(W,\partial W),W} \circ PD(V)$ is a subspace of $i^W_{\Lambda_S,Q}(K)$. It then follows that $k^* \circ i^W_{\Lambda_S,Q} \circ i_{\Lambda_S}^{(W,\partial W),W} \circ PD(V)$ is a subspace of the term in the lower right corner. Since these spaces are Q-vector spaces, multiplication by $\frac{1}{\Delta}$ makes sense. It also follows from these observations and Lemma 3.4 that the lower two evaluation maps in (3.8) are well defined. The upper two evaluation maps are well defined since induced maps commute with evaluations. The next lemma will conclude the proof of Proposition 3.7.

Lemma 3.9 *All the squares in* (3.8) *commute.*

Proof The upper left square commutes by definition of Φ, see (3.7). The middle left square, the bottom right square, and the bottom triangle all clearly commute. Let us now deal with the large rectangle on the upper right. Start with x in $\partial^{-1}(T)$. Using the definition of j, we have $j(x) = i(x_0)$, where x_0 lies in $H_2(W;\Lambda_S)$ and satisfies $i(x_0) = \Delta x$. The desired relation now follows readily:

$$\frac{1}{\Delta}(i^W_{\Lambda_S,Q} \circ PD \circ j)(x) = \frac{1}{\Delta}(i^W_{\Lambda_S,Q} \circ PD \circ i)(x_0) = (i^W_{\Lambda_S,Q} \circ PD)(x).$$

Finally, we deal with the lower left square. Let φ be in $H^2(W,\partial W;\Lambda_S)$ and let x be in $\partial^{-1}(T)$. Using once again the definition of j, we have $(i^{-1} \circ j)(x) = [x_0]$ where x_0

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lies in $H_2(W;\Lambda_S)$ and satisfies $i(x_0)=\Delta x$. Consequently, we get the relation

$$\begin{aligned}\frac{1}{\Delta^2}\langle(i^W_{\Lambda_S,Q}\circ i^{(W,\partial W),W}_{\Lambda_S})(\varphi),(i^{-1}\circ j)(x)\rangle &= \frac{1}{\Delta^2}\langle(i^W_{\Lambda_S,Q}\circ i^{(W,\partial W),W}_{\Lambda_S})(\varphi),[x_0]\rangle\\ &= \frac{1}{\Delta^2}\langle\varphi,i([x_0])\rangle=\frac{1}{\Delta}\langle\varphi,x\rangle,\end{aligned}$$

where in the second equality, we simultaneously used that induced maps commute with evaluations and the fact that $i^W_{\Lambda_S,Q}$ changes the coefficients without affecting the expression involved. On the other hand, recalling the conclusion of Remark 3.8, we can compute the other term:

$$\frac{1}{\Delta}\langle(k^*\circ i^W_{\Lambda_S,Q}\circ i^{(W,\partial W),W}_{\Lambda_S})(\varphi),x\rangle=\frac{1}{\Delta}\langle i^{(W,\partial W)}_{\Lambda_S,Q}(\varphi),x\rangle=\frac{1}{\Delta}\langle\varphi,x\rangle.$$

Combining these observations, the lower left square of (3.8) commutes. This concludes the proof the lemma and thus the proof of Proposition 3.7. ■

We are now in position to conclude the proof of Theorem 1.1.

***Proof of Theorem* 1.1** Let L be a colored link and let W be the exterior of a pushed-in totally connected C-complex for L. Recall that i denotes the inclusion induced map from $H_2(W;\Lambda_S)$ to $H_2(W,\partial W;\Lambda_S)$ and that given torsion elements $[x]$ and $[y]$ in $H_1(X_L;\Lambda_S)\cong H_1(\partial W;\Lambda_S)\cong\operatorname{coker}(i)$, there exists x_0 and y_0 in $H_2(W;\Lambda_S)$ such that $i(x_0)=\Delta x$ and $i(y_0)=\Delta y$. Using Proposition 3.6, we already know that $\mathrm{Bl}(L)([x],[y])=-\theta(x,y)$. Next, Proposition 3.7 implies that $\theta(x,y)=\psi(x,y)=\frac{1}{\Delta^2}\lambda(x_0,y_0)$. Summarizing, we have

$$\mathrm{Bl}(L)([x],[y])=-\theta(x,y)=-\psi(x,y)=-\frac{1}{\Delta^2}\lambda(x_0,y_0). \tag{3.9}$$

Note that any choice of x_0,y_0 will do since λ vanishes on $\ker(i)$; this was already noticed in the definition of ψ. Furthermore, note that (3.9) holds independently of the chosen representatives x and y for the classes $[x]$ and $[y]$. Indeed if x and x' represent $[x]$, we claim that $\psi(x,y)$ and $\psi(x',y)$ coincide in Q/Λ_S, *i.e.*, that $\psi(x-x',y)$ lies in Λ_S; the same proof will hold for the second variable. Since x and x' both represent $[x]$, there is a v in $H_2(W;\Lambda_S)$ for which $x-x'=i(v)$. Consequently $i(\Delta v)=\Delta i(v)$. Picking y_0 such that $i(y_0)=\Delta y$ and using the definition of λ, the following equalities prove our claim, since the rightmost term lies in Λ_S:

$$\psi(x-x',y)=\psi(i(v),y)=\frac{1}{\Delta^2}\lambda(\Delta v,y_0)=\frac{1}{\Delta}\langle(PD\circ i)(y_0),v\rangle=\langle PD(y),v\rangle.$$

Using [16, Theorem 1.3], we know that there are bases with respect to which the intersection pairing λ on $H_2(W;\Lambda_S)$ is represented by the C-complex matrix H described in the introduction. Furthermore, with respect to the same bases, it was observed in [16, §5.2] that the map i is represented by $\overline{H}=H^T$. Consequently, Equation (3.9) can be reformulated as follows. Let n denote the rank of the Λ_S-module $H_2(W;\Lambda_S)$. Given $[x],[y]\in TH_1(X_L;\Lambda_S)$, we have $\mathrm{Bl}(L)([x],[y])=-\frac{1}{\Delta^2}x_0^T H\overline{y_0}$ for any choice of $x_0,y_0\in\Lambda_S^n$ such that $\overline{H}x_0=\Delta x$ and $\overline{H}y_0=\Delta y$. Using the notations of the introduction, this can be written as

$$\mathrm{Bl}(L)([x],[y])=-\lambda_H([x],[y]).$$

Up to now, we always supposed that W arose by pushing in a totally connected C-complex. Thus, *a priori*, Theorem 1.1 only holds for C-complex matrices which arise from totally connected C-complexes. To conclude the proof of Theorem 1.1, it therefore only remains to check that the pairing λ_H is independent of the choice of a C-complex for L.

As explained in [10, p. 1230] (see also [9]), if F and F' are two C-complexes for isotopic links, then the corresponding C-complex matrices H and H' are related by a finite number of the following two moves:

$$H \longmapsto H \oplus (0) \quad \text{and} \quad H \longmapsto \begin{pmatrix} H & \xi & 0 \\ \xi^* & \lambda & \alpha \\ 0 & \overline{\alpha} & 0 \end{pmatrix},$$

with α a unit of Λ_S. In the first case, the Λ_S-module $\Lambda_S^n/\overline{H}\Lambda_S^n$ picks up a free rank one factor, so its torsion submodule is left unchanged. It can then be checked that λ_H and $\lambda_{H\oplus(0)}$ are canonically isometric. In the second case, since α is a unit in Λ_S, one can assume via the appropriate base change that H is transformed into $H \oplus \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. One can then once again check that the forms associated with these two Hermitian matrices are canonically isometric. ∎

The proof of Theorem 1.1 is now complete. However, we wish to emphasize an argument that we will use again later on.

Remark 3.10 It follows from [10, Corollary 3.6] (see also [16, Theorem 1.1]) that $\overline{H}$ presents the Λ_S-coefficient Alexander module $H_1(X_L; \Lambda_S)$ if H is a C-complex matrix which arises from a totally connected C-complex. However, as we saw in the proof of Theorem 1.1, $\operatorname{Tor}_{\Lambda_S}(\Lambda_S^n/\overline{H}\Lambda_S^n)$ is (possibly non-naturally) isomorphic to $TH_1(X_L; \Lambda_S)$ for *any* C-complex matrix H. Furthermore, the same argument shows that $\overline{H}$ actually presents $H_1(X_L; \Lambda_S)$ under the weaker hypothesis that H is a C-complex matrix which arises from a *connected* C-complex. Indeed, the transformation $H \mapsto H \oplus (0)$ only arises when one wishes to connect two disconnected components of a C-complex (see [10, page 1230]).

4 Applications

In this section, we provide several applications of Theorem 1.1. First, in Section 4.1 we give a new proof that the Blanchfield pairing is Hermitian. Then, in Section 4.2 we give quick proofs of some elementary properties of the Blanchfield pairing. Finally, in Section 4.3 we apply Theorem 1.1 to boundary links.

4.1 The Blanchfield Pairing Is Hermitian

In this subsection, we prove Corollary 1.2, which states that the Blanchfield pairing is Hermitian and takes value in $(\Delta_L^{\text{tor}})^{-1}\Lambda_S/\Lambda_S$, where Δ_L^{tor} denotes the first non-vanishing Alexander polynomial of L over Λ_S. Using Theorem 1.1, this reduces to showing the corresponding statement for λ_H, where H is any C-complex matrix for L. Since this is a purely algebraic statement, we will prove it in a somewhat greater generality.

Let R be an integral domain with involution and let $Q(R)$ be its field of fractions. Given an R-module V, a pairing $b: V \times V \to Q(R)/R$ is *sesquilinear* if it is linear in the first entry and antilinear in the second entry. A sesquilinear pairing b is *non-degenerate* (respectively *nonsingular*) if the adjoint map $V \to \overline{\mathrm{Hom}_R(V, Q(R)/R)}$, $p \mapsto (q \mapsto \lambda(p,q))$ is a monomorphism (respectively, an isomorphism) and *Hermitian* if $\overline{\lambda(w,v)} = \lambda(v,w)$ for any $v, w \in V$.

Henceforth, we make the additional assumption that R is Noetherian and factorial. Let H be a Hermitian $n \times n$-matrix over R, and let Δ denote the order of the R-module $\mathrm{Tor}_R(R^n/\overline{H}R^n)$. Given classes $[v]$ and $[w]$ in $\mathrm{Tor}_R(R^n/\overline{H}R^n)$, there exists v_0, w_0 in R^n such that $\Delta v = \overline{H}v_0$ and $\Delta w = \overline{H}w_0$. Proposition 4.2 will show that setting

$$\lambda_H([v],[w]) := \frac{1}{\Delta^2} v_0^T H \overline{w_0}$$

gives rise to a well-defined, Hermitian $\Delta^{-1}R/R$-valued pairing on $\mathrm{Tor}_R(R^n/\overline{H}R^n)$. Before proving this result, we explain its connection to the Blanchfield pairing.

Remark 4.1 Let M be an R-module. For $k \geq 0$, let $\Delta^{(k)}(M)$ denote the greatest common divisor of all $(m-k) \times (m-k)$ minors of an $m \times n$ presentation matrix of M. Using r to denote the rank of M, it is known that the order of $\mathrm{Tor}_R(M)$ is equal to $\Delta^{(r)}(M)$ (see [38, Lemma 4.9]). If M is presented by a Hermitian matrix H, the above discussion and the equality $\overline{H} = H^T$ guarantee that $\overline{\Delta} = \Delta$.

Taking R to be Λ_S and H to be a C-complex matrix for a link L, we now claim that Δ is equal to $\Delta_L^{\mathrm{tor}}(L)$, the first non-vanishing Alexander polynomial of L over Λ_S. First of all, note that while the Λ_S-module $\Lambda_S^n/\overline{H}\Lambda_S^n$ may not be equal to $H_1(X_L;\Lambda_S)$, their torsion parts agree (see Remark 3.10). The claim now follows from the fact that the order of $TH_1(X_L;\Lambda_S)$ is equal to the first non-vanishing Alexander polynomial of L, as mentioned above.

Combining Theorem 1.1 with Remark 4.1, the following proposition (which was suggested by David Cimasoni) will immediately imply Corollary 1.2.

Proposition 4.2 *The assignment* $(v,w) \mapsto \frac{1}{\Delta^2} v_0^T H \overline{w_0}$ *induces a well-defined pairing*

$$\lambda_H: \mathrm{Tor}_R(R^n/\overline{H}R^n) \times \mathrm{Tor}_R(R^n/\overline{H}R^n) \to \Delta^{-1}R/R$$

that is Hermitian. Furthermore, if $\det(H)$ *is non-zero, then this form is induced by the pairing* $(v,w) \mapsto v^T H^{-1} \overline{w}$.

Proof Let us first check that this definition is independent of the choice of v_0 in R^n such that $\Delta v = \overline{H}v_0$. Any other choice is of the form $v_0 + k$ with k in R^n such that $\overline{H}k = 0$. Since H is Hermitian, we have the equalities

$$\frac{1}{\Delta^2} k^T H \overline{w_0} = \frac{1}{\Delta^2} (\overline{H}k)^T \overline{w_0} = 0,$$

that give the result. A similar argument shows that the definition is independent of the choice of w_0 such that $\Delta w = \overline{H}w_0$. Next, let us check that it does not depend on the choice of v representing the class $[v]$. Any other choice is of the form $v + \overline{H}u$ where

u lies in R^n; since $\Delta(v + \overline{H}u) = \overline{H}(v_0 + \Delta u)$ and $\overline{\Delta} = \Delta$, the element

$$\frac{1}{\Delta^2}(\Delta u)^T H\overline{w_0} = \frac{1}{\Delta}u^T H\overline{w_0} = u^T\overline{w}$$

belongs to R, so the class in $Q(R)/R$ is indeed well defined. A similar argument shows that it does not depend on the choice of w representing the class $[w]$, thus concluding the proof that λ_H is well defined. The fact that λ_H is sesquilinear is clear, and it is Hermitian since H is and $\overline{\Delta} = \Delta$. To show the second claim, first note that if $\det(H)$ is non-zero, then H is invertible over $Q(R)$ so the equation $\Delta v = \overline{H}v_0$ is equivalent to $v_0 = \Delta\overline{H}^{-1}v$ (and similarly for w_0). Replacing v_0 and w_0 by these values and using the fact that H is Hermitian, we see that λ_H is indeed induced by $(v, w) \mapsto v^T H^{-1}\overline{w}$. This concludes the proof of the proposition. ∎

4.2 Some Properties of the Blanchfield Pairing

Let R be a Noetherian factorial integral domain with involution. Before dealing with the properties of the Blanchfield pairing, we start by investigating the behavior of λ_H under direct sums and multiplication by norms.

Lemma 4.3 *Let $H_1, \dots, H_\mu$ and H be Hermitian matrices and let u be a unit of R.*

(i) *Setting $B := H_1 \oplus \cdots \oplus H_\mu$, one has $\lambda_B = \bigoplus_{i=1}^{\mu} \lambda_{H_i}$.*
(ii) *The pairings $\lambda_{u\overline{u}H}$ and λ_H are isometric.*

Proof For (i), assume that each H_i is of size k_i, set $k := k_1 + \cdots + k_\mu$ and observe that $R^k/\overline{B}R^k$ is equal to $R^{k_1}/\overline{H_1}R^{k_1} \oplus R^{k_2}/\overline{H_2}R^{k_2} \oplus \cdots \oplus R^{k_\mu}/\overline{H_\mu}R^{k_\mu}$. Since the torsion of the latter direct sum is equal to the direct sum of the torsion of the $R^{k_i}/\overline{H_i}R^{k_i}$, it follows that the order of $\operatorname{Tor}_R(R^k/\overline{B}R^k)$ is equal to the product of the orders of the $\operatorname{Tor}_R(R^{k_i}/\overline{H_i}R^{k_i})$. We will write this as $\Delta = \Delta_1 \cdots \Delta_\mu$, where Δ_i denotes the order of $\operatorname{Tor}_R(R^{k_i}/\overline{H_i}R^{k_i})$.

Next, we compute the sum of the λ_{H_i}. Let $x = x^1 \oplus x^2 \oplus \cdots \oplus x^\mu$ and $y = y^1 \oplus y^2 \oplus \cdots \oplus y^\mu$ be torsion elements in $R^k/\overline{B}R^k$. Relying on the previous paragraph, the x^i and y^i are torsion in $R^{k_i}/\overline{H_i}R^{k_i}$, and so there exist x_0^i and y_0^i that satisfy $\overline{H_i}x_0^i = \Delta_i x^i$ and $\overline{H_i}y_0^i = \Delta_i y^i$. Thus, by definition we have

$$\bigoplus_{i=1}^{\mu} \lambda_{H_i}(x, y) = \sum_{i=1}^{\mu} \frac{1}{\Delta_i^2}(x_0^i)^T H_i \overline{y_0^i}. \tag{4.1}$$

In order to compute λ_B and conclude the proof we proceed as follows. We define an element x_0 in $R^k/\overline{B}R^k$ by requiring its i-th component to be equal to $\Delta\Delta_i^{-1}x_0^i$. This way, the i-th component of $\overline{B}x_0$ is $\overline{H_i}(\Delta\Delta_i^{-1}x_0^i) = \Delta\Delta_i^{-1}\overline{H_i}x_0^i = \Delta x^i$ and thus $\overline{B}x_0 = \Delta x$. We can therefore use x_0 and y_0 to compute $\lambda_B(x, y)$ and we get

$$\lambda_B(x, y) = \frac{1}{\Delta^2}x_0^T B\overline{y_0} = \frac{1}{\Delta^2}\sum_{i=1}^{n}(\Delta\Delta_i^{-1}x_0^i)^T H_i \overline{(\Delta\Delta_i^{-1}y_0^i)} = \sum_{i=1}^{n}\frac{1}{\Delta_i^2}(x_0^i)^T H_i \overline{y_0^i},$$

which agrees with (4.1). This concludes the proof of the first statement.

To deal with (ii), first observe that since u is a unit, so are $\overline{u}$ and $u\overline{u}$. Consequently $R^n/\overline{H}R^n$ is equal to $R^n/(u\overline{u}\overline{H})R^n$ and thus the corresponding torsion submodule

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supports both the pairings λ_H and $\lambda_{u\overline{u}H}$. To prove the assertion, we wish to show that the automorphism φ defined by sending x to $u^{-1}x$ provides the desired isometry. To see this, start with torsion elements x and y in the cokernel of H and let x_0, y_0 be such that $\overline{H}x_0 = \Delta x$ and $\overline{H}y_0 = \Delta y$. Since u is a unit, $(u\overline{u})^{-1}$ lies in R, and thus $(u\overline{u}\overline{H})((u\overline{u})^{-1}x_0) = \Delta x$ and similarly for y. It follows that

$$\begin{aligned}\lambda_{u\overline{u}H}(x,y) &= \frac{1}{\Delta^2}((u\overline{u})^{-1}x_0)^T(u\overline{u}H)\overline{((u\overline{u})^{-1}y_0)} = (u\overline{u})^{-1}\frac{1}{\Delta^2}x_0^T H\overline{y_0}\\ &= (u\overline{u})^{-1}\lambda_H(x,y).\end{aligned}$$

On the other hand, the sesquilinearity of λ_H immediately implies that

$$\lambda_H(\varphi(x),\varphi(y)) = \lambda_H(u^{-1}x, u^{-1}y) = (u\overline{u})^{-1}\lambda_H(x,y).$$

Consequently λ_H and $\lambda_{u\overline{u}H}$ are isometric, which concludes the proof. ∎

We can now apply Lemma 4.3 to obtain some results on the Blanchfield pairing. Before that however, given a C-complex F for a μ-colored link and a sequence $\varepsilon = (\varepsilon_1, \dots, \varepsilon_\mu)$ of ± 1, we briefly recall some terminology. Pushing curves off F_i in the ε_i-normal direction for $i = 1, \dots, \mu$ produces a map $i_\varepsilon\colon H_1(F) \to H_1(S^3 \smallsetminus F)$. The assignment $\alpha^\varepsilon(x,y) := \operatorname{lk}(i_\varepsilon(x), y)$ gives rise to a bilinear pairing on $H_1(F)$ and thus to a *generalized Seifert matrix* A^ε for L. We refer to [9, 10, 17] for details.

In the next two propositions, we will use $\mathrm{Bl}(L)(t_1, \dots, t_\mu)$ to denote the Blanchfield pairing of a μ-colored link and similarly for the C-complex matrices.

Proposition 4.4 *Let $L' = L_1 \cup \dots \cup L_{\nu-1} \cup L'_\nu$ and $L'' = L''_\nu \cup L_{\nu+1} \cup \dots \cup L_\mu$ be two colored links. Consider a colored link $L = L_1 \cup \dots \cup L_\mu$, where L_ν is a connected sum of L'_ν and L''_ν along any of their components. Then $\mathrm{Bl}(L)(t_1, \dots, t_\mu)$ is isometric to $\mathrm{Bl}(L')(t_1, \dots, t_\nu) \oplus \mathrm{Bl}(L'')(t_\nu, \dots, t_\mu)$.*

Proof Denote $\prod_{i>\nu}(1-t_i^{-1})(1-t_i)$ by u_1 and $\prod_{i<\nu}(1-t_i^{-1})(1-t_i)$ by u_2. Given a C-complex F' for L' and a C-complex F'' for L'', a C-complex for L is given by the band sum of F' and F'' along the corresponding components of F'_ν and F''_ν. Consequently, $A_F^\varepsilon = A_{F'}^{\varepsilon'} \oplus A_{F''}^{\varepsilon''}$, with $\varepsilon' = (\varepsilon_1, \dots, \varepsilon_\nu)$ and $\varepsilon'' = (\varepsilon_\nu, \dots, \varepsilon_\mu)$. It follows that a C-complex matrix H for L is given by $H = u_1 H'(t_1, \dots, t_\nu) \oplus u_2 H''(t_\nu, \dots, t_\mu)$. Denoting these matrices by H' and H'', Theorem 1.1 and Lemma 4.3 (i) imply that $\mathrm{Bl}(L)$ is isometric to $\lambda_{u_1 H'} \oplus \lambda_{u_2 H''}$. Since u_1 and u_2 are of the form $u\overline{u}$ with u a unit of Λ_S, the result follows from the second assertion of Lemma 4.3. ∎

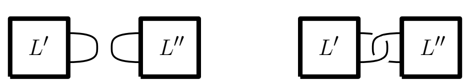

Figure 1: Performing a trivial band clasping of the links L' and L''.

A *trivial band clasping* of two links is the operation depicted in Figure 1. A proof similar to that of Proposition 4.4 yield the following result.

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Proposition 4.5 *Let $L' = L_1 \cup \cdots \cup L_\nu$ and $L'' = L_{\nu+1} \cup \cdots \cup L_\mu$ be colored links with disjoint sets of colors.*

(i) *Consider a colored link L obtained by trivially band clasping L_ν and $L_{\nu+1}$ along any of their components. Then $\mathrm{Bl}(L)(t)$ is isometric to $\mathrm{Bl}(L')(t') \oplus \mathrm{Bl}(L'')(t'')$.*

(ii) *Consider the colored link given by the disjoint sum of L' and L''. Then $\mathrm{Bl}(L)(t)$ is isometric to $\mathrm{Bl}(L')(t') \oplus \mathrm{Bl}(L'')(t'')$.*

We conclude this subsection by studying the effect of orientation reversal and taking the mirror image.

Proposition 4.6 *Let L be a colored link.*

(i) *If $\overline{L}$ denotes the mirror image of L, then $\mathrm{Bl}(\overline{L})$ is isometric to $-\mathrm{Bl}(L)$.*

(ii) *If $-L$ denotes L with the opposite orientation, then $\mathrm{Bl}(-L)$ is isometric to $\mathrm{Bl}(L)$.*

Proof If F is a C-complex for L, then the mirror image F' of F is a C-complex for $\overline{F}$. It follows that $H' = -H$. Since these two matrices present the same module, the corresponding torsion submodule supports both λ_H and λ_{-H}. We claim that the automorphism which sends x to $-x$ gives the required isometry. Indeed, if $\overline{H}x_0 = \Delta x$ and $\overline{H}y_0 = \Delta y$, then $(-\overline{H})x_0 = \Delta(-x)$ and $(-\overline{H})y_0 = \Delta(-y)$. Consequently, $\lambda_{-H}(-x,-y)$ and $-\lambda_H(x,y)$ are both equal to $-x_0^T H \overline{y_0}$. The result now follows from Theorem 1.1. The second assertion follows similarly by noting that a C-complex matrix for $-L$ is given by $\overline{H}$ and by using the fact that λ_H is Hermitian. ∎

4.3 Boundary Links

An n-component *boundary link* is a link $L = K_1 \cup \cdots \cup K_n$ whose n components bound n disjoint Seifert surfaces $F_1, \ldots, F_n$. Set $F = F_1 \sqcup \cdots \sqcup F_n$. Pushing curves off this *boundary Seifert surface* in the negative normal direction produces a homomorphism $i_-\colon H_1(F) \to H_1(S^3 \smallsetminus F)$. The assignment $\theta(x,y) := \mathrm{lk}(i_-(x), y)$ gives rise to a pairing on $H_1(F)$ and to a *boundary Seifert matrix for L* [29, p. 670]. Since $H_1(F)$ decomposes as the direct sum of the $H_1(F_i)$, the restriction of θ to $H_1(F_i) \times H_1(F_j)$ produces matrices A_{ij}. For $i \neq j$, these matrices satisfy $A_{ij} = A_{ji}^T$, while A_{ii} is nothing but a Seifert matrix for the knot K_i. Let g_i be the genus of F_i, let I_k be the $k \times k$ identity matrix, let τ be the block diagonal matrix whose diagonal blocks are $t_1 I_{2g_1}, t_2 I_{2g_2}, \ldots, t_n I_{2g_n}$, and set $g := g_1 + \cdots + g_n$. We use Theorem 1.1 in order to give a new proof of the following result, originally due to Hillman [24, pp. 122–123], see also [14, Theorem 4.2].

Theorem 4.7 *Let $L = K_1 \cup \cdots \cup K_n$ be a boundary link. Assume that A is a boundary Seifert matrix for L of size $2g$. The Blanchfield pairing of L is isometric to*

$$\begin{aligned}\Lambda_S^{2g}/(A\tau - A^T)\Lambda_S^{2g} \times \Lambda_S^{2g}/(A\tau - A^T)\Lambda_S^{2g} &\longrightarrow Q/\Lambda_S \\ (a,b) &\longmapsto a^T (A - \tau A^T)^{-1}(\tau - I_{2g})\overline{b}.\end{aligned}$$

Proof Let F be a boundary Seifert surface which gives rise to A. View F as a C-complex for L, and use A_{ij}^ε to denote the restriction of the generalized Seifert matrix A^ε to $H_1(F_i) \times H_1(F_j)$. If $i \neq j$, since L is a boundary link, A_{ij}^ε is independent of ε

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and is equal to the block A_{ij} of the boundary Seifert matrix A. Similarly, for each ε with $\varepsilon_i = -1$, the restriction of A^ε to $H_1(F_i) \times H_1(F_i)$ is equal to the block A_{ii}. Let $H_i = (1-t_i)A_{ii}^T + (1-t_i^{-1})A_{ii}$ denote the corresponding C-complex matrix for the knot K_i and let u denote $\prod_{j=1}^n (1-t_j)$. The previous considerations show that a C-complex matrix H for L is given by

$$H = \begin{bmatrix} u\overline{u}(1-t_1)^{-1}(1-t_1^{-1})^{-1}H_1 & u\overline{u}A_{12} & \cdots & u\overline{u}A_{1n} \\ u\overline{u}A_{21} & u\overline{u}(1-t_2)^{-1}(1-t_2^{-1})^{-1}H_2 & \cdots & u\overline{u}A_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ u\overline{u}A_{n1} & u\overline{u}A_{n2} & \cdots & u\overline{u}(1-t_n)^{-1}(1-t_n^{-1})^{-1}H_n \end{bmatrix}.$$

Since $H_i = (1-t_i^{-1})(A_{ii} - t_i A_{ii}^T)$, the diagonal blocks of H can be rewritten as $u\overline{u}(1-t_i)^{-1}(A_{ii} - t_i A_{ii}^T)$. Using the equation $A_{ij} = A_{ji}^T$, we see that a C-complex matrix for L is given by

$$H = u\overline{u}(I_{2g} - \tau)^{-1}(A - \tau A^T). \tag{4.2}$$

It follows that $H^T = u\overline{u}(A^T - A\tau)(I_{2g} - \tau)^{-1}$. Since u is a unit of Λ_S and $(I_{2g} - \tau)^{-1}$ is an automorphism of Λ_S^{2g}, the module presented by $\overline{H} = H^T$ is canonically isomorphic to the module presented by $A\tau - A^T$. As the isomorphism is induced by the identity of Λ_S^{2g}, we will slightly abuse notations and consider these modules as "equal" (see the second left vertical arrow in (4.3)).

Next, we claim that $\Lambda_S^{2g}/\overline{H}\Lambda_S^{2g}$ is Λ_S-torsion. Trivially band clasp F_1 with F_2, F_2 with F_3, F_i with F_{i+1}, and finally F_{n-1} with F_n. The result is a link L' that bounds a connected C-complex F' for which the associated C-complex matrix is also H. Since L has pairwise vanishing linking numbers, L' does not. Consequently, using the Torres formula, the Alexander polynomial of L' is non-zero and thus its Alexander module is torsion. As we saw in Remark 3.10, if a C-complex matrix H arises from a *connected* C-complex, $\overline{H}$ presents the Λ_S-localized Alexander module. Thus $\overline{H}$ presents the torsion module $H_1(X_{L'}; \Lambda_S)$ and the claim follows.

Now consider the following diagram:

$$\begin{tikzcd} TH_1(X_L;\Lambda_S) \times TH_1(X_L;\Lambda_S) \arrow[r, "\mathrm{Bl}(L)"] \arrow[d, "\cong"] & Q/\Lambda_S \arrow[d, "="] \\ \frac{\Lambda_S^{2g}}{\overline{H}\Lambda_S^{2g}} \times \frac{\Lambda_S^{2g}}{\overline{H}\Lambda_S^{2g}} \arrow[r, "(a,b)\mapsto -a^T H^{-1}\overline{b}"] \arrow[d, "="] & Q/\Lambda_S \arrow[d, "="] \\ \frac{\Lambda_S^{2g}}{(A\tau - A^T)\Lambda_S^{2g}} \times \frac{\Lambda_S^{2g}}{(A\tau - A^T)\Lambda_S^{2g}} \arrow[r, "(a,b)\mapsto a^T(u\overline{u})^{-1}(A-\tau A^T)^{-1}(\tau - I_{2g})\overline{b}"] \arrow[d, "(a,b)\mapsto(u^{-1}a,u^{-1}b)"] & Q/\Lambda_S \arrow[d, "="] \\ \frac{\Lambda_S^{2g}}{(A\tau - A^T)\Lambda_S^{2g}} \times \frac{\Lambda_S^{2g}}{(A\tau - A^T)\Lambda_S^{2g}} \arrow[r, "(a,b)\mapsto a^T(A-\tau A^T)^{-1}(\tau - I_{2g})\overline{b}"] & Q/\Lambda_S. \end{tikzcd} \tag{4.3}$$

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The top square commutes by Theorem 1.1. Note that Corollary 1.2 ensures that $\lambda_H(a,b) = -a^T H^{-1}\overline{b}$. Indeed, we argued above that $\Lambda_S^{2g}/\overline{H}\Lambda_S^{2g}$ is torsion. The middle rectangle commutes thanks to (4.2). Finally, the commutativity of the bottom square follows from a direct computation. ∎

A Proof of Lemma 3.2

For the reader's convenience, we recall both the set-up and the statement of Lemma 3.2. Given a commutative ring R, consider the following commutative diagram of cochain complexes of R-modules whose columns and rows are assumed to be exact:

(A.1)
$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r, "h_B"] \arrow[d, "\nu_B"'] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & D \arrow[r, "h_D"] \arrow[d, "\nu_D"'] & E \arrow[r] \arrow[d, "\nu_E"'] & F \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H \arrow[r, "h_H"] \arrow[d] & J \arrow[r, "h_J"] \arrow[d] & K \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0. &
\end{tikzcd}$$

If $0 \to S \to T \to U \to 0$ is one of the short exact sequences of cochain complexes in (A.1), we will use δ_U^ν, respectively δ_U^h, to denote the connecting homomorphism $H^*(U) \to H^{*+1}(S)$ if the sequence is depicted vertically, respectively, horizontally. For instance, there are connecting homomorphisms $\delta_K^\nu\colon H^*(K) \to H^{*+1}(C)$ and $\delta_K^h\colon H^*(K) \to H^{*+1}(H)$.

Just as in Section 3, we use the same notation for cochain maps as for the homomorphisms they induce on cohomology. Furthermore, we will write $H^*(D) \to H^*(J)$ for the homomorphism induced by any composition of the cochain maps from D to J. Also, $H^*(J) \to H^{*+1}(C)$ will denote the composition of the connecting homomorphism $\delta_J^h\colon H^*(J) \to H^{*+1}(B)$ with the homomorphism $h_B\colon H^*(B) \to H^*(C)$. Alternatively, the latter map can also be described as the composition of the homomorphism induced by the cochain map $h_J\colon H^*(J) \to H^*(K)$ with the connecting homomorphism $\delta_K^\nu\colon H^*(K) \to H^{*+1}(C)$.

Furthermore, as we argued in the discussion leading to the statement of Lemma 3.2, the connecting homomorphism δ_K^ν induces a well-defined map $\frac{H^{*-1}(K)}{\ker(\delta_K^h)} \to \frac{H^*(C)}{\mathrm{im}(H^{*-1}(J)\to H^*(C))}$, which we also denote by δ_K^ν. Additionally, there are well-defined homomorphisms $(\delta_K^h)^{-1}$ and ν_B^{-1}, whose definitions we now recall, referring to Section 3 for details.

1. There is a homomorphism $(\delta_K^h)^{-1}$ from $\nu_D \ker(H^*(D) \to H^*(J))$ to $H^{*-1}(K)/\ker(\delta_K^h)$. More precisely, $(\delta_K^h)^{-1}(\nu_D([x]))$ is defined as the class of $[k]$ in $H^{*-1}(K)/\ker(\delta_K^h)$ for any $[k]$ in $H^{*-1}(K)$ such that $\delta_K^h([k]) = \nu_D([x])$.

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2. There is a homomorphism v_B^{-1} from $h_D \ker(H^*(D) \to H^*(J))$ to $\frac{H^*(B)}{\ker(v_B)}$. More precisely, $v_B^{-1}(h_D([x]))$ is defined as the class of $[b]$ in $\frac{H^*(B)}{\ker(v_B)}$ for any $[b]$ in $H^*(B)$ such that $v_B([b]) = h_D([x])$.

The aim of this appendix is to prove Lemma 3.2, which states that the following diagram anticommutes:

$$\begin{tikzcd}
\ker(H^m(D) \to H^m(J)) \arrow[r, "v_D"] \arrow[d, "h_D"] & v_D \ker(H^m(D) \to H^m(J)) \arrow[d, "(\delta_K^h)^{-1}"] \\
h_D \ker(H^m(D) \to H^m(J)) \arrow[d, "v_B^{-1}"] & \frac{H^{m-1}(K)}{\ker(\delta_K^h)} \arrow[d, "\delta_K^v"] \\
\frac{H^m(B)}{\ker(v_B)} \arrow[r, "h_B"] & \frac{H^m(C)}{\operatorname{im}(H^{m-1}(J) \to H^m(C))}.
\end{tikzcd}$$

Proof of Lemma 3.2 The proof is structured as follows: first, we set up some notation; then we compute $h_B \circ v_B^{-1} \circ h_D$; and, finally, we show that $\delta_K^v \circ (\delta_K^h)^{-1} \circ v_D$ yields the same result.

Our first task is to write out explicitly the short exact sequences of cochain complexes displayed in (A.1). We restrict our attention to the degrees of interest (namely m and $m-1$) and omit the trivial modules that ought to appear on the extremities of the exact rows and columns. The result is the following commuting cube of R-modules in which the rows and columns are exact:

(A.2)
$$\begin{tikzcd}[row sep=small, column sep=small]
A^{m-1} \arrow[rr] \arrow[dr] \arrow[dd] & & B^{m-1} \arrow[rr] \arrow[dr] \arrow[dd] & & C^{m-1} \arrow[dr] \arrow[dd] & \\
& A^m \arrow[rr, crossing over] & & B^m \arrow[rr, crossing over] & & C^m \arrow[dd] \\
D^{m-1} \arrow[rr] \arrow[dr] \arrow[dd] & & E^{m-1} \arrow[rr] \arrow[dr] \arrow[dd] & & F^{m-1} \arrow[dr] \arrow[dd] & \\
& D^m \arrow[rr] \arrow[from=uu, crossing over] & & E^m \arrow[rr] \arrow[from=uu, crossing over] & & F^m \arrow[dd] \\
H^{m-1} \arrow[rr] \arrow[dr] & & J^{m-1} \arrow[rr] \arrow[dr] & & K^{m-1} \arrow[dr] & \\
& H^m \arrow[rr] \arrow[from=uu, crossing over] & & J^m \arrow[rr] \arrow[from=uu, crossing over] & & K^m.
\end{tikzcd}$$

Although the maps of this cube are not labeled, we systematically use the following conventions. First, the codifferential of a cochain complex T will be denoted by δ_T. Secondly, cochain maps are indexed by their domain and are named h, respectively, v, if they are horizontal, respectively, vertical. For instance, in the lower right corner of (A.2), the horizontal map is denoted by h_J, the vertical map is denoted by v_F and the diagonal map is denoted by δ_K. Additionally, recall that we use the same notation for cochain maps as for the homomorphisms they induce on cohomology.

Next, we move on to the second step of the proof: since our goal is to show that the equality

$$h_B \circ v_B^{-1} \circ h_D([x]) = -\delta_K^v \circ (\delta_K^h)^{-1} \circ v_D([x])$$

holds for all $[x] \in \ker(H^m(D) \to H^m(J))$, we now describe the map $h_B \circ \nu_B^{-1} \circ h_D$. Let $x \in D^m$ be a cocycle representing a class $[x] \in \ker(H^m(D) \to H^m(J))$. As we saw in Section 3, there exists $[b]$ in $H^m(B)$ such that $\nu_B([b]) = h_D([x])$. Fixing once and for all such a $[b]$, the definition of ν_B^{-1} implies that $\nu_B^{-1} \circ h_D([x])$ is equal to the class of $[b]$ in $\frac{H^m(B)}{\ker(\nu_B)}$. We deduce that $h_B \circ \nu_B^{-1} \circ h_D([x]) = h_B([b])$.

To carry out the third step of the proof, we must compute $\delta_K^\nu \circ (\delta_K^h)^{-1} \circ \nu_D([x])$. Consequently, we briefly recall the definition of connecting homomorphisms.

Remark A.1 Given a short exact sequence $0 \to S \xrightarrow{j} T \xrightarrow{\pi} U \to 0$ of cochain complexes, the connecting homomorphisms $\delta_{\mathrm{conn}}: H^m(U) \to H^{m+1}(S)$ are defined as follows. Since π is surjective, pick any $t \in T^m$ such that $\pi(t) = u$ is a cocycle representing a cohomology class $[u]$ in $H^m(U)$, and set $\delta_{\mathrm{conn}}([u]) := [s]$, where $s \in S^{m+1}$ is the (unique) cocycle satisfying $j(s) = \delta_T(t)$. It is well known that δ_{conn} is well defined.

In order to compute $\delta_K^\nu \circ (\delta_K^h)^{-1} \circ \nu_D([x])$, we first compute $(\delta_K^h)^{-1} \circ \nu_D([x])$. Since $h_D([x]) - \nu_B([b])$ vanishes in cohomology (by definition of $[b]$), there is a cochain $e \in E^{m-1}$ such that $\delta_E(e) = h_D(x) - \nu_B(b)$.

Claim *The class of $h_J(\nu_E([e]))$ in $\frac{H^{m-1}(K)}{\ker(\delta_K^h)}$ is equal to $(\delta_K^h)^{-1} \circ \nu_D([x])$.*

Proof By definition of $(\delta_K^h)^{-1}$, it is enough to verify that $\delta_K^h(h_J(\nu_E([e]))) = \nu_D([x])$. To check this, recall from Remark A.1 that we must show that $h_J(\nu_E(e))$ is a cocyle and that $\delta_J(\nu_E(e)) = h_H(\nu_D(x))$. To check that $h_J(\nu_E(e))$ is indeed a cocycle, we use successively the commutativity of (A.1), the definition of e, and the exactness of the lines in (A.1) to get

$$\delta_K(h_J(\nu_E(e))) = \nu_F(h_E(\delta_E(e))) = \nu_F(h_E(h_D(x) - \nu_B(b))) = -\nu_F(h_E(\nu_B(b))).$$

Using once again the commutativity of (A.1) and the exactness of its lines, we deduce the desired result, namely that

$$\delta_K(h_J(\nu_E(e))) = -\nu_F(h_E(\nu_B(b))) = -h_J(\nu_E(\nu_B(b))) = 0.$$

Next, we check the equality $\delta_J(\nu_E(e)) = h_H(\nu_D(x))$. This verification is carried out by using successively the commutativity of (A.2), the definition of e, the exactness of the columns in (A.2), and the commutativity of (A.2):

$$\delta_J(\nu_E(e)) = \nu_E(\delta_E(e)) = \nu_E(h_D(x)) - \nu_E(\nu_B(b)) = \nu_E(h_D(x)) = h_H(\nu_D(x)).$$

This concludes the proof of the claim. ∎

The conclusion of the lemma will promptly follow from the next claim.

Claim *The class of $-h_B([b])$ in $\frac{H^m(C)}{\mathrm{im}(H^{m-1}(J) \to H^m(C))}$ is equal to $\delta_K^\nu \circ (\delta_K^h)^{-1} \circ \nu_D([x])$.*

Proof We will show that the announced equality holds without having to pass to the quotient. To check this assertion, recall from Remark A.1 that we must find a cochain $f \in F^{m-1}$ such that $\nu_F(f)$ is a cocycle representing $(\delta_K^h)^{-1} \circ \nu_D([x])$. Furthermore f

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must also satisfy $\delta_F(f) = -\nu_C(h_B(b))$. We claim that $h_E(e)$ can be taken to play the role of f. We first check that $h_E(e)$ is such that $\nu_F(h_E(e))$ is a cocycle representing $(\delta_K^h)^{-1} \circ \nu_D([x])$. Since we proved in the previous claim that $(\delta_K^h)^{-1} \circ \nu_D([x])$ is (the class of) the cohomology class of $h_J(\nu_E(e))$, it is actually enough to show that $\nu_F(h_E(e)) = h_J(\nu_E(e))$. This is immediate from the commutativity of (A.2). Finally, we show that $\delta_F(h_E(e)) = -\nu_C(h_B(b))$. This follows from the commutativity of (A.2), the definition of e, the exactness of the rows in (A.2), and the commutativity of (A.2):

$$\delta_F(h_E(e)) = h_E(\delta_E(e)) = h_E(h_D(x)) - h_E(\nu_B(b)) = -h_E(\nu_B(b)) = -\nu_C(h_B(b)).$$

This concludes the proof of the claim. ∎

Summarizing, we have just shown that $-[h_B(b)]$ represents $\delta_K^\nu \circ (\delta_K^h)^{-1} \circ \nu_D([x])$. Since the second step of the proof consisted in showing that $h_B([b])$ represents $h_B \circ \nu_B^{-1} \circ h_D([x])$, the proof of the lemma is concluded. ∎

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