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CLOSED DERIVATIONS ON THE UNIT SQUARE

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In this paper we extend Kurose's structure theorem to characterize a closed derivation in the algebra of continuous functions on the unit square, under the conditions that the range is the whole algebra and the kernel is the set of all functions depend only on the second variable, as a partial derivative with respect to signed measures on the unit square.

1. Introduction.

The theory of closed derivations on the unit interval has been studied recently by many authors [1][3][4]. Kurose [4] characterizes a closed derivation in $C(I)$ as a standard abstraction of the usual differentiation. Namely, a closed derivation δ on $C(I)$ satisfying the conditions that the range of δ is $C(I)$, and the kernel of δ is the set of constant functions, can be identified with the different-iation with respect to a non-atomic signed measure on I. In this paper we will discuss the structure of closed derivations in $C(I\times I)$ and extend Kurose's result to the two dimensional case.

The algebra of all continuous real-valued functions on a compact

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space Ω is denoted by $C(\Omega)$. Any derivation on $C(\Omega)$ will be assumed to have a dense subalgebra $\mathcal{D}(\delta)$ of $C(\Omega)$ as domain, and to have range in $C(\Omega)$, and we may always assume that all constant functions belong to $\mathcal{D}(\delta)$.

2. Derivations on the Unit Square.

The following lemmas will be used on several occasions.

LEMMA 2.1. [6, Theorem 3.8.]. *Let* δ *be a closed derivation in* $C(\Omega)$ *and let* f *be a continuously differentiable real function on the real line. Then for* $a \in \mathcal{D}(\delta)$, $f(a) \in \mathcal{D}(\delta)$ *and* $\delta(f(a)) = f'(a)\delta(a)$.

LEMMA 2.2. *Let* δ *be a closed derivation in* $C(\Omega)$. *If* $f \in \mathcal{D}(\delta)$ *is constant on an open subset* U *of* Ω , *then* $\delta(f) = 0$ *on* U.

For the proof of Lemma 2.2, the reader is referred to [3, Lemma 1.1.5]; this is based on the fact that $\mathcal{D}(\delta)$ is a Silov subalgebra of $C(\Omega)$ [6, Proposition 1.4].

If K is a closed subset of Ω , the map $\delta_K(f|K) = \delta(f)|K$ for $f \in \mathcal{D}(\delta)$ need not be well-defined. For suitable K , there is an induced derivation δ_K in $C(K)$

PROPOSITION 2.3. *Let* δ *be a closed derivation in* $C(I\times I)$, *whose kernel is the set of all functions which depend only on the second variable. Let* K *be a closed subset of* $I \times I$ *of the form* $I \times [r,s]$, *where* $0 \le r < s \le 1$. *Then the map* δ_K *for* $f \in \mathcal{D}(\delta)$ *is well-defined and is a closed derivation in* $C(K)$ *with domain* $\mathcal{D}(\delta_K) = \{f|K : f \in \mathcal{D}(\delta)\}$.

Proof. Suppose f and g belong to $\mathcal{D}(\delta)$ and $f = g$ on K , then $f - g = 0$ on the open set $I \times (r,s)$. By Lemma 2.2. $\delta(f - g) = 0$ on K and hence δ_K is well-defined.

Let $\{f_n\}$ be a sequence in $\mathcal{D}(\delta)$ such that f_n converges to f and $\delta(f_n)$ converges to F in $C(K)$. Choose a function $g \in C(I)$ with support$(g) = I - (r,s)$. Then the function h defined by $h(x,y) = g(y)$ for $(x,y) \in I \times I$ belongs to ker (δ). For each n , by [1, Lemma 4.3] there exists a function $h_n \in C(R)$ with $0 \le h_n \le 1$ and $h_n(0) = 1$ such that

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$$||(h_n(h))(|f_n| + |\delta(f_n)|)|| = ||(|f_n| + |\delta(f_n)|)||_K$$
$$\leq ||f_n||_K + ||\delta(f_n)||_K \; .$$

The sequences of functions $\{(h_n(h))f_n\}$ and $\{(h_n(h))\delta(f_n)\}$ are then Cauchy sequences in $C(I \times I)$, and hence they converge to some H and G in $C(I \times I)$ respectively. Let $\{p_{nm}\}_{m=1}^{\infty}$ be a sequence of polynomials such that p_{nm} converges to h_n uniformly on the compact set $h(K)$. Then $p_{nm}(h)$ converges to $h_n(h)$ in $C(I \times I)$. By Lemma 2.1, $p_{nm}(h) \in \mathcal{D}(\delta)$ and $\delta(p_{nm}(h)) = p_{nm}{}'(h)\delta(h) = 0$. The closedness of δ implies that $h_n(h) \in \mathcal{D}(\delta)$ and $\delta(h_n(h)) = 0$. Therefore $(h_n(h))f_n \in \mathcal{D}(s)$ and

$$\delta((h_n(h))f_n) = (h_n(h))\delta(f_n) + f_n\delta(h_n(h)) = (h_n(h))\delta(f_n) \; .$$

By closedness of δ , $H \in \mathcal{D}(\delta)$ and $\delta(H) = G$. Since $h_n(h) = 1$ on K, it follows $(h_n(h))f_n|K = f_n|K$ and $\delta((h_n(h))f_n)|K = (h_n(h))\delta(f_n)|K = \delta(f_n)|K$. Hence $f = H$ on K and $F = G$ on K . Thus $f \in \mathcal{D}(\delta_K)$ and

$$\delta_K(f) = \delta_K(H|K) = \delta(H)|K = G|K = F \; .$$

This proves δ_K is closed.

Remark. In fact, the condition we need to prove the closedness of δ_K is that K is the zero set of a function in ker (δ) .

The following proposition gives a partial converse of Lemma 2.2 that is analogous to Kurose's result [4, Lemma 2.4] in the two dimensional case.

PROPOSITION 2.4. *Under the assumption of Proposition 2.3., if* $f \in \mathcal{D}(\delta)$ *and* $\delta(f)$ *vanishes on* K *then* f *depends only on the second variable on* K .

Proof. Let t and u be any two fixed numbers such that $r < t < u < s$. Choose a function $g \in \mathcal{D}(\delta)$ such that $0 \leq g \leq 1$ and

$$g = \begin{cases} 0 & \text{on} \quad I \times [t,u] \; , \\ 1 & \text{on} \quad I \times [s,1] \cup I \times [0,r] \; . \end{cases}$$

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Consider the function $h = f + 3||f||g$. We have $h \in \mathcal{D}(\delta)$ and $\delta(f) = \delta(h)$ on K , and $h \geq 2||f||$ on $I \times I - K$. Let $p(y)$ be a C^1-function on R such that

$$p(y) = \begin{cases} y & \text{if} \quad y \leq ||f|| \ , \\ 0 & \text{if} \quad y \geq 2||f|| \ . \end{cases}$$

Then by Lemma 2.1, $p(h) \in \mathcal{D}(\delta)$ and $\delta(p(h)) = p'(h)\delta(h)$. The definition of p implies that $p'(h) = 0$ on $I \times I - K$. Together with the fact that $\delta(h) = \delta(f) = 0$ on K , we have $\delta(p(h)) = 0$ on $I \times I$, and so $p(h)$ depends solely on the second variable on $I \times I$. Since $f = h$ on $I \times [t,u]$, it follows that $f = p(f) = p(h)$ on $I \times [t,u]$. Therefore f depends solely on the second variable on $I \times [t,u]$ for arbitrary t and u with $r < t < u < s$, thus f depends only on the second variable on K .

We shall later show that the result of Proposition 2.4. is valid for the degenerate subset $K = I \times \{y\}$, $y \in I$.

3. Derivations Induced by Signed Measures.

Let μ be a non-atomic signed measure on I . For a function f in $C(I \times I)$, consider the function

$$f_{\mu,\lambda}(x,y) = \lambda(y) + \int_0^x f(t,y)\, d\mu(t) \ ,$$

where $\lambda(y) \in C(I)$. The following theorem shows that a non-atomic measure gives rise to a closed derivation in $C(I \times I)$.

THEOREM 3.1. *Let* μ *be a non-atomic signed measure with support* $(\mu) = I$. *Then the map* $\delta_\mu(f_{\mu,\lambda}) = f$ *is a closed derivation in* $C(I \times I)$ *with domain* $\mathcal{D}(\delta_\mu) = \{f_{\mu,\lambda} : \lambda \in C(I) \ , \ f \in C(I \times I)\}$. *(The derivation* δ_μ *is denoted by* $\frac{\partial}{\partial\mu}$ *)* .

Note that this result has already been treated in [7, p. 72].

Proof. Since the total variation function of μ is continuous, it follows that $\int_0^x f(t,y)\, d\mu(t)$ is a continuous function on $I \times I$. The proof of the well-definedness of the map δ_μ is the same as in one-dimensional case [4, Theorem 2.2]. The following identity

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$$\int_0^x f(t,y)\,d\mu(t) \int_0^x g(t,y)\,d\mu(t) =$$

$$\int_0^x (f(s,y) \int_0^s g(r,y)\,d\mu(r) + g(s,y) \int_0^s f(r,y)\,d\mu(r))d\mu(s),$$

for every f and g in $C(I \times I)$, shows that $\mathcal{D}(\delta_\mu)$ is a subalgebra of $C(I \times I)$ and

$$\delta_\mu(f_{\mu,\lambda} g_{\mu,\xi}) = g_{\mu,\xi}\delta_\mu(f_{\mu,\lambda}) + f_{\mu,\lambda}\delta_\mu(g_{\mu,\xi}) .$$

Furthermore, $\mathcal{D}(\delta_\mu)$ separates points of $I \times I$ and contains the identity. By the Stone-Weierstrass theorem $\mathcal{D}(\delta_\mu)$ is dense in $C(I \times I)$. It is immediate from the definition that δ_μ is closed.

Note that the closed derivation δ_μ induced by a non-atomic measure μ has the properties: the range of δ_μ is $C(I \times I)$ and the kernel of δ_μ is the set of functions depending only on the second variable. We shall prove next that every closed derivation δ in $C(I \times I)$, whose range and kernel have the above properties, induces signed measures and δ can be identified with the partial derivative with respect to the signed measures. If a closed derivation δ on $C(I \times \Omega)$ is so-called well-behaved, and has kernel containing all functions depending only on the second variable, then Batty [1] introduces the self-determining subspace and characterizes the derivation δ as a partial derivative of standard limit form.

THEOREM 3.2. *Let δ be a closed derivation in $C(I \times I)$ such that range $(\delta) = C(I \times I)$ and $\ker(\delta)$ is the set of all functions depending solely on the second variable. Then for each $(x,y) \in I \times I$ there exists a unique signed measure $\mu_{x,y}$ on $I\times\{y\}$ such that $\mathcal{D}(\delta) =$ $\{F(w,z) = \lambda(z) + \int_0^1 f(t,z)\,d\mu_{w,z}(t) : \lambda \in C(I),\ f \in C(I \times I)\}$ and $\delta(F)(x,y) = f(x,y)$.*

Proof. Let $\mathcal{D}_0$ be the set of all functions $f \in \mathcal{D}(\delta)$ such that $f(0,y) = 0$ for all $y \in I$. Suppose $g \in C(I \times I)$, and there exists a function $f \in \mathcal{D}(\delta)$ such that $\delta(f) = g$. Define $h(x,y) = f(0,y)$ for $(x,y) \in I \times I$, then $h \in \mathcal{D}(\delta)$, $f-h \in \mathcal{D}_0$ and $\delta(f-h) = \delta(f) = g$.

The map $\delta_0 = \delta|\mathcal{D}_0$ is hence a closed one-to-one map from $\mathcal{D}_0$ onto $C(I\times I)$. By the closed graph theorem the inverse δ_0^{-1} is a continuous map of $C(I\times I)$ onto $\mathcal{D}_0$. Take a fixed point $(x,y) \in I\times I$. If $f,g \in C(I\times I)$ and $f = g$ on $I\times[0,y]$, then $\delta(\delta_0^{-1}(f - g)) = f - g = 0$ on $I\times[0,y]$. By Proposition 2.4, $\delta_0^{-1}(f - g)$ is constant in the first variable on $I\times[0,y]$. Hence $\delta_0^{-1}(f - g)(x,y) = \delta_0^{-1}(f - g)(0,y) = 0.$ Thus $\delta_0^{-1}(f)(x,y) = \delta_0^{-1}(g)(x,y)$. It follows that the functional

$$f \in C(I\times[0,y]) \longrightarrow \delta_0^{-1}(\tilde{f})(x,y) ,$$

where $\tilde{f}$ is a continuous extension of f to $I\times I$, is well-defined and continuous. Thus by the Riesz Representation theorem there exists a unique signed measure $\mu_{x,y}$ on $I\times[0,y]$ such that $\delta_0^{-1}(f)(x,y) = \int_{I\times[0,y]} f(s)\, d\mu_{x,y}(s)$ for every function $f \in C(I\times I)$. Hence we have

$$\mathcal{D}(\delta) = \{F(w,z) = \lambda(z) + \int_{I\times[0,z]} f(s)\, d\mu_{w,z}(s) : \lambda \in C(I) ,$$
$$f \in C(I\times I)\} \text{ and } \delta(F)(x,y) = f(x,y) .$$

Suppose $f \in C(I\times I)$ and $f = 0$ on $I\times\{y\}$. Let $\tilde{f}$ be a function in $C(I\times I)$ such that $\tilde{f}$ and f coincide on $I\times[0,y]$ and $\tilde{f}$ is constant on $I\times[y,1]$. Then by Proposition 2.4. $\delta_0^{-1}(\tilde{f})$ is constant in the first variable on $I\times[y,1]$, and so $\delta_0^{-1}(\tilde{f})(x,y) = \delta_0^{-1}(\tilde{f})(0,y) = 0$. Hence $\delta_0^{-1}(f)(x,y) = 0$. This shows that support$(\mu_{x,y})$ is contained in $I\times\{y\}$, and therefore

$$\delta_0^{-1}(f)(x,y) = \int_0^1 f(t,y)\, d\mu_{x,y}(t) .$$

COROLLARY. *Let δ be a closed derivation in $C(I \times I)$ such that range$(\delta) = C(I \times I)$ and ker(δ) is the set of all functions depending solely on the second variable. If $K = I \times \{y\}$ for some $y \in I$, then for a function $f \in \mathcal{D}(\delta)$, if $\delta(f)$ vanishes on K then f is constant on K.*

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Remark. The converse of this Corollary is false, that is, the degenerate subset K is not self-determining. See Kurose [5 Example 2.2].

The author has learned from Professor H. Kurose that there exists a closed derivation δ in $C(I\times I)$ such that Range$(\delta) = C(I\times I)$ and Ker(δ) $= \{\lambda I\}$ as follows:

$\mathcal{D}(\delta)$ is the set of all $f \in C(I\times I)$, such that f has partial derivative $\frac{\partial f}{\partial x}$ for the first variable at every point in $I\times I$ and partial derivative $\frac{\partial f}{\partial y}$ for the second variable at every point $\{0\}\times I$ and, for every y, $\frac{\partial f}{\partial x}(0,y) = \frac{\partial f}{\partial y}(0,y)$; and $\delta(f) = \frac{\partial f}{\partial x}$ for $f \in \mathcal{D}(\delta)$. Then it follows that $f \in C(I\times I)$ belongs to $\mathcal{D}(\delta)$ if and only if there exist $\lambda \in R$ and $F \in C(I\times I)$ such that

$$f(x,y) = \lambda + \int_0^y F(0,t)\,dt + \int_0^x F(u,y)\,du .$$

A characterization of such closed derivations with scalar kernels is analogous to Theorem 3.2.

THEOREM 3.3. *Let* δ *be a closed derivation in* $C(I\times I)$, *whose kernel is the set of scalar functions. Let* $K = [0,1]\times[r,s]$ *be a subset of* $I\times I$. *Then for every* $f \in \mathcal{D}(\delta)$, *if* $\delta(f)$ *vanishes on* K, *then* f *is constant on* K.

The proof of Proposition 2.4 can be applied to prove this Theorem.

THEOREM 3.4. *Let* δ *be a closed derivation in* $C(I\times I)$ *such that Range*$(\delta) = C(I\times I)$ *and Ker*$(\delta) = \{\lambda I\}$. *Then for each* $(x,y) \in I\times I$ *there exists a unique signed measure* $\mu_{x,y}$ *on* $I\times\{y\}$ *such that* $\mathcal{D}(\delta) =$ $\{F(w,z) = \lambda + \int_0^1 f(t,z)\,d\mu_{w,z}(t),\ \lambda \in R,\ f \in C(I\times I)\}$; *and* $\delta(F)(x,y) =$ $f(x,y)$.

Let $\mathcal{D}_0 = \{f \in \mathcal{D}(\delta),\ f(0,0) = 0\}$. Together with Theorem 3.3., one may proceed as in the proof of Theorem 3.2. to construct signed measures $\mu_{x,y}$, and the assertion follows easily.

A point ω of Ω is said to be well-behaved for a derivation δ in $C(\Omega)$ if $\delta(f)(\omega) = 0$ for all f in $\mathcal{D}(\delta)$ such that $f(\omega) = ||f||$; the set of all well-behaved points for δ is denoted by W_δ, δ is said to be well-behaved if $W_\delta = \Omega$, and quasi well-behaved if the interior Int(W_δ) of W_δ is dense in Ω.

THEOREM 3.5. *Let* δ *be a well-behaved closed derivation in* $C(I\times I)$ *, such that* $range(\delta) = C(I\times I)$ *and* $Ker(\delta)$ *is the set of all functions depending solely on the second variable. Then*

(1) *for each* $y \in I$ *, the map* $\delta_{I\times\{y\}}$ *is well-defined and closed,*

(2) *for each* $y \in I$ *there exists a unique non-atomic signed measure* μ_y *on* $I\times\{y\}$ *such that*

$$\mathcal{D}(\delta) = \{F(w,z) = \lambda(z) + \int_0^w f(t,z)\, d\,\mu_z(t):\ \ \lambda \in C(I)\ ,$$
$$f \in C(I\times I)\} \ \text{ and } \ \delta(F)(x,y) = f(x,y)\ .$$

Proof. Let $y \in I$ be fixed. Let g be a function in $C(I)$ such that $g(z) = 0$ only at $z = y$. Then the function $f(w,z) = g(z)$ for $(w,z) \in I\times I$ belongs to $ker(\delta)$, and $I \times \{y\}$ is the set where f vanishes. Since $I \times \{y\} \cap W_\delta$ is dense in $I \times \{y\}$, it follows by [1, Corollary 4.5] that $\delta_{I\times\{y\}}$ is well-defined and closed.

Let $\mathcal{D}_y = \{f_y(x) = f(x,y):\ f \in \mathcal{D}(\delta)\}$ and set

$$\delta_y(f_y)(x) = \delta(f)(x,y)\ .$$

Then we have the following properties:

1. $\mathcal{D}_y$ is dense in $C(I \times \{y\})$,
2. The map δ_y is well-defined and closed by part (1) ,
3. It is clear that $range(\delta_y) = C(I \times \{y\})$.
4. By the Corollary in this section $Ker(\delta_y)$ is the set of all constant functions.

Hence the characterization theorem [4, Theorem 2.3] for the closed derivation δ_y induces a non-atomic measure signed measure μ_y on $I \times \{y\}$ such that $\mathcal{D}(\delta_y) = \{F(w,y) = \lambda(y) + \int_0^w f(t,y)\, d\mu_y(t):$ $f \in C(I \times \{y\})$ and $\delta_y(F)(x,y) = f(x,y)$.

Let δ be a closed derivation in $C(I\times I)$ extending the partial derivative $\partial = \frac{\partial}{\partial x}$. Goodman [3] has determined the structure of such a derivation by its kernetl, and the kernel always has the form $\Phi^0(C(X))$, where $\Phi : I\times I \longrightarrow X$ is a generalized Cantor function (abbreviated gcf). In the following theorem, we shall examine such a derivation in terms of signed measures.

THEOREM 3.6. *Let* δ *be a closed derivation in* $C(I\times I)$ *extending* ∂ . *Then for every* $x \in X$, *there exists a unique non-atomic signed measure* μ_x *on* $\Phi^{-1}(x)$ *such that* $\delta = \frac{\partial}{\partial\mu_x}$ *on* $\Phi^{-1}(x)$, *where* Φ *is a gcf induced by* δ .

Proof. By the definition of a gcf as given in [3, p.321] , we have for every $x \in X$, $\Phi^{-1}(x)$ is a closed subinterval of $I \times \{w\}$ for some $w \in I$, and $\Phi^0(C(X)) = \{f \in C(I\times I)$, f is constant on $\Phi^{-1}(t)$ for each $t \in X\}$.

If $f \in \mathcal{D}(\delta)$ and $f = 0$ on $\Phi^{-1}(x)$ then there exists $g \in \mathcal{D}(\partial)$ such that $\delta(f) = \partial(g)$. Hence $f-g \in \mathrm{Ker}(\delta)$. By [3, Theorem 2.1.3], $\mathrm{Ker}(\delta) = \Phi^0(C(X))$, it follows that $f-g$ is constant on $\Phi^{-1}(x)$, and so g is constant on $\Phi^{-1}(x)$. Therefore $\partial(g) = 0$ on $\Phi^{-1}(x)$, thus $\delta(f) = 0$ on $\Phi^{-1}(x)$. Hence the map $\delta_{\Phi^{-1}(x)}(f) = \delta(f)|\Phi^{-1}(x)$ for $f \in \mathcal{D}(\delta)$ is well-defined. Choose a function $f \in C(X)$ such that $f^{-1}(0) = \{x\}$. Then $f\Phi$ takes the constant value $f(t)$ on each fibre $\Phi^{-1}(t)$ and $(f\Phi)^{-1}(0) = \Phi^{-1}(x)$. It follows that $f\Phi \in \mathrm{Ker}(\delta)$ and $\Phi^{-1}(x)$ is the only set where $f\Phi$ vanishes. Hence $\delta_{\Phi^{-1}(x)}$ is closed by the remark of Proposition 2.3.

If $f \in \mathcal{D}(\delta)$ and $\delta(f)|\Phi^{-1}(x) = 0$, then there is a function $g \in \mathcal{D}(\delta)$ such that $\delta(f) = \partial(g)$, $f-g$ is then in $\mathrm{Ker}(\delta)$ and $f-g$ is constant on $\Phi^{-1}(x)$. It follows that $(g)|\Phi^{-1}(x) = 0$, and g is constant on $\Phi^{-1}(x)$. Thus f is constant on $\Phi^{-1}(x)$.

If $g \in C(\Phi^{-1}(x))$ then extend g continuously to $I \times \{w\}$, and then extend this function continuously to $I\times I$ by making it constant on every vertical line. Denoted this function by $\tilde{g}$. Since $\mathrm{Range}(\delta) =$

$\mathrm{Range}(\partial) = C(I\times I)$, there exists $f \in \mathcal{D}(\delta)$ such that $\delta(f) = \tilde{g}$. Then

$$\delta_{\Phi^{-1}(x)}(f|\Phi^{-1}(x)) = \delta(f)|\Phi^{-1}(x) = \tilde{g}|\Phi^{-1}(x) = g ,$$

which shows that $\mathrm{Range}(\delta_{\Phi^{-1}(x)}) = C(\Phi^{-1}(x))$.

Finally if $f \in C(\Phi^{-1}(x))$, applying the same extension to $I\times I$ we get a function $\tilde{f} \in C(I\times I)$ extending f . The domain $\mathcal{D}(\delta)$ is dense in $C(I\times I)$, there is a sequence $f_n \in \mathcal{D}(\delta)$ such that f_n converges to $\tilde{f}$ in $C(I\times I)$, and hence $f_n|\Phi^{-1}(x)$ converges to $\tilde{f}|\Phi^{-1}(x) = f$ in $C(\Phi^{-1}(x))$, $(\delta|\Phi^{-1}(x))$ is dense.

From Kurose's structure theorem [4, Theorem 2.3] applying the closed derivation $\delta_{\Phi^{-1}(x)}$ in $C(\Phi^{-1}(x))$, we obtain a non-atomic signed measure μ_x on $\Phi^{-1}(x)$ such that if $f \in \mathcal{D}(\delta)$, $\delta(f)(t,u) = \frac{\partial f}{\partial \mu_x}(t,u)$ for all $(t,u) \in \Phi^{-1}(x)$.

4. Differentiations.

Derivations δ_1 on $C(\Omega_1)$ and δ_2 on $C(\Omega_2)$ will be said to be equivalent if there is an algebra isomorphism α of $C(\Omega_1)$ onto $C(\Omega_2)$ such that $\delta_2 = \alpha\delta_1\alpha^{-1}$.

Batty [1] has proved that any quasi-well-behaved closed derivation in $C(I)$ is equivalent to an extension of $\lambda \frac{d}{dx}$ for some $\lambda \in C(I)$. Indeed, Batty [1][2, p.336] and Sakai [6] obtain the following equivalent conditions for a closed derivation δ in $C(I)$:

(1) δ is quasi-well-behaved

(2) There is a function λ in $C(I)$ such that δ is equivalent to an extension of $\lambda \frac{d}{dx}$.

(3) $\mathcal{D}(\delta)$ contains a strictly monotone function.

Let Ω be a compact Hausdorff space. In this section, we obtain the following results which are analogous to the above equivalent conditions for a closed derivation in $C(\Omega)$.

THEOREM 4.1. *Let* δ *be a closed derivation in* $C(\Omega)$ *. Then* δ *is equivalent to a closed extension of* $\lambda \frac{d}{dx}$ *,* $\lambda \in C(I)$ *, if and only if* $\mathcal{D}(\delta)$ *contains an injective function.*

Proof. If $\mathcal{D}(\delta)$ contains an injective function h, apply the proof of [6, Proposition 1.16], let $k = (||h||1 + h)/|| \, ||h||1 + h||$, $k(t_0) = \inf_{t\in\Omega} k(t)$ and $\eta = (k - k(t_0))/||k - k(t_0)1||$, then $\eta \in \mathcal{D}(\delta)$ and η is a homeomorphism of Ω onto I. By the Banach-Stone theorem, η induces a *-isomorphism α of $C(I)$ onto $C(\Omega)$ by $\alpha(f) = f\eta$. If $f \in C'(I)$ then by [6, Theorem 3.8] $f\eta \in \mathcal{D}(\delta)$ and $\delta(f\eta) = f'(\eta)\delta(\eta)$. The following equalities

$$\alpha^{-1}\delta\alpha(f) = \alpha^{-1}\delta\alpha(\alpha^{-1}f(\eta)) = \alpha^{-1}\delta(f(\eta))$$
$$= \alpha^{-1}(f'(\eta)\delta(\eta)) = \alpha^{-1}(f'(\eta))\alpha^{-1}(\delta(\eta)) = f' \lambda$$

show that $\alpha^{-1}\delta\alpha(f) = \lambda f'$ for all $f \in C'(I)$, where $\lambda = \alpha^{-1}(\delta(\eta))$, and hence δ is equivalent to an extension of $\lambda \frac{d}{dx}$.

Conversely, if δ is equivalent to a closed extension of $\lambda \frac{d}{dx}$ then there is a *-isomorphism α of $C(\Omega)$ onto $C(I)$ such that

$$\alpha\delta\alpha^{-1}(f) = \lambda f' \quad \text{for all} \quad f \in C'(I)$$

and $\alpha^{-1}C'(I) \subset \mathcal{D}(\delta)$. Let θ be a homeomorphism of I onto Ω induced by α where $\alpha(f) = f\theta$. Choose $g \in C(\Omega)$ such that $g(\theta(x)) = x$ for all $x \in I$, then $g = \alpha^{-1}(g\theta) \in \alpha^{-1}(C'(I))$, and hence $g \in \mathcal{D}(\delta)$. If $x \neq y$ in Ω then $\theta^{-1}(x) \neq \theta^{-1}(y)$ and $g(x) = g(\theta(\theta^{-1}(x))) = \theta^{-1}(x) \neq \theta^{-1}(y) = g(\theta(\theta^{-1}(y))) = g(y)$. Hence g is injective.

THEOREM 4.2. *Let* δ *be a closed derivation in* $C(\Omega)$*. If* δ *is equivalent to an extension of* $\lambda \frac{d}{dx}$ *then* δ *is quasi well-behaved, where* $\lambda \in C(I)$ *.*

Proof. Let δ be a closed derivation in $C(\Omega)$ which is equivalent to an extension of $\lambda \frac{d}{dx}$. Then there exists a *-isomorphism ξ of

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$C(\Omega)$ onto $C(I)$ such that $\xi\delta\xi^{-1}(f) = \lambda f'$ for all $f \in C'(I)$. The morphism ξ induces a homeomorphism θ of I onto Ω such that $\xi(f) = f\theta$. Since $\xi\delta\xi^{-1}$ is a closed extension of $\lambda \frac{d}{dx}$, it follows by [1, Theorem 3.2] that $\xi\delta\xi^{-1}$ is quasi well-behaved. Let W_ξ be the set of well behaved points for $\xi\delta\xi^{-1}$. Then closure(interior$(W_\xi)) = I$. Let $\omega \in \theta(W_\xi)$ and $f \in \mathcal{D}(\delta)$ be such that $||f|| = f(\omega)$. Let $t \in W_\xi$ be such that $\theta(t) = \omega$. Since the *-isomorpfism ξ of the two C^*-algebras $C(\Omega)$ and $C(I)$ is isometric, it follows that $||\xi(f)|| = ||f|| = f(\omega)$, and $\xi(f)(t) = f(\theta(t)) = f(\omega)$. We have $||\xi(f)|| = \xi(f)(t)$. Hence $(\xi\delta\xi^{-1}(\xi(f)))(t) = 0$. Then

$$\delta(f)(\omega) = \delta(f)(\theta(t)) = (\xi(\delta(f)))(t) = (\xi\delta\xi^{-1}(\xi(f)))(t) = 0 .$$

Thus every $\omega \in \theta(W_\xi)$ is well-behaved for δ, and closure(interior $(\theta(W_\xi))= \theta$(closure(interior$(W_\xi)) = \theta(I) = \Omega$. Hence δ is quasi-well-behaved.

Batty [1] shows that quasi-well-behaved derivation in $C(I)$ and extension of $\lambda \frac{d}{dx}$ are equivalent. We have obtained in Theorem 4.2. one direction of this result in $C(\Omega)$ case. Unfortunately, the other way the quasi-well-behavedness of δ in $C(\Omega)$ does not imply that δ is equivalent to an extension of $\lambda \frac{d}{dx}$. Example: Let δ be the partial differentiation operator $\frac{\partial}{\partial x}$ on $C(I\times I)$. Then δ is quasi well-behaved by Rolle's theorem, but δ is not equivalent to an extension of $\lambda \frac{d}{dx}$ If it does, then by Theorem 4.1. $\mathcal{D}(\delta)$ contains an injective function, and hence $I\times I$ is homeomorphic to a compact interval, a contradiction.

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