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RESEARCH ARTICLE

On Namba Forcing And Minimal Collapses

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Abstract
We build on a 1990 paper of Bukovský and Copláková-Hartová. First, we remove the hypothesis of CH from one of their minimality results. Then, using a measurable cardinal, we show that there is a $|\aleph_2^V| = \aleph_1$-minimal extension that is not a $|\aleph_3^V| = \aleph_1$-extension, answering the first of their questions.

1. Introduction

Forcing extensions often involve collapsing cardinals, so it is natural to study the properties of various forcing collapses from a general perspective. Namba forcing, which was discovered independently by Bukovský and Namba [17, 2], has two notable properties that distinguish it from other forcings that collapse cardinals. First, it allows one to avoid collapsing cardinals below the target: the classical versions singularize $\aleph_2$ while preserving $\aleph_1$. There are limitations, however: Shelah proved that the classical versions collapse $\aleph_3$ [6, Theorem 4.73]. The other notable property of Namba forcing is that, like many other tree forcings, it is to some extent minimal. Bukovský and Copláková-Hartová conducted a thorough investigation into the minimality properties of Namba-like forcings [3] and their paper became a go-to reference for further work on Namba forcing [7, 9].

Bukovský and Copláková-Hartová's paper is partially known for a question about collapsing $\aleph_{\omega+1}$ (see [4]), but the authors also raised questions about how minimality and control over collapsed cardinals interact. In their notation, a forcing is $|\lambda| = \kappa$-minimal if it forces $|\lambda| = \kappa$ and has no subforcings collapsing λ to have cardinality κ. They showed that CH implies the $|\aleph_2^V| = \aleph_1$-minimality of classical Namba forcing. We will remove the assumption of CH. They also asked whether there is a $|\aleph_2^V| = \aleph_1$-minimal extension that is not a $|\aleph_3^V| = \aleph_1$-extension [3, Question 1]. Assuming the consistency of a measurable cardinal, we will answer their question positively here. Essentially, we are showing that there is more flexibility for producing various $|\lambda| = \kappa$-extensions than was previously known.

The forcing used to obtain these results is a variant of classical Namba forcing in the sense that it is a tree forcing with wide splitting, although in our case the trees will have a height equal to ω_1. An aspect of the main technical idea is present in the recent result that classical Namba forcing consistently has the weak ω_1-approximation property [15]. The crux is a sweeping argument that is used to pair the successors of splitting nodes with distinct forced values for a given forcing name. The difference with the present results is that we will use a strengthening of precipitousness, due to Laver to define the splitting behavior of our forcing. This will allow us to use the sweeping argument while ensuring that our forcing is countably closed. We should expect something like this because of the above-mentioned result of Shelah, which in fact shows that *any* extension singularizing $\aleph_2$ to have cofinality ω while preserving $\aleph_1$ will collapse $\aleph_3$.

The use of large cardinals appears necessary. First, the assumption we employ to define the forcing, which we refer to as the Laver Ideal Property, implies the consistency of a measurable cardinal [12]. An extension fitting Bukovský and Copláková-Hartová's minimality criteria would likely be a tree forcing, and joint work with Mildenberger [16] shows that tree forcings with uncountable height exhibit a number of (rather interesting) pathologies, particularly when it comes to fusion arguments, unless some regularity of their splitting behavior is enforced. Hence, it seems like we need the Laver Ideal Property as long as we expect to use tree forcings. Of course, this does not prove that the large cardinals are necessary. An argument to this effect would probably use an almost disjoint sequence that arises from the failure of a large cardinal principle, and it would need to use the notion of a *strictly* intermediate extension of an arbitrary extension.

1.1. Definitions and notation

We assume that the reader is familiar with the basics of set theory and forcing. It will also be helpful to have familiarity with tree forcings, in particular fusion arguments (see [11, Chapters 15 and 28]). Here we will clarify our notation.

Definition 1.1.

1. A *tree* T is (for our purposes) a collection of functions $f : \mathrm{ON} \to \mathrm{ON}$ with $\mathrm{dom}(f) \in \mathrm{ON}$ such that if $f \in T$ and $\alpha \in \mathrm{dom}(f)$, then $f \upharpoonright \alpha \in T$.
2. If T is a tree, we refer to an element $t \in T$ as a *node*.
3. For an ordinal α, the set $T(\alpha)$ is the set of $t \in T$ with $\mathrm{dom}(t) = \alpha$.
4. The *height* $\mathrm{ht}(T)$ of a tree T is $\min\{\alpha : T(\alpha) = \emptyset\}$.
5. We let $[T] = \{f\colon \mathrm{ht}(T) \to \kappa : \forall\alpha < \mathrm{ht}(T), f \upharpoonright \alpha \in T\}$. Elements of $[T]$ are called *cofinal branches*.
6. For $t_1, t_2 \in T \cup [T]$ we write $t_1 \sqsubseteq t_2$ if $t_2 \upharpoonright \mathrm{dom}(t_1) = t_1$. (Hence the tree order is the relation $\sqsubseteq$.) We write $t_1 \sqsubset t_2$ if $t_1 \sqsubseteq t_2$ and $t_1 \neq t_2$.
7. If $t = s \cup \{(\mathrm{dom}(s), \beta)\}$, we write $t = s^\frown\langle\beta\rangle$.
8. $T \upharpoonright \alpha = \bigcup_{\beta<\alpha} T(\beta)$.
9. $T \upharpoonright t = \{s \in T : s \sqsubseteq t \vee t \sqsubseteq s\}$.
10. For $t \in T(\alpha)$ we let $\mathrm{succ}_T(t) = \{c : c \in T(\alpha+1) \wedge c \sqsupseteq t\}$ denote the *set of immediate successors of* t, and $\mathrm{osucc}_T(t) = \{\beta : t^\frown\langle\beta\rangle \in T(\alpha+1)\}$ denote the *ordinal successor set of* t.
11. We call $t \in T$ a *splitting node* if $|\mathrm{succ}_T(t)| > 1$.
12. $\mathrm{stem}(T)$ is the $\sqsubseteq$-minimal splitting node.

For the purposes of this paper, we will define tree forcings, but this definition should not be considered in unqualified terms.

Definition 1.2. We say that $\mathbb{P}$ is a *tree forcing* if there are regular cardinals μ and κ such that for all $p \in \mathbb{P}$, $p \subseteq \mu^{<\kappa}$ is a tree of height κ and for all $p, q \in \mathbb{P}$, $p \leq_{\mathbb{P}} q$ if and only if $p \subseteq q$.

In general, tree forcings are understood to require the trees to be perfect, which means that they have splitting nodes above every node. But there are a number of ways in which this is made precise, so we will avoid the term "perfect" below.

Definition 1.3. Let $\mathbb{P}$ be a tree forcing and let κ, μ be regular cardinals such that p is a subtree of $\mu^{<\kappa}$ of height κ for all $p \in \mathbb{P}$.

1. Take $p \in \mathbb{P}$. We let $\mathrm{stem}(p)$ be, as above, the $\sqsubseteq$-minimal splitting node of p.
2. We let $\mathrm{split}(p)$ denote the set of splitting nodes of p. For $\alpha \in \kappa$, $\mathrm{split}_\alpha(p)$ is the set of α-order splitting nodes of p, that is, the set of $t \in \mathrm{split}(p)$ such that $\mathrm{ot}\{s \sqsubset t : s \in \mathrm{split}(p)\} = \alpha$.

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3. Let $p, q \in \mathbb{P}$, $\alpha < \kappa$. We write $q \leq_\alpha p$ if $q \leq p$ and $\operatorname{split}_\alpha(p) = \operatorname{split}_\alpha(q)$.
4. A sequence $\langle p_\alpha : \alpha < \delta \rangle$ such that $\delta \leq \kappa$ and for $\alpha < \gamma < \delta$, $p_\gamma \leq_\alpha p_\alpha$ is called a *fusion sequence*.

Fusion sequences are a fundamental tool for working with tree forcings. The idea is that if $\langle p_\alpha : \alpha < \kappa \rangle$ is a fusion sequence, then $p := \bigcap_{\alpha<\kappa} p_\alpha$, otherwise known as the fusion limit, should be a condition with the property that $p \leq_\alpha p_\alpha$ for all $\alpha < \kappa$. The first forcing that we use (Theorem 2.1 below) is already in the literature, so we will use fusion limits without comment. However, we will include a precise argument for fusion limits in the proof of the main theorem (Theorem 2.9 below).

2. Results

2.1. Minimality without the continuum hypothesis

We will discuss the version of Namba forcing that appears in Bukovský's treatment [2] since this is the one that appears in Jech's textbook [11]. We define it here so that there is no risk of ambiguity:

Definition 2.1. The conditions in *classical Namba forcing*, which we denote $\mathbb{P} = \mathbb{P}_{\mathsf{CNF}}$, consist of conditions that p are subsets of $\aleph_2^{<\omega}$ such that:

1. p is a tree in $\aleph_2^{<\omega}$;
2. for all $t \in p$, $|\operatorname{osucc}_t(p)| \in \{1, \aleph_2\}$;
3. and for all $t \in p$ there is some $s \sqsupseteq t$ such that $s \neq t$ and $|\operatorname{osucc}_p(s)| = \aleph_2$.

If $p, q \in \mathbb{P}$, then $p \leq_{\mathbb{P}} q$ if and only if $p \subseteq q$.

Bukovský and Copláková-Hartová showed that CH implies that $\mathbb{P}_{\mathsf{CNF}}$ is $|\aleph_2^V| = \aleph_1$-minimal [3, Corollary 1.3], and also proved a more general statement, but we will show that the hypothesis of CH can be dropped if we just want the minimality result for $\mathbb{P}_{\mathsf{CNF}}$. Part of the issue here is that $\mathbb{P}_{\mathsf{CNF}}$ adds reals if and only if CH holds: that CH suffices is shown in Jech's textbook, and if CH fails, then one can observe that the generic branch will code a new countable sequence of reals. Note that Theorem 2.2 below was specifically proved by Bukovský [2, Theorems 2 and 3] with the assumption of CH. The argument anticipates the main result, Theorem 2.14.

Lemma 2.2. *Suppose G is $\mathbb{P}_{\mathsf{CNF}}$-generic over V and suppose $f \in V[G]$ is an unbounded function $\omega \to \theta$ where $\operatorname{cf}^V(\theta) \geq \aleph_2$. Then $V[f] = V[G]$.*

Proof. Suppose that $\dot{f}$ is a $\mathbb{P}_{\mathsf{CNF}}$-name for an unbounded function $\omega \to \theta$ and this is forced by some $p \in \mathbb{P}_{\mathsf{CNF}}$. We will define a fusion sequence $\langle p_n : n < \omega \rangle$ together with an assignment $\{(t, n_t) : t \in \bigcup_{n<\omega} \operatorname{split}_n(p_n)\}$ such that:

1. for all $t \in \bigcup_{n<\omega} \operatorname{split}_n(p_n)$, $n_t \in \omega$,
2. for all $t \in \bigcup_{n<\omega} \operatorname{split}_n(p_n)$, $n_t > |t| \geq \max\{n_s : s \sqsubset t\}$.
3. for each $n < \omega$ and $t \in \operatorname{split}_n(p_n)$, there is a sequence $\langle \gamma_\alpha^t : \alpha \in \operatorname{osucc}_{p_n}(t) \rangle$ such that $p_{n+1} \upharpoonright t^\frown\langle\alpha\rangle \Vdash$ "$\dot{f}(n_t) = \gamma_\alpha^t$" and such that $\alpha \neq \beta$ implies $\gamma_\alpha^t \neq \gamma_\beta^t$.

If we define such a sequence and $\bar{p} = \bigcap_{n<\omega} p_n$, then $\bar{p} \Vdash$ "$V[\Gamma(\mathbb{P}_{\mathsf{CNF}})] = V[\dot{f}]$" (where $\Gamma(\mathbb{P}_{\mathsf{CNF}})$ is the canonical name for the generic), that is, $\bar{p}$ forces that the generic can be recovered from the evaluation of $\dot{f}$.

The recovery of the generic goes as follows: Assuming that we have such a $\bar{p}$, suppose G is $\mathbb{P}_{\mathsf{CNF}}$-generic over V with $\bar{p} \in G$, and $V \subseteq W \subseteq V[G]$ where W is a model with $g = \dot{f}_G \in W$. Then we can argue that $G \in W$. Note that it is sufficient to argue that $b := \bigcap G \in W$. We work in W. By induction on $k < \omega$, we define a sequence $\langle s_k : k < \omega \rangle \subset \bar{p}$ such that $|s_k| \geq k$ and such that for all $k < \omega$, there is $q_k \in G$ such that $s_k \sqsubseteq \operatorname{stem}(q_k)$. Let $s_0 = \emptyset$. Given s_k, let s_{k+1}^* be the $\sqsubseteq$-minimal k^{th}-order splitting node of $\bar{p}$ above s_k, meaning in particular that $s_{k+1}^* \in \operatorname{split}_k(p_k)$. Then by Item (3) above, there is a unique α_{k+1} such that $\bar{p} \upharpoonright (s_{k+1}^{*\frown}\langle\alpha_{k+1}\rangle) \Vdash \dot{f}(n_{s_{k+1}^*}) = g(n_{s_{k+1}^*})$. Then let $s_{k+1} = s_{k+1}^{*\frown}\langle\alpha_{k+1}\rangle$. It must

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be the case that $b = \bigcup_{k<\omega} s_k$. This completes the definition of $\langle s_k : k < \omega\rangle$ and the argument that the generic can be recovered from $\bar{p}$ as described.

Now we define $\langle p_n : n < \omega\rangle$. Let $p_0 = p$ and suppose we have defined p_n. Define p_{n+1} as follows: For each $t \in \mathrm{split}_n(p_n)$, we will define a sequence of ordinals $\langle \alpha^t_\xi : \xi < \aleph_2\rangle \subseteq \mathrm{osucc}_{p_n}(t)$, a sequence of conditions $\langle q^t_\xi : \xi < \aleph_2\rangle$, a sequence of ordinals $\langle \gamma^t_\xi : \xi < \aleph_2\rangle \subseteq \aleph_2$, and a sequence of natural numbers $\langle n^t_\xi : \xi < \aleph_2\rangle$ such that for all $\xi < \aleph_2$:

1. $q_\xi \le p_n \restriction (t^\frown\langle \alpha_\xi\rangle)$,
2. $q_\xi \Vdash$ "$\dot{f}(n_\xi) = \gamma_\xi$",
3. $\xi \neq \zeta$ implies $\gamma_\xi \neq \gamma_\zeta$.

Let $\bar{n} = n_s$ where s is the $\sqsubseteq$-maximal splitting node in p_n strictly below t assuming it exists; otherwise let $\bar{n} = 0$.

We define α^t_ξ's, the q^t_ξ's, the γ^t_ξ's, and the n^t_ξ's by induction on ξ. Suppose we have defined them for $\xi < \zeta < \aleph_2$. We claim that there is some $\beta \in \mathrm{osucc}_{p_n}(t) \setminus \langle \alpha^t_\xi : \xi < \zeta\rangle$, some $r \le p_n \restriction (t^\frown\langle\beta\rangle)$, some ordinal δ, and some $m > \max\{\bar{n}, |t|\}$ such that $\delta \notin \langle \gamma^t_\xi : \xi < \zeta\rangle$ and such that $r \Vdash$ "$\dot{f}(m) = \delta$". Otherwise it is the case that

$$\bigcup\{p \restriction (t^\frown\langle\alpha\rangle) : \alpha \in \mathrm{osucc}_{p_n}(t) \setminus \sup_{\xi<\zeta} \alpha^t_\xi\} \Vdash \text{“}\mathrm{range}(\dot{f} \restriction (\max\{\bar{n}, |t|\}, \omega)) \subseteq \langle \gamma^t_\xi : \xi < \zeta\rangle\text{”},$$

which contradicts the fact that p forces $\dot{f}$ to be unbounded in θ where θ has a cofinality strictly greater than $\aleph_1^V$. Hence we can let $\alpha^t_\zeta := \beta$, $\gamma^t_\zeta := \delta$, $q^t_\zeta = r$, and $n^t_\zeta := m$.

Now that the q^t_ξ's, γ^t_ξ's, and n^t_ξ's have been defined, let $k < \omega$ be such that there is an unbounded $X \subseteq \aleph_2$ and a $k < \omega$ such that $n_\xi = k$ for all $\xi \in X$. Then let $n_t = k$ and let $q_t = \bigcup_{\xi\in X} q^t_\xi$. Finally, let $p_{n+1} = \bigcup\{q_t : t \in \mathrm{split}_n(p_n)\}$. Now that we have defined $\langle p_n : n < \omega\rangle$, let $\bar{p} = \bigcap_{n<\omega} p_n$. As argued above, $\bar{p} \Vdash$ "$V[\Gamma(\mathbb{P}_{\mathsf{CNF}})] = V[\dot{f}]$". □

Theorem 2.3. ZFC *proves that* $\mathbb{P}_{\mathsf{CNF}}$ *is* $|\aleph_2^V| = \aleph_1$*-minimal.*

Proof. Let $\kappa = \aleph_2^V$ and $\lambda = \aleph_1^V$. Suppose that G is $\mathbb{P}_{\mathsf{CNF}}$-generic over V and that $V \subseteq W \subseteq V[G]$ where $W \models$ "$|\kappa| = \aleph_1$". Consider the case that $\mathrm{cf}^W(\kappa) = \lambda$ as witnessed by some increasing and cofinal $g : \lambda \to \kappa$ in W. If $f' : \omega \to \kappa$ is the cofinal function added by $\mathbb{P}_{\mathsf{CNF}}$, then in $V[G]$ one can define a cofinal function $h : \omega \to \lambda$ by setting $h(n)$ to be the least ξ such that $f'(n) < g(\xi)$. Then h is cofinal because if $\xi < \lambda$ and n is such that $g(\xi) < f'(n)$, then $h(n) > \xi$. But this implies that $V[G] \models$ "$|\lambda| = \omega$", contradicting the fact that $\mathbb{P}_{\mathsf{CNF}}$ preserves ω_1.[1] Therefore it must be the case that $\mathrm{cf}^W(\kappa) = \omega$ as witnessed by some cofinal $f \in W$, so by Theorem 2.2 we have that $V[f] \subseteq W \subseteq V[G] = V[f]$, hence $W = V[G]$. □

2.2. *Developing a version of higher Namba forcing*

We will use a notion of Laver to define the forcing.

Definition 2.4 (Laver). (See [18, Chapter X, Definition 4.10].) Given regular cardinals $\lambda \le \mu$, we write $\mathsf{LIP}(\mu, \lambda)$ if there is a μ-complete ideal $I \subset P(\mu)$ that extends the bounded ideal on μ and there is a set $D \subseteq I^+$ such that:

1. D is λ-closed in the sense that if $\langle A_i : i < \tau\rangle$ is a $\subseteq$-descending sequence of elements of D with $\tau < \lambda$, then $\bigcap_{i<\tau} A_i \in D$,
2. D is dense in I^+, that is, for all $A \in I^+$, there is some $B \subseteq A$ with $B \in I^+$ such that $B \in D$.

Fact 2.5 (Laver). If $\lambda < \mu$ where λ is regular and μ is measurable, then $\mathrm{Col}(\lambda, {<}\mu)$ forces $\mathsf{LIP}(\mu, \lambda)$.

[1]Specifically, $\mathbb{P}_{\mathsf{CNF}}$ and many other variants of Namba forcing in which the trees have height ω have the property that they preserve stationary subsets of ω_1. A careful and detailed proof for one variant appears in Krueger [14].

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Laver's original proof of Theorem 2.5 is unpublished, but a proof is provided in full by Jech [10, Theorem 3*]. The argument is similar to the one found by Galvin, Jech, and Magidor for obtaining a certain precipitous ideal on $\aleph_2$ [8]. Some variations appear in Shelah [18, Chapter X].

Now we will define a "tall" augmented version of Namba forcing.

Definition 2.6. Assume that $\kappa \leq \lambda < \mu$ are regular cardinals. Assume $\mathsf{LIP}(\mu, \lambda)$ holds and is witnessed by an ideal I and a dense set $D \subseteq I^+$. Let $\mathbb{P}^\kappa_{\mathsf{TANF}}(D)$ consist of subsets $p \subseteq {}^{<\kappa}\mu$ such that:

1. p is a tree,
2. if t is a splitting node in p then $\operatorname{osucc}_p(t) \in D$,
3. for all $t \in p$, there is some $s \sqsupset t$ such that s is a splitting node in p,
4. for all $\sqsubseteq$-increasing sequences of nodes $\langle t_i : i < j\rangle \subset p$ with $j < \kappa$, if $t^* = \bigcup_{i<j} t_i$, then
   (a) $t^* \in p$,
   (b) and if each t_i is a splitting node in p, then t^* is a splitting node in p.

For $p, q \in \mathbb{P}^\kappa_{\mathsf{TANF}}(D)$, let $p \leq_{\mathbb{P}^\kappa_{\mathsf{TANF}}(D)} q$ if and only if $p \subseteq q$.

In other words, the conditions in $\mathbb{P}^\kappa_{\mathsf{TANF}}(D)$ are Miller-style perfect trees of height κ and with club-wise vertical splitting and horizontal splitting sets in D.

Variants of this definition are found throughout the literature starting with work of Kanamori [13]. Our presentation is chosen to correspond with analogous examples in the literature (e.g., [1, Definition 74]).

We will develop $\mathbb{P}^\kappa_{\mathsf{TANF}}(D)$ in this section. Many of its properties generalize those of classical Namba forcing, but Theorem 2.11 is a delicate point. For the remainder of this section, let D witness $\mathsf{LIP}(\mu, \lambda)$ with respect to an ideal I and let $\mathbb{P} = \mathbb{P}^\kappa_{\mathsf{TANF}}(D)$.

Proposition 2.7. *$\mathbb{P}$ is κ-closed. In particular, if $\langle p_i : i < \tau\rangle$ is a $\leq_{\mathbb{P}}$-descending sequence of conditions in $\mathbb{P}$, then $\bigcap_{i<\tau} p_i \in \mathbb{P}$.*

Proof. Let $\tau < \kappa$ and suppose $\langle p_i : i < \tau\rangle$ is a descending sequence of conditions in $\mathbb{P}$. Let $p_* := \bigcap_{i<\tau} p_i$.

Claim 2.8. *For all $t \in p_*$, there is some $s \sqsupseteq t$ such that* $\operatorname{osucc}_{p_*}(s) \in D$.

Proof of Theorem 2.8. First, we need to argue the subclaim that for all $i < \tau$, if $t \in p_*$, then there is some $s \sqsupseteq t$ such that $s \in p_*$ and $s \in \operatorname{split}(p_i)$. Begin with $t \in p_*$. If $t \in \operatorname{split}(p_i)$, then we are done, so assume otherwise. Let s be $\sqsubseteq$-minimal such that $s \sqsupseteq t$ and $s \in \operatorname{split}(p_i)$.

We will argue that $s \in p_*$. Let $\alpha = \operatorname{dom}(t)$ and let $\gamma = \operatorname{dom}(s)$. We will argue by induction on $\beta \in [\alpha, \gamma]$ that $s \restriction \beta \in p_*$. For $\beta = \alpha$ this follows from $t \in p_*$. Suppose $\beta = \beta' + 1$ and we have established that $s \restriction \beta' \in p_*$. Since $\beta' < \gamma$, the minimal choice of s implies that $s \restriction \beta' \notin \operatorname{split} p_i$, which implies that $s \restriction \beta' \notin \operatorname{split} p_j$ for $j \geq i$. This means that for all j such that $i \leq j < \tau$, there is a unique $s_j \in \operatorname{succ}_{p_j}(s \restriction \beta')$. Since the sequence of p_j's is $\subseteq$-decreasing, it must be the case that the s_j's are all the same and therefore that they are all equal to $s \restriction \beta$. Thus $s \restriction \beta \in p_j$ for $j \geq i$. Of course, $s \restriction \beta \in p_j$ for $j < i$ since $p_i \subseteq p_j$ for $j < i$. Thus $s \restriction \beta \in p_*$. Suppose β is a limit and we have established that $s \restriction \beta' \in p_*$ for $\beta' < \beta$, that is, $s \restriction \beta' \in p_i$ for all $i < \tau$. Then $s \restriction \beta \in p_i$ for all $i < \tau$ by Item (4a) of Theorem 2.6, so $s \restriction \beta \in p_*$. This finishes the proof of the subclaim.

Now we can finish proving the claim. Fix $t \in p_*$. We will build a $\sqsubseteq$-increasing sequence $\langle t_i : i < \tau\rangle \subseteq p_*$ with $t_0 = t$ such that for all $i < \tau$, $t_i \in \bigcap_{j<i} \operatorname{split}(p_j)$. If t_i has been defined, apply the subclaim to find $t_{i+1} \sqsupseteq t_i$ with $t_{i+1} \in p_*$ such that $t_{i+1} \in \operatorname{split}(p_i)$ and so $t_{i+1} \in \operatorname{split}(p_j)$ for $j \leq i$. If i is a limit and t_j has been defined for $j < i$, let $t_i = \bigcup_{j<i} t_j$. Then $t_i \in p_*$ by Item (4a) of Theorem 2.6. Also, for all $j < i$ and $\ell \in (j, i)$, $t_\ell \in \operatorname{split}(p_j)$, so Item (4b) of Theorem 2.6 implies that $t_i \in \operatorname{split}(p_j)$. Having defined $\langle t_i : i < \tau\rangle$, let $s = \bigcup_{i<\tau} t_i$. Then, as in the limit case of the construction of $\langle t_i : i < \tau\rangle$, we have $s \in p_*$ and $s \in \operatorname{split}(p_i)$ for all $i < \tau$. Let $X = \bigcap_{i<\tau} \operatorname{osucc}_{p_i}(s)$. Then $X \in D$ by the closure property of $\mathsf{LIP}(\mu, \lambda)$ and $\{s^\frown\langle\alpha\rangle : \alpha \in X\} \subseteq p_*$, so we are done. □

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We are ready to argue that $p_* = \bigcap_{i<\tau} p_i$ is a condition in $\mathbb{P}$. The fact that p_* is a tree follows immediately from the fact that the p_i's are. If t is a splitting node in p_*, then it is necessarily a splitting node for all of the p_i's, and so the closure property of $\mathsf{LIP}(\mu,\lambda)$ implies that $\mathrm{osucc}_{p_*}(t) = \bigcap_{i<\tau} \mathrm{osucc}_{p_i}(t) \in D$. Item (3) from Theorem 2.6, which is about extending nodes to splitting nodes, is exactly the statement of Theorem 2.8. Item (4) of Theorem 2.6, about closure of splitting nodes and closure of nodes in general, is inherited by p_* from the p_i's. □

Proposition 2.9. *Suppose that $\langle p_i : i < \kappa\rangle$ is a fusion sequence. Then $\bigcap_{i<\kappa} p_i$ is a condition in $\mathbb{P}$.*

Proof. Let $p_* = \bigcap_{i<\kappa} p_i$. Item (1) of Theorem 2.6 holds for p_* automatically.

We obtain Item (2) if we argue that for all $t \in \mathrm{split}_\alpha(p_*)$, it follows that $\mathrm{osucc}_{p_*}(t) = \mathrm{osucc}_{p_{\alpha+1}}(t)$. First note that for all $\alpha < \kappa$ and all $\gamma \in (\alpha,\kappa)$, $\mathrm{split}_\alpha(p_\alpha) = \mathrm{split}_\alpha(p_\gamma)$, and hence for all $\alpha < \kappa$, $\mathrm{split}_\alpha(p_\alpha) = \mathrm{split}_\alpha(p_*)$. Now suppose that $t \in \mathrm{split}_\alpha(p_{\alpha+1})$ and $\beta \in \mathrm{osucc}_{p_{\alpha+1}}(t)$. Then there is $s \sqsupseteq t^\frown\langle\beta\rangle$ such that $s \in \mathrm{split}_{\alpha+1}(p_{\alpha+1}) = \mathrm{split}_{\alpha+1}(p_*)$, and therefore $\beta \in \mathrm{osucc}_{p_*}(t)$.

Moreover, $\mathrm{split}_\alpha(p_*) = \mathrm{split}_\alpha(p_\alpha)$ implies that $\mathrm{split}_\alpha(p_*)$ is nonempty above every $t \in \bigcup_{\beta<\alpha} \mathrm{split}_\beta(p_*)$ for all $\alpha < \kappa$, giving Item (3). Item (4a) is automatic and Item (4b) follows from the observation we used for Item (2). □

Proposition 2.10. $\Vdash_{\mathbb{P}}$ "$\mathrm{cf}(\mu) = \kappa$".

Proof. We will argue that

$$\dot{b} = \{\langle i, (\mathrm{stem}\, p)(i)\rangle^\vee : \mathrm{dom}(\mathrm{stem}\, p) \geq i+1\},$$

that is, the name for the generic branch added by $\mathbb{P}$, is a cofinal function from κ to μ. Since Theorem 2.7 implies that $\Vdash_{\mathbb{P}}$ "$\mathrm{cf}(\mu) \geq \kappa$", it will follow that $\Vdash_{\mathbb{P}}$ "$\mathrm{cf}(\mu) = \kappa$".

Suppose $p \in \mathbb{P}$ and let $\beta < \mu$ be arbitrary. Since the Laver ideal I extends the bounded ideal on μ, it follows that $(\mathrm{osucc}_p(\mathrm{stem}\, p) \setminus (\beta+1)) \in I^+$, and therefore there is some $X \in D$ such that $X \subseteq \mathrm{osucc}_p(\mathrm{stem}\, p) \setminus (\beta+1)$. Let $q = \bigcup\{p \restriction (\mathrm{stem}\, p^\frown\langle\alpha\rangle) : \alpha \in X\}$. Then $q \in \mathbb{P}$, $q \leq p$, and $q \Vdash$ "$\dot{b}(i) > \beta$" where $i = \mathrm{dom}(\mathrm{stem}\, p)$. Hence we have argued that $\dot{b}$ is forced to be cofinal in μ. □

The following is our main lemma. The crux is the sweeping argument in Theorem 2.12.

Lemma 2.11. $\mathbb{P}$ *is* $(\mathrm{cf}(\mu) = \kappa)$*-minimal.*

Proof. Suppose that $\dot{f}$ is a $\mathbb{P}$-name forced by the empty condition to be a cofinal function $\kappa \to \mu$.

We define the main idea of the proof presently. Let $\varphi(q,i)$ denote the formula

$$\begin{aligned} & i < \kappa \wedge q \in \mathbb{P} \wedge \exists\langle a_\alpha : \alpha \in \mathrm{osucc}_q(\mathrm{stem}(q))\rangle \text{ s.t.} \\ & \quad \forall\alpha \in \mathrm{osucc}_q(\mathrm{stem}(q)), q \restriction (\mathrm{stem}(q)^\frown\langle\alpha\rangle) \Vdash \text{`}\dot{f} \restriction i = a_\alpha\text{'} \wedge \\ & \quad \forall\alpha,\beta \in \mathrm{osucc}_q(\mathrm{stem}(q)), \alpha \neq \beta \implies a_\alpha \neq a_\beta. \end{aligned}$$

Claim 2.12. $\forall j < \kappa, \forall p \in \mathbb{P}, \exists i \in (j,\kappa), \exists q \leq p$ *s.t.* $\mathrm{stem}(p) = \mathrm{stem}(q) \wedge \varphi(q,i)$.

Proof. (Note that by κ-closure, $\mathbb{P}$ forces "$\dot{f} \restriction i \in V$" for all $i < \kappa$.)

First we establish a slightly weaker claim: for all splitting nodes $t \in p$ and all $j < \kappa$, there is a sequence $\langle(q_\alpha, i_\alpha, a_\alpha) : \alpha \in \mathrm{osucc}_p(t)\rangle$ such that:

1. $\forall\alpha \in \mathrm{osucc}_p(t), \exists i_\alpha \in (j,\kappa), \exists q_\alpha \leq p \restriction (t^\frown\langle\alpha\rangle)$, and $q_\alpha \Vdash$ "$\dot{f} \restriction i_\alpha = a_\alpha$",
2. $\alpha \neq \beta \implies a_\alpha \neq a_\beta$.

We define this sequence by induction on $\alpha \in \mathrm{osucc}_p(t)$. Suppose we have $\langle(q_\beta, i_\beta, a_\beta) : \beta \in \alpha \cap \mathrm{osucc}_p(t)\rangle$ such that (i) and (ii) hold below α. Then we can argue that there is a triple (r,i,a) such that $r \leq p \restriction (t^\frown\langle\alpha\rangle)$ and $r \Vdash$ "$\dot{f} \restriction i = a$" and $a \notin \{a_\beta : \beta \in \alpha \cap \mathrm{osucc}_p(t)\}$. If not, this means that

$$p \restriction (t^\frown\langle\alpha\rangle) \Vdash \text{“}\{\dot{f} \restriction i : i < \kappa\} \subseteq \{a_\beta : \beta \in \alpha \cap \mathrm{osucc}_p(t)\}\text{”}.$$

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But each a_β has cardinality less than κ and $|\alpha \cap \mathrm{osucc}_p(t)| < \mu$, so

$$\left|\bigcup\{a_\beta : \beta \in \alpha \cap \mathrm{osucc}_p(t)\}\right| < \mu$$

Therefore $p \restriction (t^\frown\langle\alpha\rangle)$ forces that range$(\dot{f})$ is bounded in μ, which contradicts the assumption that $\dot{f}$ is forced to be unbounded in μ. Since the triple that we want exists, we can let $(q_\alpha, i_\alpha, a_\alpha)$ be such a triple.

Now that we have established the slightly weaker claim, apply the μ-completeness of I and the fact that $\kappa < \mu$ to find some $S' \subseteq \mathrm{osucc}_p(\mathrm{stem}\, p)$ such that $S' \in I^+$ and such that there is some i with $i_\alpha = i$ for all $\alpha \in S'$. Then choose $S \subseteq S'$ such that $S \in D$ using the density property indicated by $\mathsf{LIP}(\mu, \lambda)$ and let $q = \bigcup_{\alpha \in S} q_\alpha$. □

Now that we have our claim, we can use it to construct a fusion sequence $\langle p_\xi : \xi < \kappa\rangle$ and assignment $\{(t, i_t) : t \in T\}$ where $T := \bigcup\{\mathrm{split}_\xi(p_\xi) : \xi < \kappa\}$ such that:

1. for all $t \in T$, $i_t < \kappa$,
2. for all $t \in T$, $i_t > \mathrm{dom}(t) \geq \sup_{s \sqsubset t} i_s$,
3. for each $\xi < \kappa$ and $t \in \mathrm{split}_\xi(p_\xi)$, there is some $i > \sup\{i_s : s \sqsubset t, s \in T\}$ with $i \geq \mathrm{dom}(t)$ such that $\varphi(p_{\xi+1} \restriction t, i)$ holds.

We define the fusion sequence by cases: Let p_0 be arbitrary. If $\xi < \kappa$ is a limit, then we let $p_\xi = \bigcap_{\zeta<\xi} p_j$ (using Theorem 2.7). Now suppose that $\xi = \zeta + 1$ and we have defined p_ζ. Let $t \in \mathrm{split}_\zeta(p_\zeta)$. Apply Theorem 2.12 to obtain $q_t \leq p_\zeta \restriction t$ with stem $q = t$ and some $i_t > \mathrm{dom}(t) \cup \sup\{i_s : s \sqsubset t, s \in T\}$ such that $\varphi(q_t, i_t)$. Then let $p_\xi = \bigcup\{q_t : t \in \mathrm{split}_\zeta(p_\zeta)\}$. Finally, having defined the fusion sequence, we let $p = \bigcap_{\xi<\kappa} p_\xi$.

Now we argue that $p \Vdash$ "$\Gamma(\mathbb{P}) \in V[\dot{f}]$". Let $g = \dot{f}[G]$ for some G that is $\mathbb{P}$-generic over V. We will argue that G is definable from g. Specifically, we will define a sequence $\langle s_\xi : \xi < \kappa\rangle$ such that for all $\xi < \kappa$, $\mathrm{dom}(s_\xi) \geq \xi$ and $\exists q_\xi \in G$ such that $s_\xi \sqsubseteq \mathrm{stem}(q_\xi)$. Let $s_0 = \emptyset$. Given s_ξ, let $s^*_{\xi+1}$ be $\sqsubseteq$-minimal splitting node of p above s_ξ. Then by Item (3), there is a unique $\alpha_{\xi+1}$ such that

$$p_\xi \restriction s^*_{\xi+1}{}^\frown\langle\alpha_{\xi+1}\rangle \Vdash \text{“}\dot{f} \restriction i_{s^*_{\xi+1}} = g \restriction i_{s^*_{\xi+1}}\text{”}.$$

Let $s_{\xi+1} = s^*_{\xi+1}{}^\frown\langle\alpha_{\xi+1}\rangle$. The Item (3) also implies that there is some $q_{\xi+1} \in G$ such that $\mathrm{stem}(q_{\xi+1}) \sqsupseteq s_{\xi+1}$. If ξ is a limit, let $s_\xi = \bigcup_{\eta<\xi} s_\eta$. By the closure property of Theorem 2.6, $s_\xi \in p$.

This completes the proof of minimality. □

Proposition 2.13. *If $\mu^{<\kappa} = \mu$, then $\mathbb{P}$ does not add surjections from κ to θ for any regular $\theta > \mu$.*

Proof. Suppose that we have a $\mathbb{P}$-name $\dot{f}$ for a function such that (without loss of generality) the empty condition forces $\dot{f} : \kappa \longrightarrow \theta$. We will define a fusion sequence $\langle p_i : i < \kappa\rangle$ as follows: Let p_0 be arbitrary. If i is a limit, then let $p_i = \bigcap_{j<i} p_j$. If $i = k + 1$, then for all $t \in \mathrm{split}_k(p_k)$ and $\alpha \in \mathrm{osucc}_{p_k}(\alpha)$, choose some $q_{t,\alpha} \leq p_k \restriction (t^\frown\langle\alpha\rangle)$ deciding $\dot{f}(k)$. Then let $p_i = \bigcup\{q_{t,\alpha} : t \in \mathrm{split}_k(p_k), \alpha \in \mathrm{osucc}_{p_k}(t)\}$. Let $p = \bigcap_{i<\kappa} p_i$.

If we let

$$B = \{\delta < \theta : \exists i < \lambda, \exists t \in \mathrm{split}_i(p), \exists\alpha \in \mathrm{osucc}_p(t), q_{t,\alpha} \Vdash \text{“}\dot{f}(i) = \delta\text{”}\},$$

then because $|B| \leq |p| = \mu^{<\kappa} = \mu < \theta$, it follows that $p \Vdash$ "range$(\dot{f}) \subseteq \sup(B) < \theta$". □

Now we are in a position to answer the question of Bukovský and Copláková-Hartová that was mentioned in the introduction.

Theorem 2.14. *Assuming the consistency of a measurable cardinal, there is a model V such that there is an $|\aleph_2^V| = \aleph_1$-minimal extension $W \supset V$ that is not an $|\aleph_3^V| = \aleph_1$-extension.*

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Proof. Let V' be a model in which μ is a measurable cardinal, and let $V \supset V'$ be obtained by forcing with the Lévy collapse $\mathrm{Col}(\aleph_1, < \mu)$ over V'. Then CH and $\mathsf{LIP}(\aleph_2, \aleph_1)$ hold in V (the latter by Theorem 2.5). Suppose that $\mathsf{LIP}(\aleph_2, \aleph_1)$ is witnessed by the dense set D_2 in V. Then let W be an extension of V by $\mathbb{P}^{\aleph_1}_{\mathsf{TANF}}(D_2)$. Then W is an $|\aleph_2^V| = \aleph_1$-extension by Theorem 2.10 and it is a minimal such extension by Theorem 2.11. If $\mathbb{P}^{\aleph_1}_{\mathsf{TANF}}(D_2)$ collapses $\aleph_3^V$, then it would collapse it to an ordinal of cardinality $\aleph_1^V$, but it follows from Theorem 2.13 in combination with CH that this is not possible. □

We also mention the immediate generalization of Theorem 2.14:

Theorem 2.15. *Assume that in V, ν_1, ν_2, ν_3 are cardinals with ν_1 regular, that $\nu_3 = \nu_2^+ = \nu_1^{++}$, and that* $\mathsf{LIP}(\nu_2, \nu_1)$ *and* $(\nu_1)^{<\nu_1} = \nu_1$ *hold. Then there is a $|\nu_2| = \nu_1$-minimal extension $W \supset V$ that is not a $|\nu_3| = \nu_1$-extension.*

2.3. Remaining questions

As stated in the introduction, it would be clarifying to know for sure whether there is an exact equiconsistency.

Question 2.16. Does the conclusion of Theorem 2.14 require consistency of a measurable cardinal?

There is also the question of the extent to which Theorem 2.14 can be stratified.

Question 2.17. Assuming $\mathsf{LIP}(\mu, \lambda)$, is it consistent that $\omega < \kappa < \lambda < \mu$ are regular cardinals and $\mathbb{P}^{\kappa}_{\mathsf{TANF}}$ preserves cardinals $\nu \leq \lambda$?

This question appears to rely heavily on the determinacy of the generalizations of Namba-style games to uncountable length κ (see, e.g., [18, Chapter XI] [7], [5, Fact 5]). One could pose this question in terms of (κ, ν)-distributivity, but even some tricks that allow one to merely obtain cardinal preservation from similar posets (see [15, Theorem 3]) seem to depend on these types of games (see [5, Fact 1]).

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