17
D-branes and black holes

We've seen now many examples of the ways in which D-branes can be used as probes of the non-perturbative structure of string theory, with remarkable insights, including the one that string theory is not really a theory of strings beyond perturbation theory. It should not be forgotten that strings also have the intriguing feature that they insist on describing (at least) a perturbative quantum gravity. It is considerably significant that we can get insight into string theory's non-perturbative treatment of certain questions in quantum gravity, again using D-branes to probe and model the physics of black holes. This chapter will lay the foundations for how this works.

17.1 Black hole thermodynamics

17.1.1 The path integral and the Euclidean calculus

In an attempt to construct a path integral definition of quantum gravity, one might envision the following:

$$Z = \int \mathcal{D}[g,\varphi] e^{iI[g,\varphi]}, \tag{17.1}$$

for some appropriate choice of integration measure $\mathcal{D}[g,\varphi]$ over the metric g and matter fields φ. In the early days of studying the path integral for gravity, it was noticed that the gravity action for some region $\mathcal{M}$ should be supplemented by a term evaluated on its boundary $\partial\mathcal{M}$ which allows the contribution of variations which include configurations which vanish on $\partial\mathcal{M}$, but which might have non-vanishing normal derivatives on it. The result is (in units where $G_{\rm N} = 1$):

$$I = \frac{1}{16\pi}\int_{\mathcal{M}} \sqrt{-g}\; R\; d^D x + \frac{1}{8\pi}\int_{\partial\mathcal{M}} \sqrt{-h}\; K\; d^{D-1}x, \tag{17.2}$$

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where $h_{\mu\nu}$ is the induced metric on the boundary, and K is the trace of the extrinsic curvature of the boundary. (We learned how to compute these quantities in insert 10.2.) This term is required so that upon variation with metric fixed at the boundary, the action yields the Einstein equations.

Since I is real, there is the problem that the path integral has convergence problems, since the integral is in principle oscillatory. One way this is made sense of to 'Wick rotate' the time axis by 90° by the substitution $t \to -it$, and so the path integral becomes:

$$Z = \int \mathcal{D}[g, \varphi] e^{-I^{\mathrm{E}}[g,\varphi]}, \tag{17.3}$$

where $I^{\mathrm{E}} = -iI$ is the Euclidean action, which is real for real fields, and now the integrand is seen to be a damped exponential, which improves convergence. The metric has gone from signature $(-++\cdots+)$ to signature $(+++\cdots+)$. In principle, we can evaluate our path integral on the Euclidean section and then rotate back to Lorentzian signature.

The Euclidean technology allows for the definition of the canonical thermodynamical ensemble as well. Let us see how this works. The amplitude to go from a configuration (g_1, φ_1) at time t_1 to a configuration (g_2, φ_2) at time t_2 is:

$$\langle (g_2, \varphi_2), t_2 | (g_1, \varphi_1), t_1 \rangle = \int \mathcal{D}[g, \varphi] e^{iI[g,\varphi]}.$$

This quantity has another representation, in the Schrödinger picture:

$$\langle (g_2, \varphi_2) | e^{-iHt_2} e^{iHt_1} (g_1, \varphi_1) \rangle = \langle (g_2, \varphi_2) | e^{-iH(t_2-t_1)} (g_1, \varphi_1) \rangle.$$

Let us study the situation that $(g_1, \varphi_1) = (g_2, \varphi_2)$. Writing $t_2 - t_1 = -i\beta$, and summing over a complete set of eigenstates (ψ_n, E_n) of the Hamiltonian, we get the partition function:

$$Z = \sum_n e^{-\beta E_n}. \tag{17.4}$$

The system is at temperature $T = \beta^{-1}$, and we have the standard expression for the probability, p_n, of being in the nth state:

$$p_n = \frac{1}{Z} e^{-\beta E_n}.$$

The familiar representation given in equation (17.4) represents the same system represented by the Euclidean path integral given in equation(17.3), where the fields (g, φ) are periodic in τ with period β. We shall see how to extract other physical quantities from here a little later.

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17.1.2 The semiclassical approximation

The evaluation of the entire path integral will not concern us here, since as string theorists, we take a rather different approach to the problem of quantum gravity. However, we expect from the reasoning that we have used many times already that we will arrive at a low energy action of the sort we studied above, regardless of the underlying microscopic model. So in fact, when we come to examine the macroscopic predictions of the microscopic details of our particular approach to fundamental physics (string and M-theory) – or any other approach, for that matter – they should make contact with the semiclassical results to be derived from the action above.

The expectation is that the configurations with the most dominant contribution to the path integral will be those which are near an extremum of the action, i.e. solutions to the equations of motion. This of course fits with our intuition about how the classical limit arises from the path integral approach.

In this limit, the path integral becomes

$$Z = e^{-I^{\rm E}} \equiv e^{-\beta W},$$

defining the thermodynamic (effective) potential W, which is

$$W = E - TS, \tag{17.5}$$

where T is the temperature and S is the entropy of the system. We can easily extract useful information in this limit. For example, the average energy of the system would be quite reasonably defined as the normalised quantity

$$\langle E\rangle = \frac{1}{Z}\sum_n E_n e^{-\beta E_n} = -\frac{1}{Z}\frac{\partial Z}{\partial\beta} = -\frac{\partial \log Z}{\partial\beta} = \frac{\partial I^{\rm E}}{\partial\beta}. \tag{17.6}$$

Another example of some importance is the entropy. This is defined in terms of the occupation probability p_n as:

$$\begin{aligned} S &= -\sum_n p_n \log p_n = -\frac{1}{Z}\sum_n e^{-\beta E_n}\log\left(\frac{e^{-\beta E_n}}{Z}\right)\\ &= -\frac{1}{Z}\sum_n e^{-\beta E_n}\left(-\beta E_n - \log Z\right)\\ &= \beta\frac{\partial I^{\rm E}}{\partial\beta} + \log Z = \beta\langle E\rangle - I^{\rm E}. \end{aligned} \tag{17.7}$$

The approximation will allow us to extract a number of key features of the physics. For example, the contribution of the fields φ to the effective

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action of quantum fields on various curved spacetime backgrounds will be sensitive to various features of the background and the properties of the fields themselves. Meanwhile, in the purely gravitational sector, we will find that there are dramatic effects which arise in our computations due to the non-trivial interplay of topology of the Euclidean section with the path integral[288]. An example of an immediate consequence of this is the result that black holes have an intrinsic temperature. Let us compute this for the Schwarzschild and Reissner–Nordström solutions to see how it works, since the computation in this framework is surprisingly straightforward.

17.1.3 The temperature of black holes

We begin with the Schwarzschild and Reissner–Nordström solutions which we met in given in chapter 10, and as we were instructed in the previous section, we continue the solution to Euclidean signature via $t \to -i\tau$, with period β for τ:

$$ds^2 = V d\tau^2 + V^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \qquad (17.8)$$

with

$$V = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right).$$

This solution is taken as making sense in the range $r_+ \leq r \leq \infty$, where $r_+ = M + \sqrt{M^2 - Q^2}$. Now the neighbourhood of $r = r_+$ (what was the horizon) is trying to look like $\mathbb{R}^2 \times S^2$, but sadly, there is a conical singularity there, because the coordinates (r, τ), trying to look like polar coordinates in the plane, have the wrong periodicity for τ for arbitrary β.

In fact, the problem of computing the temperature reduces to the matter of removing this 'bolt singularity'[83, 82], ensuring the 'regularity of the Euclidean section'. This is quite easy to do: one has to make sure that the infinitesimal ratio of the circumference (going around in τ) to the radius (moving in r), is in fact 2π as one approaches the origin of $\mathbb{R}^2$, which is $r = r_+ = 2M$. This boils down to:

$$2\pi = \lim_{r \to r_+} \frac{\Delta\tau}{V^{-1/2}} \frac{d(V^{1/2})}{dr} \qquad \Longrightarrow \qquad \frac{4\pi}{\beta} = V'|_{r=r_+},$$

where $\Delta\tau = \beta = 1/T$. We then add a point (equivalent to a whole S^2) to repair $r = r_+$. From this we get:

$$\frac{1}{\beta} = T = \frac{Mr_+ - Q^2}{2\pi r_+^3} = \frac{\sqrt{M^2 - Q^2}}{4\pi M(M + \sqrt{M^2 - Q^2}) - 2\pi Q^2}, \qquad (17.9)$$

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and for the case of $Q = 0$ (Schwarzschild), we have

$$T = \frac{1}{8\pi M},$$

which shows that large black holes are actually quite cold, and it is small black holes which are hot. This is actually a good thing for consistency with what we have already observed, since it means that astrophysical black holes (especially the really big ones apparently indirectly detected out there at the cores of galaxies, but even stellar-sized ones) have negligible mass loss due to this sort of radiation*. In fact, this means that asymptotically flat black holes (i.e. the sort we've been studying so far) have negative specific heat, since reducing the energy of the system (mass) increases the rate at which it is lost.

Notice furthermore that for the charged black hole, the temperature vanishes at extremality, since there $r_+ = Q = M$. This fits rather well with what we have learned previously: the extremal solution is supersymmetric and in fact a BPS state, and so zero temperature is consistent with its stability. In addition, we see that the thermodynamics protects the censorship idea, since it cannot radiate further mass away, making a sub-extremal object with a naked singularity.

In fact, the temperature can be related to a purely geometrical quantity known as the *surface gravity*, κ, of a black hole, which is a purely geometric quantity that exists at the horizon, and (crucially) is constant all over it[292]. If we had a test particle in the geometry connected to an observer at infinity by a long (light) string, the surface gravity is in fact the acceleration needed to hold the particle stationary at the horizon. It can be defined in terms of a Killing vector χ normal to the horizon:

$$\kappa^2 = -\tfrac{1}{2}(\nabla^\mu \chi^\nu)(\nabla_\mu \chi_\nu)\,|_{r=r_+}\,, \tag{17.10}$$

where we perform the evaluation at the horizon.

For our solution, we have that $\chi^\mu = \xi^\mu = \delta^\mu_t$, and from the list of the non-vanishing components of the affine connection given in equation (10.5), we can compute the only non-zero component of the covariant derivative:

$$\nabla_r \chi_t = \partial_r \chi_t - \Gamma^t_{rt}\chi_t = \frac{-2Mr + 2Q^2}{r^3} + \frac{Mr - Q^2}{r^3} = \frac{-Mr + Q^2}{r^3},$$

which gives

$$\kappa = \frac{Mr_+ - Q^2}{r_+^3},$$

* The reader can multiply by $\hbar c^3/Gk_{\mathrm{B}}$ in order to restore the physical units.

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and so we have:

$$T = \frac{\kappa}{2\pi}. \tag{17.11}$$

17.2 The Euclidean action calculus

The action is usually evaluated by computing with what is called the 'Euclidean section' of the spacetime, which arose in the previous sections. Since this removes the singularities from the integrand, it makes the integration procedure sensible[288, 290]. Furthermore, for asymptotically (locally) flat spacetimes, the action is interpreted as computed with reference to an appropriate background in order to give a finite answer. Later, we will see a different prescription in the context of asymptotically anti-de Sitter spacetimes, which allows for a computation of the action which does not require reference to another spacetime. Let us compute an example with the present methods to get used to how they work.

17.2.1 The action for Schwarzschild

The Schwarzschild spacetime is asymptotically flat, and so we can compute the action by using flat spacetime as a reference background. For both spacetimes, the Ricci scalar $R = 0$ and so the second part of the action is where we must concentrate our efforts.

We must evaluate the extrinsic curvature for both spacetimes. Let us pick for our boundary the spherical shell at $r = R$. The unit outward normal to this is (see insert 10.2, p. 229):

$$n^\mu = \frac{\delta^\mu_r}{\sqrt{G_{rr}}} = \frac{1}{\sqrt{G_{rr}}}\left(\frac{\partial}{\partial r}\right)^\mu .$$

The extrinsic curvature is

$$K_{\mu\nu} = \frac{1}{2} n_\alpha G^{\alpha\beta}\partial_\beta G_{\mu\nu},$$

which gives non-zero components

$$K_{tt} = -\frac{1}{2}\frac{1}{\left(1-\frac{2M}{r}\right)^{1/2}}\left(1-\frac{2M}{r}\right)\frac{2M}{r^2};$$

$$K_{\theta\theta} = \frac{1}{2}\frac{1}{\left(1-\frac{2M}{r}\right)^{1/2}}\left(1-\frac{2M}{r}\right)2r;$$

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$$K_{\theta\theta} = \frac{1}{2}\frac{1}{\left(1-\frac{2M}{r}\right)^{1/2}}\left(1-\frac{2M}{r}\right)2r\sin^2\theta;$$
$$K = G^{\mu\nu}K_{\mu\nu} = \frac{1}{\left(1-\frac{2M}{r}\right)^{1/2}r^2}\left[2r-3M\right], \tag{17.12}$$

and by setting $M = 0$ we get the result $K = 2/r$ for Minkowski space. The measure for integration on the boundary is

$$\sqrt{h} = r^2\left(1-\frac{2M}{r}\right)^{1/2}\sin\theta$$

and recall that the period of the imaginary time is $\Delta\tau = \beta$. So for Schwarzschild we have

$$\int\sqrt{h}\,Kd^3x = \beta 4\pi(2r-3M).$$

For Minkowski, we must be more careful. The measure is $\sqrt{h} = r^2\sin\theta$, and $K = 2/r$, but we must choose our temperature carefully. Since Minkowski is regular for any period of τ, the temperature is arbitrary, and so we must fix it to match the Schwarzschild temperature. At radius r, the temperature is not β, but it is red shifted to $\beta\left(1-2M/r\right)^{1/2}$, so that is what we should use for the result of integrating over the compact time, with the result:

$$\int\sqrt{h}\,Kd^3x = \beta 4\pi 2r\left(1-\frac{2M}{r}\right)^{1/2},$$

and so the action difference in the limit $R\to\infty$ is

$$I^{\mathrm{E}} = \beta\frac{M}{2}. \tag{17.13}$$

Let us see that we can in fact extract useful information from this result. First, we note that M is a function of β ($M = \beta/8\pi$) and so we should be careful when differentiating with respect to β. A computation of the energy, using the formula (17.6) gives:

$$\langle E\rangle = \frac{M}{2}+\frac{\beta}{2}\frac{\partial M}{\partial\beta} = M,$$

which is an extremely intuitive result. We have seen that the system has a temperature, and so we should expect to compute a non-zero 'Bekenstein–Hawking'[262, 261] entropy, using equation (17.7):

$$S = \beta M - \frac{\beta M}{2} = \frac{8\pi M^2}{2} = \frac{\mathcal{A}}{4},$$

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where $\mathcal{A}$ is the area of the black hole's horizon. So we see that these results combine nicely to confirm the expression for the thermodynamic potential

$$W \equiv \frac{I}{\beta} = \frac{M}{2} = M - TS.$$

17.2.2 The action for Reissner–Nordström

A similar computation of the gravity action can be done for the charged black hole, and in fact, the result is the same as in equation (17.13), with β now from the expression given in (17.9), which should be obvious to the reader who followed the computations. The term in the metric containing Q is subleading in a $1/r$ expansion. Now the action needs to be supplemented by a contribution from the Maxwell term, which can be manipulated into a boundary term, assuming that the equations of motion are obeyed:

$$I_{\mathrm{M}} = -\frac{1}{16\pi}\int_{\mathcal{M}} \sqrt{g} F_{\mu\nu}F^{\mu\nu}\, d^D x = -\frac{1}{8\pi}\int_{\partial\mathcal{M}} A_\mu F^{\mu\nu} dS_\nu, \tag{17.14}$$

where the latter is a boundary integral, and we have used the on-shell condition that $\nabla_\nu F^{\mu\nu} = 0$.

Notice that the usual expression for the gauge potential, written as a one-form $A = A_t dt$, where $A_t = Q/r$, is singular, since the interval dt is infinite at the horizon. We can repair this problem by defining a value for the potential at the horizon, $\Phi = Q/r_+$, and then redefining the potential by a gauge transformation:

$$A = Q\left(\frac{1}{r} - \frac{1}{r_+}\right) dt.$$

Now since the non-zero components of $F_{\mu\nu}$ are just $F_{rt} = -Q/r^2$, the boundary integral for the action is easy to compute, giving, in the limit $R \to \infty$ the result:

$$I_{\mathrm{M}} = -\frac{\beta}{2} Q\Phi.$$

So the total action turns out to be

$$I^{\mathrm{E}} = \frac{\beta}{2}[M - Q\Phi].$$

Again, in the semiclassical limit we can equate this to βW, where the thermodynamic or Gibbs potential is

$$W = M - TS - Q\Phi,$$

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since in the thermodynamic analogy, Φ is like a chemical potential for Q, the analogue of particle number. Now we can use the standard thermodynamic relations to compute:

$$\begin{aligned} E &= \left(\frac{\partial I}{\partial \beta}\right)_{\Phi} - \frac{\Phi}{\beta}\left(\frac{\partial I}{\partial \Phi}\right)_{\beta} = M; \\ S &= \beta\left(\frac{\partial I}{\partial \beta}\right)_{\Phi} - I = \frac{\mathcal{A}}{4}; \\ Q &= -\frac{1}{\beta}\left(\frac{\partial I}{\partial \Phi}\right)_{\beta} = Q, \end{aligned} \qquad (17.15)$$

where $\mathcal{A}$ is the area of the black hole's horizon. These canonical ensemble computations are best performed by working in terms of r_+ as much as possible, converting in the end to, for example, $\partial/\partial\beta = (\partial r_+/\partial\beta)\partial/\partial r_+$, etc.

17.2.3 The laws of thermodynamics

The reality of the thermodynamic behaviour of black holes begun to emerge from considering (among other things) the observation that was made by relativists that an isolated black hole's horizon area, $\mathcal{A}$, cannot be decreased by any physical process[292, 289]. This is, of course, reminiscent of the analogous law for entropy, S, in thermodynamics, where it is called the *Second Law* of thermodynamics.

Combining this with the result that there is in fact a temperature to be associated with black holes, because they are radiating their mass away quantum mechanically leads to the 'Bekenstein–Hawking' relation of the entropy to the area[262, 261], which we computed in two cases above:

$$S = \frac{\mathcal{A}}{4}. \qquad (17.16)$$

In fact, a *First Law* can be formulated for black holes as well,

$$dE = TdS + pdV \quad \Longleftrightarrow \quad dM = \frac{1}{8\pi}\kappa d\mathcal{A} + \Omega_{\rm H} dJ + Qd\Phi,$$

where on the left hand side are the usual quantities from the first law, and on the right are the analogous black hole quantities, the electric charge and potential at the horizon, and the angular velocity at the horizon $\Omega_{\rm H}$ and angular momentum J such as could be computed for a rotating black hole (the Kerr solution).

Additionally, a *Third Law* can be stated[292]. For the Reissner–Nordström black hole, we saw that the extremal case has $T = 0$. However, to achieve such a case starting from finite temperature is intuitively physically impossible since approaching the extremal case would mean opening up the infinite volume spacetime which in chapter 10 was shown to live at the horizon of the extremal black hole.

17.3 $D = 5$ Reissner–Nordström black holes

It is a remarkable and profound fact that black holes obey the laws of thermodynamics, saying that gravity has some underlying structure which has yet to be fully understood. What one needs to find is (as for ordinary thermodynamics) an underlying microscopic description from which these laws arise. This is a big problem with quantum gravity. A universal microscopic description of the required degrees of freedom is not known.

Happily, the modern era has seen remarkable progress. String theory contains a theory of quantum gravity within it which is understood well enough to make progress in at least some of these questions. So far, we have only seen signs of gravity perturbatively, but black holes are firmly in the non-perturbative sector. Now, there are powerful arguments about the behaviour of strings at high energy density which can be followed to strong coupling to achieve a sharp, but qualitative understanding of the quantum behaviour of black holes as described by strings via a 'correspondence principle'[263]. There is marked qualitative agreement with the properties we have uncovered above[7].

However, by the study of a specific but large class of black holes in string theory, it is possible to find a microscopic description of them using D-branes which firmly establishes the precise (including all crucial universal numerical factors) thermodynamic relations we discussed semiclassically above. This is remarkable progress is a good sign that string theory (and M-theory) does indeed show mature signs of having a description of non-perturbative gravity. Let us begin to uncover some aspects of this description.

We shall work with five dimensions, for the simplest example. A five dimensional analogue of the charged black hole solution (10.4) that we already studied somewhat in chapter 10 is:

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2m}{R^2} + \frac{q^2}{R^4}\right) dt^2 + \left(1 - \frac{2m}{R^2} + \frac{q^2}{R^4}\right)^{-1} dR^2 + R^2 d\Omega_3^3, \\ A_t &= \frac{q}{R^2}, \end{aligned} \tag{17.17}$$

where

$$d\Omega_3^2 = d\theta^2 + \sin^2\theta(d\phi^2 + \sin^2\phi\, d\chi^2) \tag{17.18}$$

is the metric on a round three sphere, and $(t, R, \theta, \phi, \chi)$ constitute polar coordinates in the directions $(x^0, x^1, x^2, x^3, x^4)$.

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As before, there is an outer horizon at the largest root of $G^{rr} = 0$:

$$R_{\pm}^2 = m \pm \sqrt{m^2 - q^2},$$

and a singularity at $R = 0$. From our previous discussion, we know that there is a Hawking temperature and Bekenstein–Hawking entropy set by the horizon. We would like to make a link to a microscopic description of the underlying structure of the black hole.

The challenge is therefore to attempt to embed this black hole into string theory in a manner which allows us to use some of the tricks we learned about D-branes to help us study its properties. It is useful to rewrite the hole in isotropic coordinates for this study, since we are going to build the black holes out of branes, and we have presented the supergravity solutions for them in chapter 10 in terms of such coordinates. To do this, let us write $R^2 = r^2 + R_-^2$ for some new radial coordinate r. Since we can write $-G_{tt} = G^{rr}$ as

$$\frac{1}{R^4}(R^2 - R_+^2)(R^2 - R_-^2) = \frac{1}{R^4}(r^2 - (R_+^2 - R_-^2))r^2 = \frac{r^4}{R^4}\left(1 - \frac{r_{\rm H}^2}{r^2}\right),$$

where $r_{\rm H}^2 = R_+^2 - R_-^2 = 2\sqrt{m^2 - q^2}$, we find the following pleasingly simple form:

$$\begin{aligned} ds^2 &= -\mathcal{H}^{-2} f dt^2 + \mathcal{H}\left(f^{-1} dr^2 + r^2 d\Omega_3^2\right), \\ \text{where} \quad f &\equiv 1 - \frac{r_{\rm H}^2}{r^2}, \qquad \mathcal{H} \equiv \frac{R^2}{r^2} = 1 + \frac{R_-^2}{r^2} \\ A_t &= \frac{q}{r^2 + R_-^2}, \\ &= \frac{q}{R_-^2}\left(1 - \mathcal{H}^{-1}\right), \end{aligned} \tag{17.19}$$

where the horizon is at $r = r_{\rm H}$. It has area $\mathcal{A} = 2\pi^2(r_{\rm H}^2 + R_-^2)^{3/2} = 2\pi^2 R_+^3$. The interior region of the black hole containing the singularity is not covered by these coordinates. In the extremal limit where the horizon is degenerate ($m = q$), we get $R_+^2 = R_-^2 = q$ and the solution in the original coordinates is:

$$\begin{aligned} ds^2 &= -\left(1 - \frac{q}{R^2}\right)^2 dt^2 + \left(1 - \frac{q}{R^2}\right)^{-2} dR^2 + R^2 d\Omega_3^2, \\ A_t &= \frac{q}{R^2}, \end{aligned} \tag{17.20}$$

where the horizon is at $R^2 = q$. It has area $\mathcal{A} = 2\pi^2 q^{3/2}$. In isotropic

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coordinates we get simply:

$$\begin{aligned} ds^2 &= -\mathcal{H}_{\rm e}^{-2}dt^2 + \mathcal{H}_{\rm e}\left(dr^2 + r^2 d\Omega_3^2\right), \\ A_t &= \mathcal{H}_{\rm e}^{-1} \\ \text{and} \quad \mathcal{H}_{\rm e} &\equiv 1 + \frac{Q^2}{r^2}, \end{aligned} \tag{17.21}$$

where we write $Q^2 = q$ for later notational convenience. The horizon is now at $r = 0$.

Now comes the fun part. We have to see whether any of the structure of the solution is familiar to us from what we have learned so far. It is encouraging that we get something that looks like the correct type of harmonic function that we would like to come from a brane solution, but we have to achieve a constant dilaton, and see the gauge field arise from either pure metric geometry and/or the R–R sector, if we are to connect it entirely to D-branes.

17.3.1 Making the black hole

The most obvious thing to try would have been the D5-brane solution, wrapped on T^5, which would have given (ignoring the T^5 directions):

$$\begin{aligned} ds^2 &= -H^{-1/4}dt^2 + H^{3/4}\left(dr^2 + r^2 d\Omega_3^2\right), \\ C_t^{(5)} &= H^{-1}; \qquad e^{-\frac{\Phi}{2}} = H^{1/4} \\ \text{where} \qquad H &\equiv 1 + \frac{Q^2}{r^2}. \end{aligned} \tag{17.22}$$

Compare this to the solution (17.21). This comes close in the gauge field, but fails for a number of reasons. The first is that the powers of the function $H = 1 + Q^2/r^2$ are wrong in the parallel and transverse parts of the metric, and the second is that the dilaton is not a constant.

Looking at the transverse part to see what is missing, we observe that we really need an additional $H^{1/4}$. Perhaps we can combine this solution with something which has this behaviour. This behaviour is what we would get if were to attempt to make instead a hole by dimensionally reducing the D1-brane solution (delocalised in four of its transverse directions on a $T^4 \subset T^5$, so that we use r^{-2} and not r^{-6} in H):

$$\begin{aligned} ds^2 &= -H^{-3/4}dt^2 + H^{1/4}\left(dr^2 + r^2 d\Omega_3^2\right), \\ C_t^{(1)} &= H^{-1}; \qquad e^{-\frac{\Phi}{2}} = H^{-1/4} \\ \text{where} \qquad H &\equiv 1 + \frac{Q^2}{r^2}. \end{aligned} \tag{17.23}$$

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Again, this solution on its own would have shortcomings. Notice that the dilaton goes inversely with that of the D5-brane solution, but that the reduced R–R field is again just what we want.

In fact, we can make a solution by combining these two in a manner analogous to that which we saw before in section 15.4, using the harmonic function sum rule to get a solution which has eight supercharges. The harmonic functions in the three sectors (i.e. directions transverse to both, transverse to the smaller, or parallel to both) combine by product. Ignoring the five directions of the T^5 this gives us:

$$\begin{aligned} ds^2 &= -H^{-1}dt^2 + H\left(dr^2 + r^2 d\Omega_3^2\right), \\ C_t^{(1)} &= H^{-1} = C_t^{(5)}; \qquad e^{-\frac{\Phi}{2}} = 1. \end{aligned} \tag{17.24}$$

We could take the diagonal combination of the charge sector as our gauge field (thereby averaging $C^{(1)}$ and $C^{(5)}$ and so summing the charges) and things would be perfect there. So overall, this is very nearly what we want, but it sadly it fails because the power of H in G_{tt} is not correct.

Undaunted, we must search for some new component to the solution which does not modify what we have already got correct for the transverse directions and the dilaton and charge sector, but fixes the problematic power of H in G_{tt}. Switching off the contributions from the branes temporarily, we see that we must have a constant dilaton, and a metric:

$$ds^2 = -H^{-1}dt^2 + dr^2 + r^2 d\Omega_3^2,$$

and we can still possibly allow an electric potential $A_t = H^{-1}$, since we can take a linear combination of it with the other gauge sectors.

One recourse is to appeal to pure geometry. We have only so far been considering a direct reduction on the T^5 by simply ignoring it. We can be considerably more subtle and reduce on it (or part of it) with a Kaluza–Klein twist. This could achieve our modification of the metric without modifying the dilaton, since it would come the pure geometry of the reduction. Recall that we learned from earlier Kaluza–Klein studies in chapter 4 (see also insert 12.1) that we can modify a metric component which is G_{yy}^{D+1} in $D+1$ dimensions by twisting the y direction with, say the x^5 direction along which we do the Kaluza–Klein reduction. The metric component G_{yy}^{D} in the D-dimensional metric is in fact $G_{yy}^{D+1} - G_{55}A_y^2$, and the gauge field $A_y = G_{5y}/G_{55}$. In the present case, our gauge field must be of the form (up to a gauge choice) $A_t = H^{-1}$, and so this fixes for us what we can achieve in the reduction.

N.B. Since the gauge field is electric, it must come from a metric component resulting from a twist of time t with a spatial component and so this is in fact equivalent to giving the entire solution some momentum in the internal direction x^5.

To see that this Kaluza–Klein will give the modification we need to get the five dimensional black hole metric, choose a six dimensional Kaluza–Klein ansatz (still with the D1- and D5-brane components switched off):

$$\begin{aligned} ds^2 &= -\frac{1}{H}dt^2 + dr^2 + r^2 d\Omega_3^2 + H\left[dx_5 + \left(\frac{1}{H} - 1\right)dt\right]^2 \\ &= -dt^2 + dx_5^2 + \frac{Q^2}{r^2}(dt - dx_5)^2 + dr^2 + r^2 d\Omega_3^2, \end{aligned} \tag{17.25}$$

where we have shifted the gauge potential $A_t = H^{-1}$ by unity (this is just a gauge choice), and labelled the Kaluza–Klein dimension as x_5.

We see that the solution looks very simple as a six dimensional metric, but when written in the Kaluza–Klein ansatz, with the appropriate gauge field, we can achieve the desired modification of the coefficient of dt^2 which will appear in the reduced metric. When we introduce the D1 and D5 harmonic functions into the full solution, they will be multiplied back in according to the manner we have seen above, not modifying this structure at all.

Before writing the full solution, note that we can introduce orthogonal coordinates $\sqrt{2}u = x_5 - t$ and $\sqrt{2}v = x_5 + t$ and write the solution as

$$ds^2 = 2dudv + \frac{2Q^2}{r^2}du^2 + dr^2 + r^2 d\Omega_3^2.$$

There is a null vector with components $l_\mu = \partial_\mu u$, which is in fact covariantly conserved. This shows that the solution (H is independent of the u, v directions and can have a variety of dependences on the transverse ones) is in fact a 'plane-fronted' wave, which has parallel wave fronts. It is often called a 'pp-wave' for this reason. (See insert 17.1 for a discussion.)

So we have in fact succeeded in our goal. By superposing these three components according to the sum rules, we can construct the five dimensional extremal black hole. To recapitulate, it corresponds to a D5-brane wrapped on a T^4 (in directions x_6, x_7, x_8, x_9) to make a string lying in the x_5-direction. This string is combined with a D1-brane also lying in x_5. We know from previous chapters that this is supersymmetric. Finally, we

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Insert 17.1. pp-Waves as boosted Schwarzschild

Observe that we can write the pp-wave given in equation (17.25) in a manner in which is clearly a limit of a non-extremal form:

$$\begin{aligned} ds^2 \;&=\; -dt^2 + dx_5^2 + \frac{r_{\rm H}^2}{r^2}(\cosh\beta dt - \sinh\beta dx_5)^2 + \left(1 - \frac{r_{\rm H}^2}{r^2}\right)^{-1} dr^2 \\ &\quad + r^2 d\Omega_3^2 + \sum_{i=6}^{9} dx_i^2. \end{aligned} \tag{17.26}$$

This is written as a sort of boost, rather like we did for the p-brane solutions in subsection 10.2.2. It is actually a Lorentz boost of a familiar solution in the (t, x_5) plane. The supersymmetric solution we wrote previously is the limit of infinite boost, $\beta \to \infty$, with $r_{\rm H} \to 0$ holding the combination $r_P^2 = r_{\rm H}^2 e^{2\beta}/4$ held fixed, just like the infinite boost gives the supersymmetric extremal p-branes. The infinite boost gives a special supersymmetric solution with a null Killing vector $\partial/\partial u$, where $\sqrt{2}\,u = (x_5 - t)$. This is a momentum in the x_5 direction, as discussed in the main text. The correctly normalised value of r_P is

$$r_P^2 = g_{\rm s}^2 \alpha' \frac{V_*}{V} \frac{\alpha'}{R^2} Q_P,$$

where Q_P is an integer, R is the radius of x_5, and V is the volume of the T^4. We were able to compute the momentum in this direction by Kaluza–Klein reduction to be $P = Q/R = RV/(g_{\rm s}^2 \alpha'^4)$, where R is the length of x_5 and V is the volume of the T^4 on which we could put $x_6, \ldots, x_9$. More generally, we have now

$$P = \frac{Q_P^L - Q_P^R}{R} = \frac{RV r_{\rm H}^2}{g_{\rm s}^2 \alpha'^4} \frac{\sinh 2\beta}{2} = \frac{RV r_{\rm H}^2}{g_{\rm s}^2 \alpha'^4} \left(\frac{e^{2\beta} - e^{-2\beta}}{4}\right).$$

So we see that the supersymmetric limit is to have only a left-moving momentum excited. The general solution has both left and right momentum excited. What was it we boosted? Well, taking $\beta \to 0$:

$$ds^2 = -\left(1 - \frac{r_{\rm H}^2}{r^2}\right) dt^2 + dx_5^2 + \left(1 - \frac{r_{\rm H}^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_3^2 + \sum_{i=6}^{9} dx_i^2,$$

simply the five dimensional Schwarzschild solution, times a T^5.

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combine this with a third element, a wave in the x_5 direction. Compactifying on x_5 to five dimensions, we get a pointlike object, the extremal Reissner–Nordström black hole, where the $U(1)$ charge is in fact a diagonal combination of the $U(1)$s from the two R–R sector charges and the Kaluza–Klein charge of the momentum!

We can now be a bit more general. There is no reason why we cannot consider having different amounts of the various charges from the three independent sectors, since it is only their orientations which matter for the amount of preserved supersymmetry. So we can have Q_5, Q_1 and Q_P as three independent integers representing the number of D5-branes, D1-branes, and momentum in the compact x_5, respectively. Let us introduce the correctly normalised harmonic functions and write the solution representing this. The metric is (in Einstein frame)

$$g_s^{1/2}ds^2 = H_1^{-3/4}H_5^{-1/4}\left(-dt^2+dx_5^2+H_P\,(dt-dx_5)^2\right) \\ +H_1^{1/4}H_5^{3/4}\left(dr^2+r^2\,d\Omega_3^2\right)+V^{1/2}H_1^{1/4}H_5^{-1/4}ds^2_{T^4}, \tag{17.27}$$

where $ds^2_{T^4}$, in the (x^6,x^7,x^8,x^9) directions, is the metric on a T^4 with unit volume. Notice that given the orientations of the constituent branes, we can replace the T^4 by a K3 and preserve the same amount of supersymmetry. The results for the entropy count will turn out to be the same, but we will do it more carefully in a later section, since wrapping branes on K3 produces interesting subtleties, due to the enhançon mechanism which we discussed in chapter 15. The x_5 direction is compact with period $2\pi R$. The dilaton and Ramond–Ramond (R–R) fields are given by

$$e^{2\Phi}=g_s^2\frac{H_1}{H_5},\qquad F^{(3)}_{rtz}=\partial_r H_1^{-1},\qquad F^{(3)}_{\theta\phi\chi}=2\,r_5^2\,\sin^2\theta\sin\phi. \tag{17.28}$$

The harmonic functions are given by

$$H_1=1+\frac{r_1^2}{r^2},\qquad H_5=1+\frac{r_5^2}{r^2},\qquad H_P=\frac{r_P^2}{r^2}, \tag{17.29}$$

where the various scales are set by

$$r_5^2=g_s\alpha'\,Q_5,\qquad r_1^2=g_s\alpha'\,\frac{V_*}{V}\,Q_1,\qquad r_P^2=g_s^2\alpha'\,\frac{V_*}{V}\,\frac{\alpha'}{R^2}\,Q_P, \tag{17.30}$$

where $V_*=(2\pi\sqrt{\alpha'})^4$. The properties of the event horizon at $r=0$ can be computed (which the reader should do), yielding a vanishing surface gravity (and hence Hawking temperature) and a non-vanishing area and hence Bekenstein–Hawking entropy:

$$\mathcal{A}_{\rm H}=4\pi^3 V R_z\,r_1r_5r_p,\qquad S=2\pi\sqrt{Q_1\,Q_5\,Q_P}. \tag{17.31}$$

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Our goal is to find a microscopic description of this, and we do this next. Notice that the mass of the black hole is computed to be:

$$M = \frac{Q_P}{R} + \frac{Q_1 R}{g_s \alpha'} + \frac{Q_5 RV}{g_s \alpha'^3}, \tag{17.32}$$

which is just the sum of the Kaluza–Klein mass and the constituent brane charges normalised by the appropriate volume factors arising from where they are wrapped. That there is no interaction energy is consistent with the fact that we are constructing this black hole out of BPS constituents.

Notice that, inevitably, there is an explicit dependence of the mass on the embedding parameters. This is in contrast to the entropy, which is independent of the embedding parameters and so appears to be much more universal. We shall see a reason for this much later.

17.3.2 Microscopic entropy and a 2D field theory

Now we can follow the logic which we used in chapter 10. This geometry is entirely constructed with R–R charged objects, with some momentum. We have established that D-branes are the smallest possible objects carrying those charges, and so we must be able to make the black hole out of D-branes, with some momentum[7].

The case which we consider here is a compactification in which Q_5 D5-branes wrap a T^4, appearing as strings in six dimensions, forming a composite with Q_1 D1-branes. The D1 can only move within the D5-brane world-volume, and so this configuration should remind us of the D1–D5 bound state, which preserves 1/4 of the spacetime supersymmetries. Adding BPS momentum (i.e. purely right-moving) to such a configuration breaks a further 1/2 of the supersymmetries, and so we have a total of four supercharges.

Let us consider the case of $g_s Q \ll 1$, where Q is any of the charges in the solution. Then from the form of the harmonic functions (17.30), it is clear that in this limit we are studying the weakly coupled system of D-branes in flat space. We shall perform the study of the system in this limit initially. The case of $g_s Q > 1$ is where we have a macroscopic black hole, and as we shall see, our results for the counting of the entropy will apply to this case as well. This will appear to be simply due to the fact that we are counting BPS states, but later we shall see that things are more robust than that.

The configuration yields the following decomposition of the spacetime Lorentz group:

$$SO(1,9) \supset SO(1,1) \otimes SO(4) \otimes SO(4), \tag{17.33}$$

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where the first factor acts along the D-string world sheet (t, x^5), the third acts in the rest of the D5-brane world-volume (x^6, x^7, x^8, x^9) and the second in the rest of spacetime (x^1, x^2, x^3, x^4). From the point of view of the D5-brane gauge theory, the D1-branes are bound states in the 'Higgs branch', in which the D1-branes are instantons inside the D5-branes (see section 13.4). This branch is parametrised by the vacuum expectation values (vevs) of 1–5 open strings, which give $4Q_1Q_5$ bosonic and fermionic states, simply the dimension of instanton moduli space. The 'Coulomb branch' of the gauge theory is the situation where the D1-branes become pointlike instantons and then leave the D5-branes[130], ceasing to be bound states. This branch is parameterised by the vevs of 1–1 and 5–5 strings, which ultimately separate the individual D-branes from each other. This takes us away from the black hole, the state of most degeneracy. So we study the 1–5 and 5–1 open string sector, i.e. oriented strings stretching between the D1– and D5-branes. From the counting in section 13.4, we know that we have $4Q_1Q_5$ boson-fermion ground states[7].

N.B. Another way of thinking of this theory is as follows. At strong coupling, it will flow to the infra-red and become a non-trivial conformal field theory (see insert 3.1). It turns out (this is essentially a property of the superconformal algebra) that the number of boson–fermion ground states is directly related to the central charge of the conformal field theory, which in turn is equal to the difference in the number of hypermultiplets and vector multiplets $n_{\rm H} - n_{\rm V}$. In this case (things will be different in the case of K3 wrapping later in section 17.5) the number of 1–1 and 5–5 hypermultiplets exactly cancel the number of 1–1 and 5–5 vector multiplets, leaving Q_1Q_5.

Our configuration must be made to carry momentum Q_P in the x^5 direction around which the D-string is wrapped. What we really have is an effective 2D field theory in the (t, x_5) directions on the world-volume of the effective string. The Hamiltonian is $H = Q_P/R$. We are trying to distribute this total momentum amongst the $4Q_1Q_5$ bosons and fermions. This should remind the reader of earlier studies in chapters 3 and 4. It is just like being at level n and trying to distribute the energy among the bosons and fermions in the two dimensional conformal field theories we discussed in chapter (see insert 3.4, p. 92). Here, we have a supersymmetric string moving in the $4Q_1Q_5$ dimensions of the moduli space.

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The number of ways, $d(Q_P)$, of distributing a total momentum Q_P amongst the 1–5 and 5–1 strings is given by the partition function:

$$\sum d(Q_P)q^{Q_P} = \left(\prod_{n=1}^{\infty} \frac{1+q^n}{1-q^n}\right)^{4Q_1Q_5}. \qquad (17.34)$$

For large Q_P, this gives $d(Q_P) \sim \exp(2\pi\sqrt{Q_1Q_5Q_P})$, and Boltzmann's relation $S = \ln d(Q_P)$ yields precisely the entropy (17.31) we computed for our black hole using the Bekenstein–Hawking area law, in the previous section[7].

Let us pause to admire this result. We have actually counted the degeneracy of BPS states in the limit $g_sQ \ll 1$ where we have D-branes in flat space. When we go to $g_sQ > 1$ and the geometry of the branes will take over, making the black hole with geometry given in (17.27), we can be assured that the degeneracy will be precisely the *same*, because this is not renormalised by any quantum effect. So we have actually found a microscopic description of the black holes, at least for the purposes of counting the entropy. This works for black holes in four dimensions too[268], and with other properties like spin, etc. There are excellent reviews of this in the literature[278]. In fact, as we shall see, it is not really supersymmetry that is protecting us from an awful mismatch between the strong and weak coupling limits, but an important universal structure which will be uncovered later in chapter 18. A sign of this is to perform the counting successfully for a non-extremal black hole[269], which we shall do next.

17.3.3 Non-extremality and a 2D dilute gas limit

A non-extremal generalisation[269] of the solution can be written by exploiting the boost forms of the various components which we noted in subsection 10.2.2 (see equation (17.26)) and insert 17.1, with the following result:

$$\begin{aligned} g_s^{1/2}ds^2 &= Z_1^{-3/4}Z_5^{-1/4}\left(-dt^2 + dx_5^2 + \frac{r_{\rm H}^2}{r^2}(\cosh\beta dt - \sinh\beta dx_5)^2\right) \\ &\quad + Z_1^{1/4}Z_5^{3/4}\left(\left(1-\frac{r_{\rm H}^2}{r^2}\right)^{-1}dr^2 + r^2\,d\Omega_3^2\right) + V^{1/2}Z_1^{1/4}Z_5^{-1/4}ds^2_{T^4}, \\ e^{2\Phi} &= g_s^2 Z_1/Z_5, \end{aligned} \qquad (17.35)$$

where*

$$Z_1 = 1 + \sinh^2\beta_1\frac{r_{\rm H}^2}{r^2}; \quad Z_5 = 1 + \sinh^2\beta_5\frac{r_{\rm H}^2}{r^2}.$$

* The reader might find it worth checking that in the case that all of the R–R charges and the momentum are the same, a reduction to five dimensions gives the isotropic form of the five dimensional Reissner–Nordström black hole given in equation (17.19).

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The R–R charges of this solution are as before, while as we learned in insert 17.1, there is now both left- and right-moving momentum in the x_5 direction, creating the non-extremality.

$$P = \frac{Q_{\rm P}^{\rm L} - Q_{\rm P}^{\rm R}}{R} = \frac{RVr_{\rm H}^2}{g_{\rm s}^2\alpha'^4}\frac{\sinh 2\beta}{2} = \frac{RVr_{\rm H}^2}{g_{\rm s}^2\alpha'^4}\left(\frac{e^{2\beta} - e^{-2\beta}}{4}\right).$$

The mass of this solution is

$$\widehat{M} = \frac{RVr_{\rm H}^2}{g_{\rm s}^2\alpha'^4}\left(\frac{\cosh 2\beta_1}{2} + \frac{\cosh 2\beta_5}{2} + \frac{\cosh 2\beta}{2}\right).$$

Now we can compute the entropy of the solution by computing the area of the horizon at $r = r_{\rm H}$:

$$S = \frac{2\pi RVr_{\rm H}^3}{g_{\rm s}^2\alpha'^4}(\cosh\beta_1\cosh\beta_5\cosh\beta).$$

Now we study an interesting limit. We take the R–R charge densities to be greater than the momentum densities which in turn is larger than the string scale:

$$r_1^2, r_5^2 \gg r_P^2 \gg \alpha', \tag{17.36}$$

which has the effect of keeping the D-brane component close to extremality but allowing both left and right momenta to survive. We can check this by seeing that the energy above the amount at extremality, computed in equation (17.32), becomes:

$$\widehat{M} - M \simeq \frac{RVr_{\rm H}^2}{g_{\rm s}^2\alpha'^4}\frac{e^{2\beta}}{4} = \frac{Q_{\rm P}^{\rm L}}{R},$$

and so we see that the extra energy coming from the left-moving sector is simply additive, as though the left- and right-moving components of the system are non-interacting, despite the fact that we are non-extremal. This is called the 'dilute gas' limit, since in the 1+1 dimensional model, a 'gas' of $4Q_1Q_5$ boson-fermion pairs, there is no interaction between the left- and right-moving parts.

A little algebra shows that in this limit we get for the entropy

$$S = 2\pi\left(\sqrt{Q_1Q_2Q_P^{\rm L}} + \sqrt{Q_1Q_5Q_P^{\rm R}}\right). \tag{17.37}$$

The microscopic computation for the statistical entropy is just like the one we had before, but with both left- and right-moving sectors. In this dilute limit, since they are decoupled the result is just the sum of the entropies of the two sectors, as we have seen coming from the supergravity.

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So again, we have exactly verified with a microscopic computation the entropy of a black hole, now even without the help of supersymmetry.

17.4 Near horizon geometry

Recall that in our earliest examination of extremal black holes in chapter 10, we found that the geometry of the horizon was an interesting place, since the geometry was highly symmetric. The extremal horizon size was controlled entirely by the asymptotic charge at infinity, and not by the details of the embedding of the solution into the supergravity. In fact, there are other special properties of the black hole apparent when the system is embedded in the supergravity.

Just as we saw in the case of the solution for the D6-brane wrapped on K3 in section 15.4, the parameters of the compact solution are just the asymptotic values of fields – the moduli – in the full supergravity. There, we studied a solution where the volume of K3. Here, the radius R of the x_5 circle, and the volume V of the T^4, are the asymptotic values of scalars. In fact, these scalars approach fixed universal values at the black hole horizon, due to what is called the *'attractor mechanism'*[267]. The values are fixed by the underlying U-duality algebraic structure of the supergravity. In particular, the area of the horizon itself is fixed in terms of the $E_{6,(6)}$ U-duality invariant, and the parameters which make it up are determined only by the charges measured at infinity and not the details of the geometry or the embedding. In particular, the entropy itself is an $E_{6,(6)}$ U-duality invariant.

We won't study this general issue in any detail here, but refer the reader to the literature[267]. Let us instead directly examine the near-horizon geometry of the black hole that we constructed in the previous sections. Consider the non-extremal black hole solution given in equation (17.35), but in string frame:

$$\begin{aligned} ds^2 &= Z_1^{-1/2} Z_5^{-1/2}\left(-dt^2+dx_5^2+\frac{r_{\rm H}^2}{r^2}(\cosh\beta dt-\sinh\beta dx_5)^2\right)\\ &\quad + Z_1^{1/2}Z_5^{1/2}\left(\left(1-\frac{r_{\rm H}^2}{r^2}\right)^{-1}dr^2+r^2\,d\Omega_3^2\right)+V^{1/2}Z_1^{1/2}Z_5^{-1/2}ds^2_{T^4},\\ e^{2\Phi} &= g_s^2 Z_1/Z_5. \end{aligned} \tag{17.38}$$

The limit we will take is that $g_{\rm s}Q_1, g_{\rm s}Q_5$ are large, but Q_P are arbitrary. This means that $r^2 < r_1^2$ and r_5^2, and so we can neglect the 1 in the harmonic functions in which they appear. So we see that the volume of the T^4 has become fixed to Vr_1^2/r_5^2, and the dilaton has gone to $e^\Phi = g_s r_1/r_5$.

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In the limit, we get:

$$ds^2 = \frac{r^2}{r_1 r_5}\left(-dt^2 + dx_5^2 + \frac{r_{\rm H}^2}{r^2}(\cosh\beta dt - \sinh\beta dx_5)^2\right) + \frac{r_1 r_5}{r^2}\left(\left(1 - \frac{r_H^2}{r^2}\right)^{-1} dr^2 + r^2\, d\Omega_3^2\right) + \frac{r_1}{r_5} ds_{T^4}^2. \quad (17.39)$$

It is useful to define

$$\rho_+^2 \equiv r_{\rm H}^2 \cosh^2\!\beta, \quad \rho_-^2 \equiv r_{\rm H}^2 \sinh^2\!\beta, \quad (17.40)$$

and we get

$$ds^2 = \frac{r^2}{r_1 r_5}\left[-\left(1 - \frac{\rho_+^2}{r^2}\right)dt^2 + \left(1 + \frac{\rho_-^2}{r^2}\right)dx_5^2 + 2\frac{\rho_+\rho_-}{r^2}dt dx_5\right] + \frac{r_1 r_5}{r^2}\left(1 - \frac{(\rho_+^2 - \rho_-^2)}{r^2}\right)^{-1} dr^2 + r_1 r_5 d\Omega_3^2 + \frac{r_1}{r_5} ds_{T^4}^2. \quad (17.41)$$

Finally, after a change of coordinates to $\rho^2 = r^2 + \rho_-^2$, the metric is:

$$ds^2 = -\frac{(\rho^2 - \rho_+^2)(\rho^2 - \rho_-^2)}{r_1 r_5 \rho^2} dt^2 + \frac{r_1 r_5 \rho^2}{(\rho^2 - \rho_+^2)(\rho^2 - \rho_-^2)} d\rho^2 + \rho^2\left(dx_5 + \frac{\rho_+\rho_-}{\rho^2}dt\right)^2 + r_1 r_5 d\Omega_3^2 + \frac{r_1}{r_5} ds_{T^4}^2, \quad (17.42)$$

which can be recognised[294, 295] as a three dimensional black hole solution called the *'BTZ black hole'*[296] multiplied by an S^3 and T^4. In fact, the black hole solution can be seen to be asymptotically AdS_3, with a length scale ℓ set by $\ell^2 = r_1 r_5$. See insert 17.2. The case $\rho_+ = \rho_-$, gives the extremal 5D black hole, and the near-horizon metric becomes locally $\mathrm{AdS}_3 \times S^3 \times T^4$, with an identification on the x_5 circle. This is a situation that we have seen before, where the extreme black hole has a simple, highly symmetric spacetime in the near-horizon limit, with the size of the solution controlled by the asymptotic charges.

The fact that the near-horizon geometry of the black hole is actually AdS_3, (times fixed compact spaces) with a black hole in it is interesting. As we shall see in the next chapter, there is remarkable duality proposed which – if correct – ensures that the physics of the 1+1 dimensional theory which was controlling our entropy count is captured entirely by the AdS_3 physics. Especially in the case of AdS_3, the aspects of the duality relevant to our problem are quite well understood. It is this AdS/CFT duality which seems to ensure that the entropy count was correct, even away from extremality. See insert 17.2.

Insert 17.2. The BTZ black hole

Consider the action for (2+1)-dimensional gravity with negative cosmological constant $\Lambda = -1/\ell^2$:

$$S = \frac{1}{16\pi G_3}\int d^3x\sqrt{-g}\,(R - 2\Lambda). \tag{17.43}$$

There is an interesting solution, whose metric is:

$$\begin{aligned} ds^2_{\rm BTZ} &= -V(\rho)dt^2 + V(\rho)^{-1}d\rho^2 + \rho^2\left(d\varphi + \frac{4G_3J}{\rho^2}dt\right)^2, \\ V(\rho) &= \left(-8G_3M + \frac{\rho^2}{\ell^2} + \frac{16G_3^2J^2}{\rho^2}\right), \end{aligned} \tag{17.44}$$

where φ is periodic, with period 2π. This is the 'BTZ black hole' solution[296], and there are two event horizons are at $\rho = \rho_\pm$, in terms of which we arrived at the solution (17.42), and $\ell^2 = r_1r_5$ there. The mass and angular momentum of this solution are given by

$$M = \frac{\rho_+^2 + \rho_-^2}{8\ell^2G_3}, \qquad J = \frac{\rho_+\rho_-}{4\ell G_3}.$$

Notice that the case $M = -1/8G_3, J = 0$ gives us AdS_3 in global coordinates, as given in equation (10.29). The case $M = 0, J = 0$ is also AdS_3, but now in local coordinates. In fact, the BTZ spacetime is locally AdS_3 everywhere. Since φ is compact, there is a global difference which makes it a non-trivial solution for arbitrary M and J.

Using the techniques presented at the beginning of this chapter, the entropy and temperature may be computed to be

$$S = \frac{2\pi\rho_+}{4G_3} = \frac{\mathcal{A}}{4G_3}, \qquad T = \frac{\rho_+^2 - \rho_-^2}{2\pi\ell^2\rho_+}.$$

The AdS/CFT correspondence, to be discussed in the next chapter, associates a dual (1+1)–dimensional CFT to the physics of $AdS_3\times S^3$, with[297] $c = 3\ell/2G_3$. In fact, the $M = 0$ and $M = -1/8G_3$ cases can be identified[298] with the NS–NS and R–R ground states of the theory, with energy $E = 0$ or $E = -\ell/8G_3$, where the fermions are either periodic or antiperiodic around φ. (The factor of ℓ results from a conformal rescaling, see section 18.1.3.) Computations we know how to do from chapters 2 and 4 show that the zero point energy difference is $c/12$, which is the result one would get from converting $\ell/8G_3$.

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17.5 Replacing T^4 with K3

An important variation on the constructions above is to replace the T^4 in the $x_6, \ldots, x_9$ directions by the K3 manifold instead. In fact, this does not break any more supersymmetries than the D5–D1 orientation, and so in principle, everything should go through trivially. However, as we know from chapters 9 and 15, the wrapping of the D5-branes on the K3 should change things considerably, since the enhançon mechanism ought to modify the geometry significantly in the limit of large charges where the black hole becomes manifest. In fact the original reference considered K3 first[7], and did not take into account the subtleties introduces by K3 in the macroscopic geometry. Our goal in this section is to examine this physics carefully[299]. Their answer for the entropy was not wrong, however, for reasons we shall see. Our careful analysis will produce a new result, however, since it will become clear that the enhançon mechanism works in precise conjunction with the second law of thermodynamics.

17.5.1 The geometry

The Einstein frame metric is:

$$ds^2 = H_1^{-3/4} H_5^{-1/4} \left(-dt^2 + dx_5^2 + H_P(dt - dx_5)^2\right) + H_1^{1/4} H_5^{3/4} \left(dr^2 + r^2\, d\Omega_3^2\right) + V^{1/2} H_1^{1/4} H_5^{-1/4} ds_{\rm K3}^2, \quad (17.45)$$

where $ds_{\rm K3}^2$ is the metric on a K3 manifold with unit volume. The other fields and harmonic functions are the same as those listed in equations (17.30).

Of course, the integers Q_1, Q_5 and Q_P appearing in the harmonic functions measure the asymptotic charges associated with the electric and magnetic R–R fluxes and the internal x_5-momentum, respectively. We must, however, introduce another set of integers, N_1 and N_5 to denote the number of D1-branes and D5-branes, respectively, in the system. Clearly we have $N_5 = Q_5$. However, as discussed in chapter 9 and in detail in section 15.4, wrapping the D5-branes on K3 induces a negative D1-brane charge and so we have $Q_1 = N_1 - N_5$ or alternatively $N_1 = Q_1 + Q_5$.

Just like in section 15.4, the volume of the K3 manifold (measured by the string frame metric) is:

$$V(r) = \frac{H_1}{H_5} V, \quad (17.46)$$

where V is the asymptotic volume of the K3. At the horizon, it is:

$$V_{\rm H} \equiv V(r=0) = \frac{r_1^2}{r_5^2} V = \frac{Q_1}{Q_5} V_* = \frac{N_1 - N_5}{N_5} V_*, \quad (17.47)$$

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and so if $r_1 < r_5$, then $V_{\rm H} < V$. So we see that as long as $r_1 < r_5$, that the volume K3 is shrinks as we move in from $r \to \infty$. When we reach $V(r) = V_*$ at some radius, new physics will come into play, and this is the 'enhançon' locus we discovered in section 15.4. This radius is easily computed:

$$\hat{r}_{\rm e}^2 = \frac{g_{\rm s}\alpha' V_*}{(V - V_*)}(2N_5 - N_1), \qquad \begin{cases} > 0 & \text{for} \quad 2N_5 > N_1 \\ < 0 & \text{for} \quad 2N_5 < N_1 \end{cases}, \tag{17.48}$$

where $\hat{r}_{\rm e}^2 < 0$ simply indicates that the K3 volume reaches V_* inside the event horizon. Therefore we see that we can have the enhançon appearing either above or below the horizon, depending upon how we choose the parameters.

Let us consider the case of $\hat{r}_{\rm e}^2 > 0$. Now when the K3 volume reaches V_*, at the enhançon radius, $\hat{r}_{\rm e}$, the wrapped D5-branes will be unable to proceed supersymmetrically into smaller radius, due to the fact that their effective tensions are going through zero there. They are therefore forced to form an enhançon sphere at radius $\hat{r}_{\rm e}$. By contrast, D1-branes and momentum modes can movie inside of $r = \hat{r}_{\rm e}$: they are not wrapped on K3 and therefore do not care that it is approaching a special radius there. However, notice that the geometry can be made of D1–D5-bound states. The corrections of $-\tau_1$ to the effective tension of the wrapped D5-brane is precisely compensated by the $+\tau_1$ coming from the marginally bound D1-brane. Therefore we can make the above geometry in equations (17.45–17.29) by binding N_5 D1-branes to N_5 the D5-branes we wish to include in the geometry, and bring the resulting N_5 D1–D5 bound states in from infinity, together with Q_1 extra D1-branes.

17.5.2 The microscopic entropy

In the microscopic model we have some modifications to the T^4 situation. We have an effective 1+1 dimensional gauge theory on the effective D-string formed by wrapping the D5-branes and binding it with D1-branes. At strong coupling the theory will flow to a conformal field theory in the infra-red (see insert 3.1, p. 84). The important feature of the conformal field theory is its central charge, which can be computed from the gauge theory as proportional to $n_{\rm H} - n_{\rm V}$, the difference between the numbers of hypermultiplets and the number of vector multiplets. Counting the bosonic parts, the D1-branes contribute N_1^2 vectors and N_1^2 hypers, the latter coming from (x^6, x^7, x^8, x^9) fluctuations. The D5-branes contribute N_5^2 vectors, but there are no massless modes coming from oscillator excitations in the (x^6, x^7, x^8, x^9) (K3) directions. There are, in addition, 1–5 strings which give $N_1 N_5$ hypermultiplets. Evaluating the

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difference gives: $N_1 N_5 - N_5^2 = Q_1 Q_5$ hypermultiplets. Hence in total, there are $4Q_1Q_5$ bosonic excitations and an equal number of fermions, since a hypermultiplet contains four scalars and their superpartners.

In another language all that we have done is evaluated the dimension the Higgs branch of the D5-brane moduli space of vacua, where the N_1 D1-branes can become instanton strings of the $U(N_5)$ gauge theory on the world-volume of the D5-branes. The vacuum expectation values of the 1–5 strings is precisely what constitutes this branch. In this language, the absence of hypers coming from the 5–5 sector corresponds to the absence of Wilson lines on the K3 surface (there are no non-trivial one-cycles). The entropy count then goes precisely along the lines of section 17.3.2.

Let's close this discussion by observing that we have a mild apparent conflict with the microscopic description. For $N_1 < 2N_5$, we know from the analysis of the previous section that, at any given value of the momentum, the entropy can be maximised by using only $N_1/2$ of the D5-branes in the problem. So, from the field theory point of view it appears to be favourable to Higgs the $U(N_5)$ gauge theory leaving massless only a $U(N_1/2)$ subgroup. But since all of these supersymmetric vacua are degenerate, all black holes appear to be on the same footing.

This is really an artifact of the thermodynamically peculiar situation that we are at zero temperature while having a finite entropy, so the entropy strictly has a meaning as a degeneracy of ground states. Processes which maximise the entropy require dynamics, and so we must take the system away from extremality in order that it can explore configuration space, and find the maximal entropy black hole.

17.5.3 Probing the black hole with branes

Let us illustrate the above statements with some probe computations[299]. Both D1- and D5-branes are natural probes of the geometry[266], since they preserve the same supersymmetries. Consider a composite probe brane consisting of n_5 D5-branes and n_1 D1-branes. It is important for the physics of the following that this composite probe is in the D5-branes' Higgs phase. That is, this composite probe is *not* simply a collection of individual D5-branes and D1-branes moving together, but rather the D1-branes have been absorbed as instanton strings lying along the z-direction in the D5-brane world-volume. These instantons are maximally smeared over the K3 directions and that we have chosen the orientation of the vevs of the hypermultiplets arising from 1–5 strings such that the instantons are of maximal rank in the $U(n_5)$ gauge theory. In this phase, the composite probe brane is then a true bound state, i.e. the fields describing the relative separation of the branes in the Coulomb phase are all massive.

The effective action for the composite brane probe regarded as an effective string is

$$S = -\int_{\Sigma} d^2\xi \; e^{-\Phi(r)}(n_5\tau_5 V(r) + (n_1 - n_5)\tau_1)(-\det g_{ab})^{1/2}$$
$$+ n_5\tau_5 \int_{\Sigma\times \mathrm{K3}} C^{(6)} + (n_1 - n_5)\tau_1 \int_{\Sigma} C^{(2)}, \qquad (17.49)$$

where Σ is the unwrapped part of the brane's world-volume, with coordinates $\xi^{0,1}$. Remember in the above action that the wrapping of the D5-branes on the K3 introduces negative contributions to both the tension and two-form R–R charge terms. Recall that g_{ab} is the pull-back of the string-frame spacetime metric. The background fields in which the probe moves are those of the black hole solution given in equation (17.28). The corresponding R–R potentials may be written as

$$C^{(6)} = g_{\mathrm{s}}^{-1} H_5^{-1} dx^0 \wedge dx^5 \wedge \varepsilon_{\mathrm{K3}}, \quad C^{(2)} = g_{\mathrm{s}}^{-1} H_1^{-1} dx^0 \wedge dx^5, \qquad (17.50)$$

where $\varepsilon_{\mathrm{K3}}$ denotes the volume four-form on the K3 space with unit volume. These R–R potentials do *not* vanish asymptotically because we choose a gauge which eliminates a constant contribution to the energy which would otherwise appear.

We will now choose static gauge, aligning the coordinates of the effective probe string with the x^5 direction and letting it move in the directions transverse to K3 while freezing and smearing the degrees of freedom on the K3:

$$\xi^0 = x^0 \equiv t, \quad x^i = x^i(t), \quad i = 1, 2, 3, 4. \qquad (17.51)$$

The result can be written as an effective Lagrangian $\mathcal{L}$ for a particle moving in the (x^1, x^2, x^3, x^4) directions:

$$\mathcal{L} = \frac{1}{2}\left(n_5\tau_5 V H_1 + (n_1 - n_5)\tau_1 H_5\right)(1 + H_P)\left[\dot{r}^2 + r^2\dot{\Omega}_3^2\right], \qquad (17.52)$$

where, as usual, a dot is used to denote $\partial/\partial t$, and

$$\dot{\Omega}_3^2 = \dot{\theta}^2 + \sin^2\theta(\dot{\phi}^2 + \sin^2\phi\,\dot{\chi}^2).$$

As should be expected by now, here is no non-trivial potential, since supersymmetry cancelled the mass against the R–R charge as in previous computations of this type.

The effective tension of the probe is given by the prefactor in equation (17.52). We can already see that there is the possibility that the tension will go negative when $n_5 > n_1$.

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Putting in the definitions of the harmonic functions given in equations (17.29) and (17.30), we see that the tension remains positive as long as

$$(n_5\tau_5 V H_1 + (n_1 - n_5)\tau_1 H_5) > 0 \tag{17.53}$$

which translates into

$$r^2 > \tilde{r}_{\rm e}^2 = g_{\rm s} l_{\rm s}^2 \, V_* \frac{(2N_5 - N_1)n_5 - N_5 n_1}{(V - V_*)n_5 + V_* n_1}. \tag{17.54}$$

It is worth considering some special cases of this result. If we remove all of the D5-branes, the result for pure D1-brane probes is quite simple, as setting n_5 to zero in the above result gives:

$$\mathcal{L}_{\rm D1} = \frac{1}{2} n_1 \tau_1 H_5 (1 + H_P) \left[\dot{r}^2 + r^2 \dot{\Omega}_3^2\right], \tag{17.55}$$

since the D1-brane is not wrapped on the K3 and so its tension is positive everywhere. It simply floats past the enhançon radius on its way to the origin without seeing anything particularly interesting there.

Note that the result (17.55) is the same as would have been obtained in the case of probing for a T^4 compactification, considering only motion in the directions transverse to the torus. Similarly in the case that $n_1 = n_5$, we get:

$$\mathcal{L} = \frac{1}{2} n_5 \tau_5 H_1 (1 + H_P) \left[\dot{r}^2 + r^2 \dot{\Omega}_3^2\right], \tag{17.56}$$

which is the same as the result for pure D5-brane probes in the case where they are wrapped on T^4. The cancellation of the induced tensions from K3 wrapping and non-trivial instanton number in constructing the bound state probe provided a simple result: the wrapped D5-branes, when appropriately dressed with instantons, can indeed pass through the enhançon shell.

If we instead remove all of the D1-branes, we just get the familiar result of section 15.4 that the probe, made of pure D5-branes, hangs up at the enhançon radius $\hat{r}_{\rm e}$ given by equation (17.48). Now we discover that our earlier enhançon result is just a special case of a more general result: whenever there are more D5-branes than D1-branes making up the probe (i.e. $n_5 > n_1$), there is a generalisation of the enhançon radius, $\tilde{r}_{\rm e}^2$, where the composite probe will become tensionless and must stop. Notice that this happens in a 'substringy' regime where $V_{\rm K3} < V_*$.

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17.5.4 The enhançon and the second law

The entropy and area of the black holes which we construct are given by the formula

$$S = \frac{\mathcal{A}}{4G} = 2\pi\sqrt{Q_1 Q_5 Q_P} = 2\pi\sqrt{(N_1 - N_5)N_5 Q_P}, \qquad (17.57)$$

where in the second equality we have written it in terms of the number of physical branes of each type. Let us consider the dependence of the entropy on the number of D5-branes. Fixing N_1 and Q_P, we see that it gives a half an ellipse, as depicted in figure 17.1. We see that there are *maximal entropy* black holes that we can make, (corresponding to the apside of the ellipse) which are those for which $N_5 = N_1/2$, or in other words $Q_1 = Q_5$.

If we wish to consider the maximum entropy that can be achieved for a given set of parameters, N_1, N_5 and Q_P, we observe that the behaviour of this entropy changes at precisely $N_1 = 2N_5$. In figure 17.2 is a plot of the (square of the) maximal entropy as a function of Q_1 for fixed N_5 and Q_P. For a 'large' number of D1-branes ($N_1 > 2N_5$), the maximal area squared is simply proportional to Q_1, as expected from equation (17.57). However, for a 'small' number of D1-branes ($N_1 < 2N_5$), the entropy is maximised if only $N_5' = N_1/2$ of the available D5-branes participate in the formation of the black hole. In this regime, we have

$$\mathcal{A}^2_{\rm max} \propto N_1^2 = (Q_1 + Q_5)^2 \qquad (17.58)$$

and so the curve becomes a parabola which only reaches zero at $Q_1 = -Q_5$.

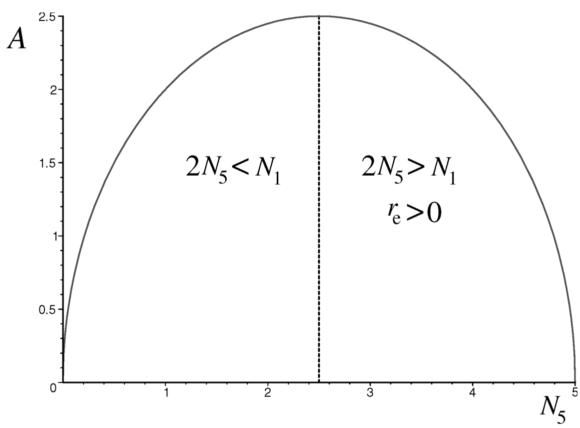

Fig. 17.1. The horizon area as a function of N_5, for fixed $Q_P\,(= 1)$ and $N_1\,(= 5)$, which forms half of an ellipse. As the number of D5-branes increases past $N_1/2$, the area decreases.

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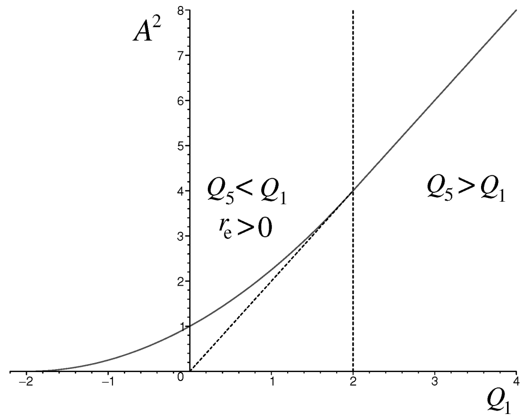

Fig. 17.2. The square of the maximal horizon area as a function of Q_1, for fixed Q_P (=1) and Q_5 (=2). For $N_1 > 2N_5$, the (area)^2 increases linearly. For $N_1 < 2N_5$, to maximise the area, one must use only $N_1/2$ of the available D5-branes (see figure 17.1), and therefore the dependence on N_1 is quadratic.

Note that in this regime, the maximum entropy is greater than one would calculate from equation (17.57). Assuming the excess D5-branes have accumulated in an enhançon shell around the black hole, the maximum entropy configuration corresponds to precisely that where the K3 volume is frozen at V_* throughout the interior region.

Let us return to the curve in figure 17.1. If we were to begin with a black hole with a 'large' number of D1-branes, we would be on a point on the left hand side of the ellipse in the figure. We may now consider increasing the number of D5-branes in the system by bringing them one at a time from infinity. As a result the black hole moves up the ellipse to the extremum at $N_5 = N_1/2$. At this point, however, if we were to add one more D5-brane, we we see that we will in fact *decrease* the horizon area, and hence the entropy of the resulting system. In principle we can bring this D5-brane up to the black hole horizon as slowly as we like, and so we have found a way of reducing the entropy of the hole by an adiabatic process. This is a violation of the second law of thermodynamics.

Actually, there is a very satisfying resolution of this problem[299]. It is precisely for this class of black holes that *the enhançon appears above the horizon.* So an attempt to bring our extra D5-brane into the hole is thwarted by the fact that it will be forced to stop at the enhançon radius $\hat{r}_e$ just above the horizon. We could bind the extra D5-brane with an extra D1-brane to bring it in, but in this case Q_1 remains fixed while

Q_5 increases. Therefore adding the D1/D5 bound state to the black hole increases the entropy.

If we begin with a black hole on the right half of the ellipse ($N_1/2 < N_5 < N_1$), the enhançon again ensures that we cannot move further to the right decreasing the horizon area by dropping D5-branes into the black hole. These were configurations where the black hole is already surrounded by a region where $V(r) < V_*$ and hence the extra D5-branes are restrained from reaching the horizon by the enhançon mechanism.

However, we have seen in section 17.5.3 that D1/D5 bound states can move through such regions where $V(r) < V_*$ and so we must still investigate if we are able to decrease the entropy by sending in a bound D1/D5 probe brane. Adopting the previous notation, let the probe consist of a bound state with n_1 D1-branes and n_5 D5-branes. Assuming that the black hole already contains many more of each type of brane, i.e. $n_1, n_5 << N_1, N_5$, dropping in such a probe would cause an infinitesimal shift in the entropy (squared) given by

$$\delta S^2 = 4\pi^2 Q_P \left(N_5 n_1 + (N_1 - 2N_5)n_5\right). \tag{17.59}$$

Note that implicitly we are assuming $N_1, N_5, n_1, n_5 > 0$. Even so the expression in parentheses has the potential to be negative, which would signal a decrease in the black hole entropy. However, we found that this expression also appears in the numerator of equation (17.54) for the radius of vanishing probe tension, but with the opposite sign. Hence the probe-brane finds no obstacle to dropping inside the horizon only in those situations where the entropy increases. Precisely in those cases where second law would be violated, the enhançon locus filters out the wrong type of D1–D5 bound states from reaching the event horizon. Thus the enhançon provides string theory with precisely the mechanism needed to maintain consistency with the second law of black hole thermodynamics[299, 300].

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