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REGULAR SEMIGROUPS WITH NORMAL IDEMPOTENTS

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Abstract

In this paper, we investigate regular semigroups that possess a normal idempotent. First, we construct a nonorthodox nonidempotent-generated regular semigroup which has a normal idempotent. Furthermore, normal idempotents are described in several different ways and their properties are discussed. These results enable us to provide conditions under which a regular semigroup having a normal idempotent must be orthodox. Finally, we obtain a simple method for constructing all regular semigroups that contain a normal idempotent.

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1. Introduction

Let S be a regular semigroup with the set E of idempotents and let $\overline{E}$ be the subsemigroup generated by E. An idempotent u of S is called a *medial idempotent* if, for every element $x \in \overline{E}$, $xux = x$. A medial idempotent u is said to be *normal* if $u\overline{E}u$ is a semilattice. This notation appeared in [3].

The purpose of this paper is to characterize normal idempotents of a regular semigroup in various ways and to develop a method to construct a regular semigroup having a normal idempotent. The results we obtained are different from those provided in [3].

In fact, Blyth and McFadden gave an example to show that there exists a nonorthodox idempotent-generated regular semigroup which contains a normal idempotent. Then they described a normal idempotent by Green's relations and got a condition under which a regular semigroup having a normal idempotent is orthodox. By contrast, in Section 2, we first construct a nonorthodox nonidempotent-generated regular semigroup which contains a normal idempotent. It helps us explore different types of nonorthodox regular semigroups which have a normal idempotent. Next,

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all normal idempotents of a regular semigroup are characterized in various ways via subsets of the idempotent set and some inverse elements. In particular, we prove that an idempotent u of a regular semigroup S is normal if and only if uSu is a multiplicative inverse transversal for S. The result leads us to claim that a regular semigroup having a normal idempotent is locally inverse. Furthermore, several conditions under which a regular semigroup having a normal idempotent is orthodox are established.

Recall from [3] that every regular semigroup that contains a normal idempotent was described in terms of an idempotent-generated regular semigroup having a normal idempotent and an inverse semigroup with an identity. Naturally, there is a question here about the description of all idempotent-generated regular semigroups having a normal idempotent. In Section 3, we focus on investigating the structure of any regular semigroup having a normal idempotent. Actually, we establish a straightforward way of constructing such a regular semigroup: that is, we characterize it by means of a left inverse semigroup and a right inverse semigroup.

Refer to [2] and [4] for useful notation and terminology not defined in this paper. For convenience, we list some basic definitions as follows.

A semigroup S° is an *inverse transversal* for a regular semigroup S if S° is a subsemigroup of S and if, for any $x \in S$, $|V_{S^\circ}(x)| = 1$. In this case, the unique inverse of x is always denoted by x°.

If $S^\circ S S^\circ \subseteq S^\circ$, then S° is called a *quasi-ideal transversal* for S.

Let $I = \{aa^\circ |\ a \in S,\ a^\circ \in V_{S^\circ}(a)\}$, $\Lambda = \{a^\circ a |\ a \in S,\ a^\circ \in V_{S^\circ}(a)\}$ and E° be the set of idempotents of S°. If $\Lambda I \subseteq E^\circ$, then an inverse transversal S° is said to be *multiplicative*. It is known that, in this case, I and Λ are bands and $I \cap \Lambda = E^\circ$.

2. Normal idempotents

From now on, let S be a regular semigroup. Denote by $E(S)$ the set of idempotents of S and by $\overline{E(S)}$ the regular semigroup generated by $E(S)$. If there are no ambiguities, we would write them as E and $\overline{E}$, respectively.

In [3], a nonorthodox idempotent-generated regular semigroup having a normal idempotent is provided. Blyth and McFadden then described a normal idempotent of S by the Green's relations on S and claimed that S having a normal idempotent u is orthodox if and only if u is a *middle unit*, that is, $xux = x^2$ for all x in E.

However, in this section, we obtain a nonorthodox nonidempotent-generated regular semigroup which contains a normal idempotent. Moreover, we characterize normal idempotents of S, alternatively, according to some subsets of E and $V(e)$ for all idempotents e, where $V(e)$ is the set of all inverse elements of e. In addition, we prove that an idempotent u of a regular semigroup S is normal if and only uSu is a multiplicative inverse transversal for S. By applying this result, we deduce that a regular semigroup having a normal idempotent is locally inverse. Lastly, several different conditions under which a regular semigroup having a normal idempotent is orthodox are obtained. These conditions are actually equivalent to the condition that Blyth and McFadden provided.

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Example 2.1. Let B denote the monoid

$$\langle p, q \mid qp = 1\rangle = \{q^m p^n : m, n \geq 0\},$$

and let $T = M[B; \{1,2\}, \{1,2\}; P]$, where $P = \begin{pmatrix} q & 1 \\ 1 & p \end{pmatrix}$. Then T is regular and $E(T) = \{(2,1,1), (1,1,2), (1,p,1), (2,q,2)\}$. By computing, $(1,1,2)$ is a normal idempotent but T is not orthodox.

In the following theorem, all normal idempotents of a regular semigroup are described in alternative ways.

Theorem 2.2. *Let $u \in E$. For any $e, f \in E$, the following statements are equivalent.*

(1) *u is normal.*
(2) *uEu is a semilattice and $V(e) \cap uSu \neq \emptyset$ and $uefu \in E$.*
(3) *Eu is a left normal band and $V(e) \cap uS \neq \emptyset$ and $efu \in E$.*
(4) *uE is a right normal band and $V(e) \cap Su \neq \emptyset$ and $uef \in E$.*
(5) *$\overline{E}u$ is a left normal band and $V(e) \cap uS \neq \emptyset$.*
(6) *$u\overline{E}$ is a left normal band and $V(e) \cap Su \neq \emptyset$.*
(7) *$u\overline{E}u$ is a semilattice and $V(e) \cap uSu \neq \emptyset$.*

Proof. We only proof (1) ⇒ (2) ⇒ (3) ⇒ (5) ⇒ (7) ⇒ (1).

(1) ⇒ (2) With the given information, $u\overline{E}u$ is a semilattice. It follows immediately that $uEu = u\overline{E}u$ is a semilattice and $uefu \in E$. On the other hand, $ueueueu = ueu$ and $eueue = e$. Then $ueu \in V(e) \cap uS$.

(2) ⇒ (3) Obviously, $V(e) \cap uS \neq \emptyset$. Suppose that $e' \in V(e) \cap uSu$. Then $e = ee'ue = ee'ue$. This means that $e \mathcal{L} ue$. As $\mathcal{L}$ is a right congruence, $efu \mathcal{L} uefu$. Since $uefu \in E$, $efu = efu(uefu) = (efu)^2$. Let $g \in E$. Then $eufugu = e(ugufu) = eugufu$, so $(eufu)^2 = eufueufu = eueufu = eufu$. Hence Eu is a left normal band.

(3) ⇒ (5) Suppose that $f' \in V(f) \cap uS$. Then $f = ff'f = fuf'f$. It means that $f \mathcal{R} fu$. As $\mathcal{R}$ is a left congruence, $efu \mathcal{R} efu$. Since $efu \in E$, $ef = efuef$. For any $g \in E$, $efgu = efufgu \in Eu$. Therefore, by mathematical induction, $xu \in Eu$ for any $x \in \overline{E}$. As a result, $\overline{E} = Eu$ is a left normal band.

(5) ⇒ (7) In view of the above proof, $efuef = ef$. Suppose that $e' \in V(e) \cap uS$. Then $e(e'u)e = ee'(ee'uee')e = ee'e = e$ and $e'uee'u = e'ee'uee'u = e'u$. These imply that $e'u \in V(e) \cap uSu$. On the other hand, let $x, y \in \overline{E}$. Since $\overline{E}u$ is a left normal band, $uxuyu = u(uyuxu) = uyuxu$. This means that $u\overline{E}u$ is a semilattice.

(7) ⇒ (1) Similarly to the proof of (3) ⇒ (5), $efu \mathcal{R} ef$ and $xu \in E$ for any $x \in \overline{E}$. Since $\mathcal{R}$ is a left congruence, $xux = x$, which, together with $u\overline{E}u$ being a semilattice, implies that u is normal. □

Corollary 2.3. *If u is a normal idempotent of S then:*

(1) *$\overline{E}u = Eu$ is a left normal band;*
(2) *$u\overline{E} = uE$ is a right normal band; and*
(3) *$u\overline{E}u = uEu$ is a semilattice.*

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PROOF. This follows from Theorem 2.2 immediately. □

Here we investigate several interesting properties of a regular semigroup possessing a normal idempotent. This enables us to intensively look into such regular semigroups.

If u is an idempotent of S, then uSu is obviously a subsemigroup of S.

Let $I = \{aa^\circ \mid a \in S,\ a^\circ \in V_{uSu}(a)\}$ and $\Lambda = \{a^\circ a \mid a \in S,\ a^\circ \in V_{uSu}(a)\}$.

PROPOSITION 2.4. *If u is a normal idempotent of S then:*

(1) $(\forall x \in S,\ \forall x' \in V(x))$

$$ux'u \in V_{uSu}(x) \cap V_{uSu}(uxu) \cap V_{uSu}(xu) \cap V_{uSu}(ux);$$

(2) $(\forall x \in S)\ |V_{uSu}(x)| = 1$;

(3) *uSu is an inverse transversal for S;*

(4) *$I = Eu$ and $\Lambda = uE$;*

(5) *$\Lambda I = uEu$;*

(6) *$I\Lambda = \overline{E}$;*

(7) *$(\forall e, f \in E)\ e\,\mathcal{R}\,f \Leftrightarrow eu = fu$; and*

(8) *$(\forall e, f \in E)\ e\,\mathcal{L}\,f \Leftrightarrow ue = uf$.*

PROOF. (1) Notice that $ux'uxux'u = ux'xx'uxx'xux'xx'u = ux'xx'u = ux'u$ and $xux'ux = xx'xux'xx'uxx'x = xx'x = x$. We have $ux'u \in V_{uSu}(x)$. The remainder can be proved similarly.

(2), (3) Let $x', x^\circ \in V_{uSu}(x)$. Then $x', x^\circ \in V_{uSu}(uxu)$. Since $u\overline{E}u = E(uSu)$, uSu is an inverse semigroup. Therefore $x' = x^\circ$, so $|V_{uSu}(x)| = 1$.

(4) By Theorem 2.1, $\emptyset \neq I \subseteq Eu$. Let $e \in E$. Then $ueu \in V_{uSu}(eu)$ and $eu = e(ueu)$. It follows that $eu \in I$. Consequently, $I = Eu$. Similarly, $\Lambda = uE$.

(5) For any $e, f \in E$, $uefu \in u\overline{E}u = uEu$, while $uEu \subseteq \Lambda I$ is obvious. So $uEu = \Lambda I$ is as required.

(6) Suppose that $x \in \overline{E}$. Then, by Corollary 2.3, $xu, ux \in E$. So $xu \in I$ and $ux \in \Lambda$. From $x = xuux$ it follows that $x \in I\Lambda$. Therefore $\overline{E} \subseteq I\Lambda$, together with $I\Lambda = EuE \subseteq \overline{E}$, implies that $I\Lambda = \overline{E}$.

(7) Since $eue = e$ for any $e \in E$, $e\,\mathcal{R}\,eu$. Then $eu\,\mathcal{R}\,e\,\mathcal{R}\,f\,\mathcal{R}\,fu$. As Eu is left normal, $eu = fu$. The converse part follows from $e\,\mathcal{R}\,eu = fu\,\mathcal{R}\,f$.

(8) This is obtained by a similar argument to that of (7). □

Up to now, we have shown that if u is a normal idempotent of S, then uSu is a multiplicative inverse transversal. Actually, the reverse is also true.

THEOREM 2.5. *For any $u \in E$, u is a normal idempotent if and only if uSu is a multiplicative inverse transversal.*

PROOF. The forward direction of this theorem is immediate from the above proposition. For the reverse direction, we take the Theorem 2.2(2) into consideration. By the hypothesis, $V_{uSu}(x) \neq \emptyset$ for any $x \in S$. In particular, if $x \in E$, then $V_{uSu}(x) \subseteq E(uSu)$, where $E(uSu)$ is the set of idempotents of uSu. We next need to show that uEu is a semilattice and that, for any $e, f \in E$, $uefu \in uEu$. In fact, it is easy to check that $V_{uSu}(e) = V_{uSu}(eu)$. Suppose that $e^\circ \in V_{uSu}(e)$. Then $ee^\circ \in I$ and $e^\circ eu \in E(uSu)$. Since we know that $E(uSu) \subseteq I$ and I is a band, $eu = (ee^\circ)e^\circ eu \in I$. Together with $I \subseteq Eu$, we obtain $I = Eu$. By applying similar arguments, $\Lambda = uE$. So Eu and uE are bands. We conclude that $uEu = E(uSu)$ is a semilattice. Again by the hypothesis, $\Lambda I \subseteq E(uSu)$: that is, $uEEu \subseteq uEu$. Therefore $uefu \in uEu$, as required. □

Corollary 2.6. *Let S° be a multiplicative inverse transversal for S. Then the identity of S° is a normal idempotent of S.*

Proof. Let $e \in E$ be the identity of S°. Then $eSe = S^\circ$ is a multiplicative inverse transversal for S, so e is normal. □

Recall from [1] that a regular semigroup S with an inverse transversal S° is locally inverse if and only if S° is a quasi-ideal of S. Together with Theorem 2.5, we claim that a regular semigroup having a normal idempotent must be locally inverse. In addition, we have the following proposition.

Proposition 2.7. *If u is a normal idempotent of S then:*

(1) $(\forall e \in E)$ *Se is a left inverse semigroup;*
(2) $(\forall e \in E)$ *eS is a right inverse semigroup; and*
(3) $(\forall e \in E)$ *eSe is an inverse semigroup.*

Proof. We only prove (1); the remainder will be obtained by a similar argument. Let $E(Se)$ be the set of idempotents of Se. Suppose that $g, h, k \in E(Se)$. Then

$$\begin{aligned}(gh)^2 = (gehe)^2 &= [g(eue)h(eue)]^2\\ &= (geu)(ehu)(egu)(ehu)e\\ &= (geu)(egu)(ehu)(ehu)e\\ &= guehue = gh.\end{aligned}$$

This means that $E(Se)$ is a band. On the other hand,

$$\begin{aligned}ghk = geheke &= geueheuekeue\\ &= ge(uehueku)e = ge(uekuehu)e\\ &= gkh.\end{aligned}$$

So $E(Se)$ is a left normal band and then Se is a left inverse semigroup. □

According to the above proposition, a normal idempotent of a regular semigroup must induce some orthodox subsemigroups. We now consider the conditions under which each regular semigroup having a normal idempotent must be orthodox.

Theorem 2.8. *Let u be a normal idempotent of S. Then the following statements are equivalent.*

(1) *S is an orthodox semigroup.*
(2) $(\forall x \in S)$ *$uxu \in E \Rightarrow x \in E$.*
(3) $(\forall e, f \in E)$ *$uefu \in E \Rightarrow ef \in E$.*

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(4) $(\forall i \in I, \forall \lambda \in \Lambda)\ i\lambda \in E$.
(5) $(\forall i \in I, \forall \lambda \in \Lambda)\ \lambda ui = \lambda i$.
(6) $(\forall x, y \in S)\ xy = xuy$.

Proof. (1) ⇒ (2) By the hypotheses, we know that E is a band. For any $x \in S$, suppose that $x' \in V(x) \cap uSu$. Since $x = xx'uxux'x$ and $uxu \in E$, $x \in E$.

(2) ⇒ (3) This is trivial.

(3) ⇒ (4) We have shown that if u is normal, then uEu is a semilattice. Therefore, $ui\lambda u \in (uEu)(uEu) \subseteq E$ implies that $i\lambda \in E$.

(4) ⇒ (5) With the given information, uSu is a multiplicative inverse transversal. Then $\lambda ui, \lambda i \in \Lambda I \subseteq E$. It follows that

$$(\lambda i)(\lambda ui)(\lambda i) = \lambda(i\lambda ui\lambda)i = \lambda(i\lambda)i = \lambda i \quad \text{and} \quad (\lambda ui)(\lambda i)(\lambda ui) = \lambda ui.$$

So $\lambda ui \in V_{uSu}(i\lambda)$. Easily, $\lambda i \in V_{uSu}(i\lambda)$. Notice that uSu is an inverse transversal; $\lambda ui = \lambda i$.

(5) ⇒ (6) Review the proof of Theorem 2.4; $I = Eu$ and $\Lambda = uE$. Let $x' \in V(x) \cap uSu$ and $y' \in V(y) \cap uSu$. Since $x'x = ux'x \in \Lambda$ and $yy' = yy'u \in I$, $xuy = x(x'xuyy')y = xx'xyy'y = xy$.

(6) ⇒ (1) For any $e, f \in E$, $ef = efuef = efef$. So $ef \in E$. □

In view of this theorem, we say that the conditions are equivalent to u being a middle unit.

3. A structure theorem

Blyth and McFadden in [3] described every regular semigroup that contains a normal idempotent in terms of an idempotent-generated regular semigroup with a normal idempotent and an inverse semigroup with an identity. In reality, we also wonder about the characterization of all idempotent-generated regular semigroups having a normal idempotent.

The objective of this section is to construct all regular semigroups having a normal idempotent by some simpler building bricks.

Let M be a left inverse semigroup and let N be a right inverse semigroup. Suppose that M and N have a common element u as their right identity and left identity, respectively. Then $uMu = uM$ and $uNu = Nu$. Also, assume that $uM \cong Nu$. In this case, for convenience, denote uM and Nu by S°. Then S° is an inverse monoid such that $S^\circ \subseteq M \cap N$.

As M is a left inverse semigroup, for any $x \in M$, there is an unique idempotent $e \in E(M)$ such that $e \mathcal{R} x$. We denote it by x^+. Similarly, for any $y \in N$, let y^* be the unique idempotent such that $y \mathcal{L} y^* \in E(N)$.

Define a map $\circ : N \times M \to S^\circ$ by $(y, x) \mapsto y \circ x$ for any $x \in M, y \in N$.

The quadruple $(S^\circ; M, N; \circ)$ is said to be *permissible* if:

$(P1)$ $(\forall x \in M,\ \forall y \in N,\ \forall s \in S^\circ)$ $s(y \circ x) = (sy) \circ x$ and $(y \circ x)s = y \circ (xs)$;

$(P2)$ $(\forall x \in M,\ \forall y \in N,\ \forall s \in S^\circ)$ $y \circ s = ys$ and $s \circ x = sx$; and

$(P3)$ $(\forall e \in E(M),\ \forall f \in E(N))$ $f \circ e \in E(S^\circ)$.

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Let $P(S^\circ; M, N; \circ) = \{(x, y) \mid x \in M, y \in N,\ ux = yu\}$.

Denote $P(S^\circ; M, N; \circ)$ by P and define a multiplication on P as

$$(x, y)(a, b) = (x^+(y \circ a), (y \circ a)b^*).$$

Lemma 3.1. *P is a regular semigroup.*

Proof. It is easy to check that $ux^+u \mathrel{\mathcal{R}} uxu = uyu \mathrel{\mathcal{R}} (uyu)^+$. Since S° is an inverse semigroup, $ux^+u = (uyu)^+$. Then $ux^+(y \circ a)u = ux^+u(yu \circ a)u = (ux^+uyu) \circ a)u = u(y \circ a)u$. As a dual, $u(y \circ a)a^*u = u(y \circ a)ua^*u = u(y \circ aua^*u) = u(y \circ a)u$. Hence $ux^+(y \circ a)u = u(y \circ a)a^*u$. This means that the above multiplication on P is well defined.

Let $(c, d) \in P$. Then

$$\begin{aligned}[(x, y)(a, b)](c, d) &= (x^+(y \circ a), (y \circ a)b^*)(c, d)\\ &= ([x^+(y \circ a)]^+(y \circ a)(b^* \circ c), (y \circ a)(b^* \circ c)d^*)\\ &= ([x^+(y \circ a)^+](y \circ a)(b^* \circ c), (y \circ a)(b^* \circ c)d^*)\\ &= ([x^+(y \circ a)^+(y \circ a)](b^* \circ c), (y \circ a)(b^* \circ c)d^*)\\ &= (x^+(y \circ a)(b^* \circ c), (y \circ a)(b^* \circ c)d^*)\end{aligned}$$

and

$$\begin{aligned}(x, y)[(a, b)(c, d)] &= (x, y)(a^+(b \circ c), (b \circ c)d^*)\\ &= (x^+[y \circ (a^+(b \circ c))], [y \circ (a^+(b \circ c))]d^*).\end{aligned}$$

Suppose that $a' \in V_M(a)$. Then $a^+ = aa' = aua'u$. Since $uau = ubu$, $b^* = ua'ub$. It follows that

$$\begin{aligned}y \circ (a^+(b \circ c)) &= y \circ (aua'u(b \circ c))\\ &= y \circ (a(ua'ub \circ c))\\ &= y \circ (a(b^* \circ c))\\ &= (y \circ a)(b^* \circ c).\end{aligned}$$

In conclusion,

$$[(x, y)(a, b)](c, d) = (x, y)[(a, b)(c, d)].$$

Therefore P is a semigroup.

Let $x' \in V_M(x)$ and $y' \in V_N(y)$. Then $ux'u \in V_{S^\circ}(x) = V_{S^\circ}(uxu)$ and $uy'u \in V_{S^\circ}(y) = V_{S^\circ}(uyu)$. Notice that $uxu = uyu$. Since S° is an inverse semigroup, $ux'u = uy'u$. It follows that $(ux'u, uy'u) \in P$.

$$\begin{aligned}(x, y)(ux'u, uy'u)(x, y) &= (x^+(yux'u), (yux'u)(uy'u)^*)(x, y)\\ &= (x^+(uyux'u), yux'u)(x, y)\\ &= (x^+(uxu)^+, uyux'u)(x, y)\\ &= (x^+, yux'u)(x, y)\\ &= (x^+(uyux'ux), (uyux'ux)y^*)\\ &= (x, (uyux'uxu)y^*)\\ &= (x, y)\end{aligned}$$

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and

$$\begin{aligned}(ux'u, uy'u)(x, y)(ux'u, uy'u) &= ((ux'u)^+(uy'ux), (uy'ux)y^*)(ux'u, uy'u)\\ &= ((ux'ux), (uy'uxu)(uy'uy))(ux'u, uy'u)\\ &= ((ux'ux), uy'uy)(ux'u, uy'u)\\ &= ((ux'uxu)(uy'uyuy'u), (uy'uyuy'u)(uy'u)^*)\\ &= (ux'u, uy'u).\end{aligned}$$

Therefore $(ux'u, uy'u) \in V((x, y))$, so P is regular. □

Lemma 3.2. $E(P) = \{(x, y) \in P \mid ux = yu \in E(S^\circ)\}$.

Proof. It is trivial to check that

$$(x, y) = (x, y)(x, y) \Leftrightarrow x^+(y \circ x) = x, (y \circ x)y^* = y.$$

Since $x^+(y \circ x) = x$, $ux^+(y \circ x) = ux$. This implies that

$$ux = ux^+(y \circ x) = ux^+u(y \circ x) = (ux^+uyuy^*) \circ x = ux^2.$$

Conversely, $ux = ux^2$ also implies that $ux = ux^+(y \circ x)$. Notice that

$$x = x^+ux = x^+ux^+(y \circ x) = x^+(y \circ x).$$

We conclude that $x^+(y \circ x) = x$ if and only if $ux = ux^2$, if and only if $ux \in E(S^\circ)$. Similarly, $(y \circ x)y^* = y$ if and only if $yu = y^2u$, if and only if $yu \in E(S^\circ)$. Therefore $E(W) = \{(x, y) \in W \mid ux = yu \in E(S^\circ)\}$. □

In what follows, we will use the alternative description of a normal idempotent, obtained in Section 2, to prove that P contains a normal idempotent.

Lemma 3.3. *Denote the element (u, u) by $\bar{u}$. Then $V((x, y)) \cap \bar{u}P\bar{u} \neq \emptyset$ for all $(x, y) \in E(P)$.*

Proof. According to the proof of Lemma 3.1, $(ux'u, uy'u) \in V((x, y))$. Since $(ux'u, uy'u) = \bar{u}(x', y')\bar{u}$, $V((x, y)) \cap \bar{u}P\bar{u} \neq \emptyset$. □

Lemma 3.4. *$E(P)\bar{u}$ is a left normal band.*

Proof. Let $(x, y) \in E(P)$. Then $(x, y)\bar{u} = (x^+(y \circ u), (y \circ u)u) = (x^+(yu), yu) = (x, yu)$. Since

$$\begin{aligned}(x, yu)^2 &= (x^+(ux^2), (y^2u)(yu)^*)\\ &= (x^+ux, (yu)(yu)^*) = (x, yu),\end{aligned}$$

$E(P)\bar{u} \subseteq E(P)$. Let $(a, b) \in E(P)$. Then

$$\begin{aligned}(x, y)\bar{u}(a, b)\bar{u} &= (x, yu)(a, bu) = (x^+(yua), (yua)(bu)^*)\\ &= (xua, (ybu)(bu)^*) = (xua, ybu)\\ &= (xua, xua).\end{aligned}$$

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As $uxua \in E(S^\circ)$, $E(P)\bar{u}$ is a band. Next, notice that $\bar{u}(a,b)\bar{u} = (ua, bu)$ and $\bar{u}(c,d)\bar{u} = (uc, du)$. Hence

$$\begin{aligned}\bar{u}(a,b)\bar{u}\bar{u}(c,d)\bar{u} &= (ua,bu)(uc,du)\\ &= ((ua)^+(buc),(buc)(du)^*) = (uauc, budu)\\ &= (ucua, dubu) = ((uc)^+(dua),(dua)(bu)^*)\\ &= (uc,du)(ua,bu)\\ &= \bar{u}(c,d)\bar{u}\bar{u}(a,b)\bar{u}.\end{aligned}$$

Therefore $(x,y)\bar{u}(a,b)\bar{u}(c,d)\bar{u} = (x,y)\bar{u}(c,d)\bar{u}(a,b)\bar{u}$, so $E(P)\bar{u}$ is a left normal band. □

Theorem 3.5. *Suppose that $(S^\circ; M, N; \circ)$ is a permissible quadruple. Then $P(S^\circ; M, N; \circ)$ is a regular semigroup having a normal idempotent. Conversely, any regular semigroup that contains a normal idempotent can be constructed in this way.*

Proof. According to Lemmas 3.3 and 3.4 and Theorem 2.2, we only need to prove this for any $(x,y),(a,b) \in E(P)$, $(x,y)(a,b)\bar{u} \in E(P)$. It is easy to check that $ux\ \mathcal{L}\ x$. Since $ux \in E(S^\circ)$, $x^2 = xux = x$. This implies that $x \in E(M)$. Similarly, $a \in E(M)$ and $y, b \in E(N)$. Then $y \circ a \in E(S^\circ)$ by (P3). As we know, M is a left inverse semigroup and N is a right inverse semigroup. So $x = x^+$ and $b^* = b$. Finally, by trivial computing, $(x,y)(a,b)\bar{u} \in E(P)$ is true.

Conversely, let S be any regular semigroup that contains a normal idempotent u. Then, by Proposition 2.7, Su is a left inverse semigroup and uS is a right inverse semigroup. For all $x \in Su$ and $y \in uS$, let $y \circ x$ be yx. We claim that $(uSu; Su, uS; \circ)$ is a permissible quadruple. Suppose that $(xu, uy) \in P(uSu; Su, uS; \circ)$. Then $uxu = uyu$ and $xu = (x^+uy)u$, $uy = u(x^+uy)$. Denote x^+uy by z. We know that $(xu, uy) = (zu, uz)$, so $P(uSu; Su, uS; \circ) = \{(tu, ut) \mid t \in S\}$. Define a function

$$\tau : S \to P(uSu; Su, uS; \circ),\ s \mapsto (su, us) \quad \text{for all } s \in S.$$

Obviously, τ is surjective. Assume that $(su, us) = (tu, ut)$. Then $s = (su)^+us = (tu)^+ut = t$. Hence τ is also injective. On the other hand, $\tau(s)\tau(t) = (su, us)(tu, ut) = (stu, ust) = \tau(st)$. Therefore τ is an isomorphism. □

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