

## CONTINUITY OF ROOTS, REVISITED

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### Abstract

The aim of this note is to give a simple topological proof of the well-known result concerning continuity of roots of polynomials. We also consider a more general case with polynomials of a higher degree approaching a given polynomial. We then examine the continuous dependence of solutions of linear differential equations with constant coefficients.

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### 1. Introduction

It is well known that the roots of a monic polynomial  $z^n + a_{n-1}z^{n-1} + \dots + a_1z + a_0$  in the complex plane depend continuously on the coefficients. A classical proof uses Rouché's theorem [1, page 153], but there are other proofs. The proof in [6, page 10] uses estimates of the absolute values of roots in terms of the parameters  $a_{n-1}, \dots, a_0$ , while the proofs in [3, 4] and [5] treat the space of roots as a metric space and apply topological machinery. Finally, the proof in [2] relies on the notion of projective space and gives continuity of the roots for polynomials which are not necessarily monic (see Theorem 3.3 below or, equivalently, [2, Theorem 3]).

In the first part of this note (Section 2), we give a proof of continuity of the roots for monic polynomials by characterising convergence of sequences in the quotient space of  $n$ -roots vectors, in which we identify two vectors if their coordinates may be reordered so as to obtain one vector from the other. Such a quotient space (a symmetric space) has also been used in [1–5] and we just take it as a starting point to conduct a purely topological proof without any reference to metrisability of the topological space under consideration. In Section 3, we use a simple trick to get a general continuity result (Theorem 3.3) from its ‘monic version’ presented in Corollary 2.9.

In the second part of this note, we examine the continuous dependence of solutions of linear differential equations with constant coefficients. The case of linear differential equations of a fixed order is obvious as a consequence of Theorem 2.8 and the form of

the general solution of the equations under consideration. On the other hand, when we consider linear differential equations of higher order than the limit equation, various conclusions are possible, as we show in Example 4.1.

**Notation.** Let  $\mathbb{N}$  denote the set of positive integers,  $\mathbb{C}$  the set of complex numbers,  $\mathbb{R}$  the set of real numbers and  $\mathbb{R}_+$  the set of nonnegative elements of  $\mathbb{R}$ . We write  $[n] := \{1, \dots, n\}$  for any  $n \in \mathbb{N}$ . Let  $\mathbb{S}_n$  denote the set of all permutations of elements of the set  $[n]$  for  $n \in \mathbb{N}$ .

We define the (Euclidean) norm of a vector  $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n$  by  $|\xi| := \sqrt{\sum_{i=1}^n \|\xi_i\|^2}$ , where  $\|\xi_i\|$  is the modulus of  $\xi_i \in \mathbb{C}$ . For  $\varepsilon > 0$ , let  $B_\varepsilon := \{\xi \in \mathbb{C}^n : |\xi| < \varepsilon\}$ . For nonempty sets  $A, B \subset \mathbb{C}^n$ , their Minkowski sum is the set  $A + B := \{a + b : a \in A, b \in B\}$ . For  $\xi \in \mathbb{C}^n$  we write simply  $\xi + B$  instead of  $\{\xi\} + B$ . If  $P$  is a polynomial, we denote its degree by  $\deg(P)$ .

## 2. Results for monic (normed) polynomials

Fix  $n \in \mathbb{N}$ . Let  $\sigma : \mathbb{C}^n \rightarrow \mathbb{C}^n$  be defined by  $\sigma(\xi) := (\sigma_1(\xi), \dots, \sigma_n(\xi))$ , where  $\xi = (\xi_1, \dots, \xi_n)$ ,  $\xi_i \in \mathbb{C}$  for  $i \in [n]$ ,  $\sigma_i(\xi) = a_i \in \mathbb{C}$ ,  $i \in [n]$ , and

$$\prod_{i=1}^n (z - \xi_i) = z^n + a_n z^{n-1} + \dots + a_2 z + a_1, \quad z \in \mathbb{C}.$$

Viète's formulas ensure that  $\sigma$  is a well-defined continuous function.

For  $\xi \in \mathbb{C}^n$  and  $s \in \mathbb{S}_n$ , we define  $s(\xi)$  (abusing notation) by  $s(\xi) := (\xi_{s_1}, \dots, \xi_{s_n})$ . Note that  $s(\xi + \xi') = s(\xi) + s(\xi')$  for  $\xi, \xi' \in \mathbb{C}^n$  and  $s \in \mathbb{S}_n$ .

We introduce an equivalence relation  $\sim$  on  $\mathbb{C}^n$  as in [4], that is, for  $\xi, \xi' \in \mathbb{C}^n$ ,  $\xi \sim \xi'$  if and only if  $\xi = s(\xi')$  for some  $s \in \mathbb{S}_n$ . For  $\xi \in \mathbb{C}^n$ , let  $\pi(\xi) := \{\xi' \in \mathbb{C}^n : \xi' \sim \xi\}$  and denote the quotient set induced by  $\sim$  on  $\mathbb{C}^n$  by

$$\mathbb{C}^n / \sim := \{\pi(\xi) : \xi \in \mathbb{C}^n\}.$$

The mapping  $\pi$  is called the quotient projection of  $\mathbb{C}^n$  onto  $\mathbb{C}^n / \sim$ . The quotient space induced by the relation  $\sim$  on  $\mathbb{C}^n$  is the set  $\mathbb{C}^n / \sim$  endowed with the (quotient) topology:  $A \subset \mathbb{C}^n / \sim$  is open if and only if  $\pi^{-1}(A) := \{x \in \mathbb{C}^n : \pi(x) \in A\}$  is open in  $\mathbb{C}^n$  endowed with the Euclidean topology.

**LEMMA 2.1.** *Let  $\varepsilon > 0$  and  $\xi \in \mathbb{C}^n$  be fixed. Then  $s(\xi + B_\varepsilon) = s(\xi) + B_\varepsilon$  for  $s \in \mathbb{S}_n$ .*

**PROOF.** Let  $b \in s(\xi + B_\varepsilon)$ , that is, there exists  $b' \in B_\varepsilon$  with  $b = s(\xi + b') = s(\xi) + s(b')$ . However,  $|s(b')| = |b'| < \varepsilon$ , so  $b \in s(\xi) + B_\varepsilon$ . Let  $s^{-1} \in \mathbb{S}_n$  be the inverse of  $s$ . If  $b \in s(\xi) + B_\varepsilon$ , then  $b = s(\xi) + s(s^{-1}(b')) = s(\xi + s^{-1}(b')) \in s(\xi + B_\varepsilon)$  for some  $b' \in B_\varepsilon$ .  $\square$

**COROLLARY 2.2.** *Let  $\varepsilon > 0$  and  $\xi, \xi' \in \mathbb{C}^n$  be fixed. Then  $\pi(\xi') \in \pi(\xi + B_\varepsilon)$  if and only if  $\xi' \in s(\xi) + B_\varepsilon$  for some  $s \in \mathbb{S}_n$ .*

**LEMMA 2.3.** *Let  $\varepsilon > 0$  and  $\xi \in \mathbb{C}^n$  be fixed. The set  $\pi^{-1}(\pi(\xi + B_\varepsilon)) \subset \mathbb{C}^n$  is open.*

**PROOF.** By the definition,  $\pi^{-1}(\pi(\xi + B_\varepsilon)) = \{\xi' \in \mathbb{C}^n : \pi(\xi') \in \pi(\xi + B_\varepsilon)\}$ . The claim  $\pi(\xi') \in \pi(\xi + B_\varepsilon)$  is equivalent to  $\{s(\xi') : s \in \mathbb{S}_n\} \in \{\{s(\xi + b) : s \in \mathbb{S}_n\} : b \in B_\varepsilon\}$ . Since

$$\begin{aligned} & \{s(\xi') : s \in \mathbb{S}_n\} \in \{\{s(\xi + b) : s \in \mathbb{S}_n\} : b \in B_\varepsilon\} \\ & \Leftrightarrow \{s(\xi') : s \in \mathbb{S}_n\} = \{s(\xi + b) : s \in \mathbb{S}_n\} \text{ for some } b \in B_\varepsilon \\ & \Leftrightarrow \xi' = s(\xi + b) \text{ for some } s \in \mathbb{S}_n, b \in B_\varepsilon \\ & \Leftrightarrow \xi' = s(\xi) + b \text{ for some } s \in \mathbb{S}_n, b \in B_\varepsilon \\ & \Leftrightarrow \xi' \in s(\xi) + B_\varepsilon \text{ for some } s \in \mathbb{S}_n \Leftrightarrow \xi' \in \bigcup_{s \in \mathbb{S}_n} (s(\xi) + B_\varepsilon) \end{aligned}$$

and  $\bigcup_{s \in \mathbb{S}_n} (s(\xi) + B_\varepsilon)$  is an open subset of  $\mathbb{C}^n$ , the claim follows. □

**COROLLARY 2.4.** *Let  $\varepsilon > 0$  and  $\xi \in \mathbb{C}^n$  be fixed. Then the set  $\pi(\xi + B_\varepsilon) \subset \mathbb{C}^n/\sim$  is open.*

**LEMMA 2.5.** *Let  $U \subset \mathbb{C}^n/\sim$  be a nonempty and open set. If  $\pi(\xi) \in U$  for some  $\xi \in \mathbb{C}^n$ , then there exists  $\varepsilon > 0$  such that  $\pi(\xi + B_\varepsilon) \subset U$ .*

**PROOF.** Let  $U \subset \mathbb{C}^n/\sim$  be a nonempty, open set. By the definition of the quotient topology,  $\{\xi \in \mathbb{C}^n : \pi(\xi) \in U\} \subset \mathbb{C}^n$  is open in  $\mathbb{C}^n$ . Therefore, if  $\pi(\xi) \in U$ , then there exists  $\varepsilon > 0$  such that  $\pi(\xi + B_\varepsilon) \subset U$ . □

The next corollary follows from Lemma 2.5 and Corollary 2.4.

**COROLLARY 2.6.** *A nonempty set  $U \subset \mathbb{C}^n/\sim$  is open if and only if for each  $\xi \in \mathbb{C}^n$ ,  $\pi(\xi) \in U$ , there exists  $\varepsilon > 0$  such that  $\pi(\xi + B_\varepsilon) \subset U$ .*

By Lemma 2.3 and Corollary 2.6, we conclude that for any  $\pi(\xi) \in \mathbb{C}^n/\sim$  there is a countable neighbourhood base of  $\pi(\xi)$ , so the space  $\mathbb{C}^n/\sim$  endowed with the quotient topology is a first-countable topological space. Moreover, in view of the finiteness of  $\pi(\xi), \xi \in \mathbb{C}^n$ , Corollary 2.6 has the following corollary.

**COROLLARY 2.7.** *A sequence  $(\pi(\xi^q)) \in \mathbb{C}^n/\sim, \xi^q \in \mathbb{C}^n, q \in \mathbb{N}$ , converges to  $\pi(\xi) \in \mathbb{C}^n/\sim$ , as  $n \rightarrow \infty$ , if and only if the set  $\bigcup_{q \in \mathbb{N}} \pi(\xi^q) \subset \mathbb{C}^n$  is bounded in  $\mathbb{C}^n$  and each convergent subsequence of a sequence  $\hat{\xi}^q \in \pi(\xi^q) \subset \mathbb{C}^n, q \in \mathbb{N}$ , has its limit in  $\pi(\xi) \subset \mathbb{C}^n$ .*

Let us now define a function  $\hat{\sigma} : \mathbb{C}^n/\sim \rightarrow \mathbb{C}^n$  by  $\hat{\sigma}(\pi(\xi)) = \sigma(\xi), \xi \in \mathbb{C}^n$ ; observe that  $\sigma(\xi)$  is independent of  $\xi' \in \pi(\xi)$ . The fundamental theorem of algebra guarantees that  $\hat{\sigma}$  is a surjection. Since  $\pi$  and  $\sigma$  are continuous functions, it easily follows from Corollary 2.7 that  $\hat{\sigma}$  is continuous. Moreover, by the fundamental theorem of algebra and Bézout’s theorem,  $\hat{\sigma}$  is injective. Consequently,  $\hat{\sigma}$  is a continuous bijection from  $\mathbb{C}^n/\sim$  onto  $\mathbb{C}^n$  (compare with [1, page 153] or [4]).

**THEOREM 2.8.**  *$\hat{\sigma}$  is a homeomorphism.*

**PROOF.** Let  $\tau : \mathbb{C}^n \rightarrow \mathbb{C}^n/\sim$  denote the inverse function of  $\hat{\sigma}$ :  $\tau(\hat{\sigma}(\pi(\xi))) = \pi(\xi), \xi \in \mathbb{C}^n$ . It suffices to show that if  $\mathbb{C}^n \ni a^q \rightarrow a \in \mathbb{C}^n$ , as  $q \rightarrow \infty$ , then  $\tau(a^q) \rightarrow_q \tau(a)$ . Let  $\xi \in \mathbb{C}^n$

be such that  $\pi(\xi) = \tau(a)$  and let  $\xi^q = (\xi_1^q, \dots, \xi_n^q) \in \mathbb{C}^n$  meet  $\pi(\xi^q) = \tau(a^q)$  for  $q \in \mathbb{N}$ . Let  $\hat{\xi}^q \in \pi(\xi^q)$ ,  $q \in \mathbb{N}$ . Then, for  $q \in \mathbb{N}$ ,  $i \in [n]$ ,

$$(\hat{\xi}_i^q)^n + a_n^q(\hat{\xi}_i^q)^{n-1} + \dots + a_2^q(\hat{\xi}_i^q) + a_1^q = 0$$

and, since  $(a^q)_{q \in \mathbb{N}}$  converges, the sequence  $(\hat{\xi}^q)_{q \in \mathbb{N}}$  is bounded. Since one can easily see that each convergent subsequence of  $(\hat{\xi}^q)_{q \in \mathbb{N}}$  has its limit in  $\pi(\xi)$ , by Corollary 2.7, the proof is complete.  $\square$

Since  $\pi(\xi + B_\varepsilon)$ ,  $\varepsilon > 0$ , is open in  $\mathbb{C}^n/\sim$  and, by Corollary 2.2,  $\pi(\xi') \in \pi(\xi + B_\varepsilon)$  implies that  $|\xi' - s(\xi)| < \varepsilon$  for some  $s \in \mathbb{S}_n$ , we can now state the following consequence of Theorem 2.8.

**COROLLARY 2.9.** *Let  $a_i \in \mathbb{C}$ ,  $i \in [n]$ , and  $P(z) := z^n + a_n z^{n-1} + \dots + a_2 z + a_1$ ,  $z \in \mathbb{C}$ . For each  $\varepsilon > 0$ , there exists  $\delta > 0$  such that, for  $a' \in \mathbb{C}^n$  with  $|a' - a| < \delta$ , it follows that  $|\xi' - s(\xi)| < \varepsilon$  for some  $s \in \mathbb{S}_n$ , where  $\xi \in \mathbb{C}^n$  is a vector of all roots of  $P$  and  $\xi'$  is a vector of roots of  $Q(z) := z^n + a'_n z^{n-1} + \dots + a'_2 z + a'_1$ ,  $z \in \mathbb{C}$ .<sup>1</sup>*

### 3. The general case

In this section, it is convenient to denote the coordinates of  $a \in \mathbb{C}^{n+1}$  by means of  $a = (a_n, a_{n-1}, \dots, a_1, a_0)$ . For  $a \in \mathbb{C}^{n+1}$ , the polynomial  $P_a$  with coefficients  $a$  is

$$P_a(z) := a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0, \quad z \in \mathbb{C}.$$

If  $a \in \mathbb{C}^{n+1}$ ,  $a_n \neq 0$ , by Corollary 2.9 we easily deduce that roots of  $P$  vary continuously in  $\mathbb{C}^n/\sim$  as coefficients change in a neighbourhood of  $a$ . What happens with the roots of  $P_{a'}$ ,  $a' \in \mathbb{C}^{n+1}$ , if  $a_n = 0$  and  $\deg(P_{a'}) > \deg(P_a)$ , as  $a'$  approaches  $a$ ? This section is devoted to answering that question.

Let  $\langle a \rangle := (a_0, a_1, \dots, a_n)$ ,  $a \in \mathbb{C}^{n+1}$ . For  $a \in \mathbb{C}^{n+1}$ ,  $0 \neq z \in \mathbb{C}$ ,

$$P_{\langle a \rangle}(z) = z^n P_a(z^{-1})$$

(cf. [6, Remark 1.3.2]). We state the following without proof.

**LEMMA 3.1.** *Let  $a \in \mathbb{C}^{n+1}$ ,  $a_n = \dots = a_{k+1} = 0$ ,  $0 < k \leq n$  and  $a_0 \neq 0$ . Then:*

- (1)  $P_a, P_{\langle a \rangle}$  are polynomials of degrees  $k$  and  $n$ , respectively;
- (2) all  $k$  roots of  $P_a$  are nonzero and exactly  $n - k$  out of  $n$  roots of  $P_{\langle a \rangle}$  vanish;
- (3) for  $0 \neq z \in \mathbb{C}$ , we have  $P_{\langle a \rangle}(z) = 0 \Leftrightarrow P_a(z^{-1}) = 0$ .

Applying Corollary 2.9 and Lemma 3.1 to  $P_{\langle a \rangle}$ , we obtain the following lemma.

**LEMMA 3.2.** *Let  $a \in \mathbb{C}^{n+1}$  be as in Lemma 3.1. For each sufficiently small  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for  $a' \in \mathbb{C}^{n+1}$  with  $|a' - a| < \delta$ , it follows that  $|\xi' - s(\xi)| < \varepsilon$  for some  $s \in \mathbb{S}_n$ , where  $\xi \in \mathbb{C}^n$  and  $\xi' \in \mathbb{C}^n$  are vectors of all roots of  $P_{\langle a \rangle}$  and  $P_{\langle a' \rangle}$ , respectively.*

<sup>1</sup>Here and hereafter, roots are counted with their multiplicities.

The general continuity result is contained in the following theorem.

**THEOREM 3.3.** *Let  $a = (a_n, a_{n-1}, \dots, a_1, a_0) \in \mathbb{C}^{n+1}$ ,  $a \neq 0$ ,  $a_n = \dots = a_{k+1} = 0$  and  $a_k \neq 0$  with  $k > 0$ . For each sufficiently small  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for  $a' \in \mathbb{C}^{n+1}$  with  $|a' - a| < \delta$ , exactly  $\deg(P_{a'}) - k$  roots of  $P_{a'}$  have moduli greater than  $1/\varepsilon$  and the other  $k$  roots, formed in a vector  $\xi'$ , satisfy  $|\xi' - s(\xi)| < \varepsilon$  for some  $s \in \mathbb{S}_k$ , where  $\xi \in \mathbb{C}^k$  is a vector of all roots of  $P_a$ .*

**PROOF.** Let  $\varepsilon > 0$  be fixed. Suppose that  $a_0 \neq 0$  and let us consider a sequence  $\mathbb{C}^{n+1} \ni a^q \rightarrow a$ ,  $q \in \mathbb{N}$ . Note that  $\langle a^q \rangle \rightarrow \langle a \rangle$  and that  $P_{a^q}(0) \neq 0$  for sufficiently large  $q$ . Thus, for sufficiently large  $q$ ,  $\deg(P_{\langle a^q \rangle}) = n$  and, if  $\xi^q = (\xi_1^q, \dots, \xi_n^q) \in \mathbb{C}^n$  is a vector of all roots of  $P_{\langle a^q \rangle}$ , then, by Lemma 3.2, exactly  $n - k$  of the roots are arbitrarily close to 0,  $(n - \deg(P_{a^q}))$  of the roots, say  $\xi_1^q, \dots, \xi_{n-\deg P_{a^q}}^q$ , are 0 and the other  $(\deg(P_{a^q}) - k)$  roots, say  $\xi_{n-\deg P_{a^q}+1}^q, \dots, \xi_{n-k}^q$ , are different from 0. Moreover, for sufficiently large  $q$ , the remaining  $k$  roots,  $\xi_{n-k+1}^q, \dots, \xi_n^q$ , are at a positive distance from 0 with

$$\liminf_{q \rightarrow \infty} \min\{|\xi_i^q| : i = n - k + 1, \dots, n - k\} > 0$$

(because  $a_0 \neq 0$ ). For  $i = n - \deg P_{a^q} + 1, \dots, n$ , we have  $\xi_i^q \neq 0$ , which, by Lemma 3.1(3), implies that  $P_{a^q}((\xi_i^q)^{-1}) = 0$ . Hence, the complete list of  $\deg(P_{a^q})$  roots of  $P_{a^q}$  is  $(\xi_{n-\deg P_{a^q}+1}^q)^{-1}, \dots, (\xi_n^q)^{-1}$ . Further, for sufficiently large  $q$ ,  $|(\xi_i^q)^{-1}| > 1/\varepsilon$ ,  $i = n - \deg P_{a^q} + 1, \dots, n - k$ , and, by Corollary 2.9, Lemma 3.1(3), the remark made at the beginning of this section and continuity of the mapping  $\mathbb{C} \ni z \mapsto 1/z$  at  $z \neq 0$ , we have  $|(\xi_{n-i}^q)^{-1} - s^q(\xi_i)| < \varepsilon$ ,  $i \in [k]$ , where  $\xi = (\xi_1, \dots, \xi_k)$  is a vector of all roots of  $P_a$  and  $s^q \in \mathbb{S}_k$ . The case  $a_0 = 0$  follows.

Suppose now that  $a_0 = 0$ . Since  $a \neq 0$ , there exists  $\xi \in \mathbb{C}$  with  $P_a(\xi) \neq 0$ . Let  $P_a^\xi(z) := P_a(z + \xi)$  for  $z \in \mathbb{C}$ . The degree of the polynomial  $P_a^\xi$  equals the degree of  $P_a$ . Since  $P_a^\xi(0) = P_a(\xi) \neq 0$ , it follows that 0 is not a root of  $P_a^\xi$ , so its constant term is nonzero. Moreover, the coefficients of  $P_a^\xi$  depend continuously on the coefficients  $a$ , and  $z \in \mathbb{C}$  is a root of  $P_a^\xi$  if and only if  $z + \xi$  is a root of  $P_a$ . It now suffices to apply the conclusion from the first part of the proof to  $P_a^\xi$  and use the fact that the sum  $z + \xi$  depends continuously on  $z$ . □

### 4. Applications to linear differential equations

For simplicity, we investigate linear differential equations of the second order with constant coefficients. First, let us consider the problem

$$y''(x) + a_1y'(x) + a_0y = 0, \quad y(0) = x_0, \quad y'(0) = x'_0, \tag{4.1}$$

where  $a_1, a_0 \in \mathbb{R}$ . Denote by  $P(\lambda)$  the characteristic polynomial of the equation (4.1) and suppose that it has two distinct roots. The unique solution to this problem has the form

$$y(x) = \frac{x'_0 - \lambda_2 x_0}{\lambda_1 - \lambda_2} e^{\lambda_1 x} + \frac{\lambda_1 x_0 - x'_0}{\lambda_1 - \lambda_2} e^{\lambda_2 x} \quad \text{for } x \geq 0.$$

Now, let us consider the problem

$$a_2^n y''(x) + a_1^n y'(x) + a_0^n y(x) = 0, \quad y_n(0) = x_0, \quad y'_n(0) = x'_0, \tag{4.2}$$

where  $a_2^n, a_1^n, a_0^n \in \mathbb{R}$  for  $n \in \mathbb{N}$  and suppose that  $a_i^n \rightarrow a_i$  as  $n \rightarrow \infty$  for  $i = 0, 1, 2$  (obviously here  $a_2 = 1$ ). Denote by  $y_n(x)$  the unique solution to the problem (4.2) for  $n \in \mathbb{N}$ . Theorem 2.8 implies that  $y_n \rightarrow y$  as  $n \rightarrow \infty$ , pointwise on  $\mathbb{R}_+$ .

The considerations of Section 3 motivate the following question. What can one say about the convergence of solutions to the problem (4.2) in the case when  $a_2^n \rightarrow 0$  as  $n \rightarrow \infty$ ? To answer that question, we consider the following example.

**EXAMPLE 4.1.** Fix real numbers  $x_0 \neq 0$  and  $a_0$  and consider the following equation:

$$\frac{1}{n}y''(x) + 2y'(x) + a_0y(x) = 0. \tag{4.3}$$

Note that the solution to the problem

$$2y'(x) + a_0y(x) = 0, \quad y(0) = x_0,$$

is given by  $y(x) = x_0e^{-a_0x/2}$ , so  $y(0) = x_0$  and  $y'(0) = -\frac{1}{2}a_0x_0$  and therefore it seems natural to endow the equation (4.3) with the initial conditions:

$$y(0) = x_0 \quad \text{and} \quad y'(0) = -\frac{1}{2}a_0x_0. \tag{4.4}$$

For sufficiently large  $n \in \mathbb{N}$ , there are two distinct real roots of the characteristic equation corresponding to the equation (4.3), namely

$$\lambda_1^n = -n\left(1 + \sqrt{1 - \frac{a_0}{n}}\right) \quad \text{and} \quad \lambda_2^n = -n\left(1 - \sqrt{1 - \frac{a_0}{n}}\right).$$

A general solution to the equation (4.3) is of the form

$$y_n(x) = c_1^n e^{\lambda_1^n x} + c_2^n e^{\lambda_2^n x}, \quad c_1^n, c_2^n \in \mathbb{R} \tag{4.5}$$

and thus the unique solution to the equation (4.3), satisfying the imposed conditions (4.4), is given by

$$y_n(x) = \frac{x_0(\lambda_2^n + \frac{1}{2}a_0)}{2n\sqrt{1 - a_0/n}} e^{-n(1 + \sqrt{1 - a_0/n})x} + \frac{x_0(\lambda_1^n + \frac{1}{2}a_0)}{-2n\sqrt{1 - a_0/n}} e^{-n(1 - \sqrt{1 - a_0/n})x}$$

for  $x \geq 0$ . It can be checked that, for  $x > 0$ ,

$$\lim_{n \rightarrow \infty} e^{-n(1 + \sqrt{1 - a_0/n})x} = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{x_0(\lambda_2^n + \frac{1}{2}a_0)}{2n\sqrt{1 - a_0/n}} = 0,$$

so

$$\frac{x_0(\lambda_2^n + \frac{1}{2}a_0)}{2n\sqrt{1 - a_0/n}} e^{\lambda_1^n x} \rightarrow 0,$$

pointwise for  $x > 0$ , as  $n \rightarrow \infty$ . Similarly, it can be checked that, for  $x > 0$ ,

$$\lim_{n \rightarrow \infty} e^{-n(1 - \sqrt{1 - a_0/n})x} = e^{-a_0x/2} \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{x_0(\lambda_1^n + \frac{1}{2}a_0)}{-2n\sqrt{1 - a_0/n}} = x_0,$$

so

$$\frac{x_0(\lambda_1^n + \frac{1}{2}a_0)}{-2n\sqrt{1 - a_0/n}} e^{\lambda_2^n x} \rightarrow x_0 e^{-a_0 x/2},$$

pointwise for  $x > 0$ , as  $n \rightarrow \infty$ .

Now, let us consider the equation

$$-\frac{1}{n}y''(x) + 2y'(x) + a_0y(x) = 0 \quad (4.6)$$

along with the initial conditions (4.4). Again, for sufficiently large  $n \in \mathbb{N}$ , the characteristic equation to (4.6) possesses two distinct roots:

$$\lambda_1^n = n\left(1 + \sqrt{1 + \frac{a_0}{n}}\right) \quad \text{and} \quad \lambda_2^n = n\left(1 - \sqrt{1 + \frac{a_0}{n}}\right).$$

A general solution to the equation (4.6) is of the form (4.5) and thus the unique solution to the equation (4.6) is given by

$$y_n(x) = \frac{-x_0(\lambda_2^n + \frac{1}{2}a_0)}{2n\sqrt{1 + a_0/n}} e^{\lambda_1^n x} + \frac{x_0(\lambda_1^n + \frac{1}{2}a_0)}{2n\sqrt{1 + a_0/n}} e^{\lambda_2^n x}$$

for  $x > 0$ . It can be checked that, for  $x > 0$ ,

$$\lim_{n \rightarrow \infty} e^{n(1 + \sqrt{1 + a_0/n})x} = +\infty \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{-x_0(\lambda_2^n + \frac{1}{2}a_0)}{2n\sqrt{1 + a_0/n}} = 0;$$

however, one can prove that

$$\left| \frac{-x_0(\lambda_2^n + \frac{1}{2}a_0)}{2n\sqrt{1 + a_0/n}} e^{\lambda_1^n x} \right| \rightarrow +\infty,$$

pointwise for  $x > 0$ , as  $n \rightarrow \infty$ . Moreover,

$$\lim_{n \rightarrow \infty} e^{n(1 - \sqrt{1 + a_0/n})x} = e^{-a_0 x/2} \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{x_0(\lambda_1^n + \frac{1}{2}a_0)}{2n\sqrt{1 + a_0/n}} = x_0,$$

so

$$\frac{x_0(\lambda_1^n + \frac{1}{2}a_0)}{2n\sqrt{1 + a_0/n}} e^{\lambda_2^n x} \rightarrow x_0 e^{-a_0 x/2},$$

pointwise for  $x > 0$ , as  $n \rightarrow \infty$ . Therefore,  $|y_n(x)| \rightarrow +\infty$ , pointwise for  $x > 0$ , as  $n \rightarrow \infty$ .

## 5. Final comments

In Section 3 of this paper, we presented Theorem 3.3 describing the behaviour of roots of polynomials  $f^q$  approaching a polynomial  $f$  of degree, say  $k$ , not greater than the degrees of  $f^q$ ,  $q \in \mathbb{N}$ . According to that theorem, exactly  $k$  roots of  $f^q$  tend to roots of  $f$ , while the other  $\deg f^q - k$  roots (if any) diverge to  $+\infty$  (in modulus),

as  $q \rightarrow \infty$ . Theorem 3.3 concerns the situation when  $\sup\{\deg f^q : q \in \mathbb{N}\}$  is finite. Could we generalise the assertion of Theorem 3.3 to the case  $\sup\{\deg f^q : q \in \mathbb{N}\} = +\infty$ ? The answer is no and the reason for that is as follows. Suppose that there is given a polynomial  $f$  of degree  $k$ , all the roots of which have positive moduli different from 1. For  $q \in \mathbb{N}$  and  $z \in \mathbb{C}$ , define three sequences of polynomials:

$$f^q(z) := (1 - z^q)f(z), \quad g^q(z) := (1 - q^q z^q)f(z), \quad h^q(z) := \left(1 - \frac{1}{q^q} z^q\right)f(z).$$

It is clear that the roots of  $f$  are among the roots of  $f^q, g^q, h^q$ . It is also clear that  $\sup\{\deg f^q : q \in \mathbb{N}\} = \sup\{\deg g^q : q \in \mathbb{N}\} = \sup\{\deg h^q : q \in \mathbb{N}\} = +\infty$ . At the same time, for large  $q \in \mathbb{N}$ , we see that  $q$  roots of  $f^q$  have unit moduli,  $q$  roots of  $g^q$ , which are not the roots of  $f$ , have absolute value  $1/q$ , while  $q$  roots of  $h^q$  have absolute value  $q$ .

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