OSCULATING SPACES

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In this paper an attempt is made to prove some of the basic theorems on the osculating spaces of a curve under minimum assumptions. The natural approach seems to be the projective one. A duality yields the corresponding results for the characteristic spaces of a family of hyperplanes. A duality theorem for such a family and its characteristic curve also is proved. Finally the results are applied to osculating hyperspheres of curves in a conformal space.

The analytical tools are collected in the first three sections. Some of them may be of independent interest.

1. On Taylor's theorem. The following version of Taylor's theorem should be known. For the convenience of the reader, we include a proof.

In this paper, the symbol I always denotes an interval on the real axis. It may be open or closed. If $t_0 \in I$, put

$$J = \{h | t_0 + h \in I\}; \qquad \text{thus } 0 \in J.$$

"Neighbourhoods" are neighbourhoods on I respectively J.

THEOREM 1.1. *Let $f(t)$ be defined in I and p-times differentiable at $t_0 \in I$; $p > 0$. Then*

$$f(t_0 + h) = f(t_0) + \frac{h}{1!}f'(t_0) + \ldots + \frac{h^{p-1}}{(p-1)!}f^{(p-1)}(t_0) + \frac{h^p}{p!}(f^{(p)}(t_0) + \epsilon(h)) \; ; \quad \lim_{h\to 0} \epsilon(h) = 0.$$

Proof. The function

$$\phi(h) = f(t_0 + h) - \left(f(t_0) + \frac{h}{1!}f'(t_0) + \ldots + \frac{h^p}{p!}f^{(p)}(t_0)\right)$$

is defined in J and p-times differentiable at $h = 0$. It satisfies

$$\phi(0) = \phi'(0) = \ldots = \phi^{(p)}(0) = 0. \tag{1.1}$$

Apply Taylor's theorem to $\phi(h)$ with $p - 1$ instead of p. Thus there exists a $\theta = \theta(h)$ with $0 < \theta < 1$ such that

$$\phi(h) = \frac{h^{p-1}}{(p-1)!}\phi^{(p-1)}(\theta h).$$

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Since $\phi^{(p-1)}(h)$ is still differentiable at $h = 0$, (1.1) implies

$$\phi^{(p-1)}(h) = \phi^{(p-1)}(0) + h\eta(h) = h\eta(h)$$

where

$$\lim_{h\to 0} \eta(h) = \phi^{(p)}(0) = 0.$$

Replacing h by θh we obtain

$$\phi(h) = \frac{h^{p-1}}{(p-1)!}\cdot \theta h \cdot \eta(\theta h) = \frac{h^p}{p!}\epsilon(h), \qquad \lim_{h\to 0}\epsilon(h) = 0.$$

This proves Theorem 1.1.

If we put $\epsilon(0) = 0$, the function $\epsilon(h)$ will be continuous in J. The same applies to the functions

$$\epsilon_m(h) = h^m\epsilon(h); \qquad m = 0, 1, \ldots, p.$$

The function

$$\epsilon_p(h) = p!\phi(h)$$

was p-times differentiable at $h = 0$ and satisfied

$$(1.2) \qquad \epsilon_p(0) = \epsilon_p'(0) = \ldots = \epsilon_p^{(p)}(0) = 0.$$

It will be differentiable in some neighbourhood of the origin.

We require the case $m = p - 1$ of the following remark.

THEOREM 1.2. *Let $p > 1$, $1 \leqslant m \leqslant p - 1$. Then $\epsilon_m(h)$ is m-times continuously differentiable at $h = 0$ and satisfies*

$$\epsilon_m(0) = \epsilon_m'(0) = \ldots = \epsilon_m^{(m)}(0) = 0.$$

Proof. Applying Theorem 1.1 to $\epsilon_p'(h)$, we obtain on account of (1.2)

$$\epsilon_p'(h) = h^{p-1}\delta(h) \qquad \text{where} \qquad \lim_{h\to 0}\delta(h) = \delta(0) = 0.$$

Put

$$\delta_m(h) = h^m\delta(h); \qquad m = 0, 1, \ldots, p - 1.$$

We first verify that in some neighbourhood of the origin

$$(1.3) \qquad \epsilon_m'(h) = \delta_{m-1}(h) - (p - m)\epsilon_{m-1}(h); \qquad m = 1, 2, \ldots, p - 1.$$

The right-hand term vanishes at $h = 0$. On the other hand

$$\epsilon_m'(0) = \lim_{h\to 0}\frac{\epsilon_m(h) - \epsilon_m(0)}{h} = \lim_{h\to 0}\frac{\epsilon_m(h)}{h} = \lim_{h\to 0}\epsilon_{m-1}(h) = 0.$$

Now let $h \neq 0$. Then

$$\epsilon_m'(h) = \left(\frac{1}{h^{p-m}}\epsilon_p(h)\right)' = \frac{1}{h^{p-m}}\epsilon_p'(h) - \frac{p-m}{h^{p-m+1}}\epsilon_p(h)$$
$$= h^{m-1}\delta(h) - (p - m)h^{m-1}\epsilon(h).$$

This yields (1.3).

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For $m = 1$, (1.3) implies

$$\epsilon_1'(h) = \delta(h) - (p-1)\epsilon(h).$$

The right-hand term being continuous and zero at the origin, the same holds true of $\epsilon_1'(h)$.

Suppose Theorem 1.2 has been proved up to $m - 1$. Then either of the two functions in the right-hand term of (1.3) is $(m-1)$-times continuously differentiable at $h = 0$ and vanishes there together with its derivatives up to the order $m - 1$. The same will therefore apply to $\epsilon_m'(h)$. This proves our theorem for m.

2. Divided differences. Suppose the function $f(t)$ is defined in the interval I; $t_0, t_1, \ldots$ lie in I and are mutually distinct. The divided differences of $f(t)$ are defined through

$$(2.1)\qquad \begin{cases} [t_0] = f(t_0) \\ [t_0 t_1 \ldots t_p] = \dfrac{[t_0 t_1 \ldots t_{p-1}] - [t_1 \ldots t_{p-1} t_p]}{t_0 - t_p}; \qquad p = 1, 2, \ldots . \end{cases}$$

The divided differences of another function $g(t)$ are denoted by

$$[t_0 t_1 \ldots t_p]_g.$$

The following well-known formula is readily verified by induction:

$$(2.2)\qquad [t_0 t_1 \ldots t_m] = \sum_{k=0}^{m} \left\{ [t_k] \Big/ \prod_{\substack{i=0 \\ i \neq k}}^{m} (t_k - t_i) \right\}; \qquad m = 1, 2, \ldots .$$

The following mean value theorem also is known: Let $f(t)$ be p-times differentiable in I. Then

$$(2.3)\qquad \begin{cases} [t_1 \ldots t_{p+1}] = f^{(p)}(\tau)/p! \\ \text{Min}\,(t_1, \ldots, t_{p+1}) < \tau < \text{Max}\,(t_1, \ldots, t_{p+1}); \end{cases}$$

cf. **(1)**.

We need

THEOREM 2.1. *Let $f(t)$ be p-times differentiable at t_0; $p > 0$. Then*

$$\lim_{t_1, \ldots, t_p \to t_0} [t_0 t_1 \ldots t_p] = \frac{f^{(p)}(t_0)}{p!}.$$

Proof. We may assume $p > 1$. Put

$$g(t) = \begin{cases} \dfrac{f(t) - f(t_0)}{t - t_0} & \text{if } t \neq t_0 \\ f'(t_0) & \text{if } t = t_0 \end{cases}.$$

By Theorem 1.1

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$$g(t_0 + h) = \sum_1^p \frac{h^{n-1}}{n!} f^{(n)}(t_0) + \frac{h^{p-1}}{p!}\epsilon(h).$$

By Theorem 1.2, the function $h^{p-1}\epsilon(h)$ is $(p-1)$-times continuously differentiable at $h = 0$. It vanishes there together with its derivatives up to the order $p - 1$. Hence $g(t)$ is $(p-1)$-times continuously differentiable at t_0 and

$$g^{(p-1)}(t_0) = \frac{1}{p} f^{(p)}(t_0). \tag{2.4}$$

We readily verify by induction that

$$[t_1 \ldots t_m]_g = [t_0 t_1 \ldots t_m]; \qquad m = 1, 2, \ldots .$$

Replacing f by g and p by $p - 1$ in (2.3), we therefore obtain

$$[t_0 t_1 \ldots t_p] = [t_1 \ldots t_p]_g = \frac{g^{(p-1)}(\tau)}{(p-1)!},$$

$$\operatorname{Min}(t_1, \ldots, t_p) < \tau < \operatorname{Max}(t_1, \ldots, t_p).$$

Let $t_1, \ldots, t_p$ tend to t_0. Then τ will also converge to t_0 and we obtain on account of (2.4)

$$\lim_{t_1, \ldots, t_p \to t_0} [t_0 t_1 \ldots t_p] = \lim_{\tau \to t_0} \frac{g^{(p-1)}(\tau)}{(p-1)!}$$

$$= \frac{g^{(p-1)}(t_0)}{(p-1)!} = \frac{f^{(p)}(t_0)}{p!}.$$

Obviously, (2.1), (2.2) and Theorem 2.1 may be applied to vector valued functions.

3. Some mean-values and limits. In the following let $n > 0$ be fixed. The vector function

$$\mathfrak{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t))$$

is defined in the interval I. Let $0 < m \leqslant n$. The parameter values $t_1, \ldots, t_m$ are mutually distinct. Let $\mathfrak{a}_{m+1}, \ldots, \mathfrak{a}_n$ be fixed vectors, say

$$\mathfrak{a}_k = (a_{k1}, \ldots, a_{kn});\ k = m + 1, \ldots, n.$$

Put

$$(\mathfrak{x}(t_1), \mathfrak{x}(t_2), \ldots, \mathfrak{x}(t_m), \mathfrak{a}_{m+1}, \ldots, \mathfrak{a}_n)$$

$$= \begin{vmatrix} x_1(t_1) & x_1(t_2) & \ldots & x_1(t_m) & a_{m+1,1} & \ldots & a_{n1} \\ x_2(t_1) & x_2(t_2) & \ldots & x_2(t_m) & a_{m+1,2} & \ldots & a_{n2} \\ \ldots \\ x_n(t_1) & x_n(t_2) & \ldots & x_n(t_m) & a_{m+1,n} & \ldots & a_{nn} \end{vmatrix}.$$

Let

$$\Delta_m = \frac{(\mathfrak{x}(t_1), \mathfrak{x}(t_2), \ldots, \mathfrak{x}(t_m), \mathfrak{a}_{m+1}, \ldots, \mathfrak{a}_n)}{\prod_{1 \leqslant i < k \leqslant m} (t_k - t_i)}. \tag{3.1}$$

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Formula (2.2) readily implies

(3.2) $$\Delta_m = ([t_1], [t_1t_2], \ldots, [t_1 \ldots t_m], \mathfrak{a}_{m+1}, \ldots, \mathfrak{a}_n)$$

where the divided differences are those of $\mathfrak{x}(t)$.

THEOREM 3.1. *Let $\mathfrak{x}(t)$ be $(m-1)$-times differentiable at t_1. Then*

$$\lim_{t_2,\ldots,t_m\to t_1} \Delta_m = \frac{(\mathfrak{x}(t_1), \mathfrak{x}'(t_1), \ldots, \mathfrak{x}^{(m-1)}(t_1), \mathfrak{a}_{m+1}, \ldots, \mathfrak{a}_n)}{1!\,2!\ldots(m-1)!}.$$

Proof. Write

$$[t_1t_2 \ldots t_p]_{x_i} = [t_1t_2 \ldots t_p]_i.$$

Thus this number is the ith component of the vector $[t_1t_2 \ldots t_p]$.
By (3.2)

(3.3) $$\Delta_m = \begin{vmatrix} [t_1]_1 & [t_1t_2]_1 & \ldots & [t_1 \ldots t_m]_1 & a_{m+1,1} & \ldots & a_{n1} \\ [t_1]_2 & [t_1t_2]_2 & \ldots & [t_1 \ldots t_m]_2 & a_{m+1,2} & \ldots & a_{n2} \\ \ldots \\ [t_1]_n & [t_1t_2]_n & \ldots & [t_1 \ldots t_m]_n & a_{m+1,n} & \ldots & a_{nn} \end{vmatrix}.$$

By Theorem 2.1

$$\lim_{t_2,\ldots,t_p\to t_1} [t_1 \ldots t_p]_i = \frac{x_i^{(p-1)}(t_1)}{(p-1)!}.$$

The determinant being a continuous function of its elements, (3.3) therefore readily implies our assertion.

THEOREM 3.2. *Let $\mathfrak{x}(t)$ be $(m-1)$-times differentiable in I. Then there are m numbers $\tau_1 = t_1, \tau_2, \ldots, \tau_m$ such that*

$$\Delta_m = \frac{(\mathfrak{x}(\tau_1), \mathfrak{x}'(\tau_2), \ldots, \mathfrak{x}^{(m-1)}(\tau_m), \mathfrak{a}_{m+1}, \ldots, \mathfrak{a}_n)}{1!\,2!\ldots(m-1)!},$$

$$\operatorname{Min}(t_1, \ldots, t_k) < \tau_k < \operatorname{Max}(t_1, \ldots, t_k)\ ;\ k = 2, \ldots, m.$$

In order to prove this statement, we generalize it. Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ be constant vectors. For each k let $t_{k1}, \ldots, t_{kk}$ lie in I and be mutually distinct. Put

$$\Gamma_k = ([t_{11}], [t_{21}t_{22}], \ldots, [t_{k1} \ldots t_{kk}], \mathfrak{a}_{k+1}, \ldots, \mathfrak{a}_n),$$
$$f(t) = ([t_{11}], [t_{21}t_{22}], \ldots, [t_{k-1,1} \ldots t_{k-1,k-1}], \mathfrak{x}(t), \mathfrak{a}_{k+1}, \ldots, \mathfrak{a}_n).$$

Thus the $(k-1)$st divided difference

$$[t_{k1} \ldots t_{kk}]_f$$

of f is equal to Γ_k. By (2.3) with $p = k-1$, there exists a τ_k satisfying

(3.4) $$\operatorname{Min}(t_{k1}, \ldots, t_{kk}) < \tau_k < \operatorname{Max}(t_{k1}, \ldots, t_{kk})$$

such that

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$$[t_{k1} \ldots t_{kk}]_f = f^{(k-1)}(\tau_k)/(k-1)!$$

or

$$\Gamma_k = \left([t_{11}], [t_{21}t_{22}], \ldots, [t_{k-1,1} \ldots t_{k-1,k-1}], \frac{\mathfrak{x}^{(k-1)}(\tau_k)}{(k-1)!}, \mathfrak{a}_{k+1}, \ldots, \mathfrak{a}_n\right).$$

Applying this result consecutively with $k = m, m-1, \ldots, 2$, we obtain

$$\Gamma_m = \frac{(\mathfrak{x}(\tau_1), \mathfrak{x}'(\tau_2), \ldots, \mathfrak{x}^{(m-1)}(\tau_m), \mathfrak{a}_{m+1}, \ldots, \mathfrak{a}_n)}{1!\,2!\ldots(m-1)!}$$

where $\tau_1 = t_{11}$ and where the τ_k satisfy (3.4) if $2 \leqslant k \leqslant m$.

The case $m = n$ of Theorem 3.2 is a slight refinement of a mean-value theorem for determinants due to Schwarz. He developed it for similar purposes; cf. **(2)**. We note the following corollary.

THEOREM 3.3. *Suppose $\mathfrak{x}(t)$ is $(m-1)$-times continuously differentiable at t_0. Then*

$$\lim_{t_1, t_2, \ldots, t_m \to t_0} \Delta_m = \frac{(\mathfrak{x}(t_0), \mathfrak{x}'(t_0), \ldots, \mathfrak{x}^{(m-1)}(t_0), \mathfrak{a}_{m+1}, \ldots, \mathfrak{a}_n)}{1!\,2!\ldots(m-1)!}.$$

4. A definition of the osculating spaces. Existence. A *curve* C in projective n-space R_n is the continuous image of an interval I. Thus C can be described through a vector function

$$C\colon \mathfrak{x} = \mathfrak{x}(t); \qquad t \in I.$$

We do not distinguish between a point and its—homogeneous—co-ordinate vector.

Let $t_0 \in I$ be fixed. Put $L_0(t_0) = \mathfrak{x}(t_0)$. Suppose $L_0(t_0), \ldots, L_{k-1}(t_0)$ have been defined and they exist. Let $t \in I$, $t \neq t_0$. It can happen that the $(k-1)$-space $L_{k-1}(t_0)$ and $\mathfrak{x}(t)$ span a k-space whenever t is sufficiently close to t_0, and that this k-space converges if t tends to t_0. The limit space $L_k(t_0)$ is then called the *osculating k-space* of C at t_0.

THEOREM 4.1. *Let $0 < m < n$. Let C be m-times differentiable at t_0,*

$$\mathfrak{x}(t_0) \wedge \mathfrak{x}'(t_0) \wedge \ldots \wedge \mathfrak{x}^{(m)}(t_0) \neq 0. \tag{4.1}$$

Then C has osculating k-spaces at t_0 for $0 \leqslant k \leqslant m$, and $L_m(t_0)$ is given by

$$\mathfrak{y} \wedge \mathfrak{x}(t_0) \wedge \mathfrak{x}'(t_0) \wedge \ldots \wedge \mathfrak{x}^{(m)}(t_0) = 0. \tag{4.2}$$

Formula (4.1) states that $\mathfrak{x}(t_0), \ldots, \mathfrak{x}^{(m)}(t_0)$ are linearly independent. By (4.2), these points span $L_m(t_0)$.

We prove Theorem 4.1 by induction. In the case $m = 1$ we have

$$\lim_{t\to t_0} \mathfrak{x}(t_0) \wedge \frac{\mathfrak{x}(t) - \mathfrak{x}(t_0)}{t - t_0} = \mathfrak{x}(t_0) \wedge \lim_{t\to t_0} \frac{\mathfrak{x}(t) - \mathfrak{x}(t_0)}{t - t_0} = \mathfrak{x}(t_0) \wedge \mathfrak{x}'(t_0) \neq 0.$$

Thus

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$$\mathfrak{x}(t_0) \wedge \frac{\mathfrak{x}(t) - \mathfrak{x}(t_0)}{t - t_0} \neq 0 \tag{4.3}$$

if $|t - t_0|$ is sufficiently small. But the straight line through $\mathfrak{x}(t_0)$ and $\mathfrak{x}(t)$ is spanned by the bivector (4.3). Thus the last two formulae prove the case $m = 1$.

Suppose Theorem 4.1 has been proved up to $m - 1$. Put $h = t - t_0$. By Theorem 1.1,

$$\mathfrak{x}(t_0 + h) = \mathfrak{x}(t_0) + \frac{h}{1!}\mathfrak{x}'(t_0) + \ldots + \frac{h^{m-1}}{(m-1)!}\mathfrak{x}^{(m-1)}(t_0) + \frac{h^m}{m!}\mathfrak{x}_m(h),$$

$$\lim_{h\to 0} \mathfrak{x}_m(h) = \mathfrak{x}^{(m)}(t_0).$$

By (4.1),

$$\mathfrak{x}(t_0) \wedge \mathfrak{x}'(t_0) \wedge \ldots \wedge \mathfrak{x}^{(m-1)}(t_0) \neq 0. \tag{4.4}$$

Hence by our induction assumption, $L_{m-1}(t_0)$ exists and is given by the m-vector (4.4). From the above

$$\frac{m!}{h^m}\mathfrak{x}(t_0) \wedge \mathfrak{x}'(t_0) \wedge \ldots \wedge \mathfrak{x}^{(m-1)}(t_0) \wedge \mathfrak{x}(t_0 + h)$$

$$= \mathfrak{x}(t_0) \wedge \mathfrak{x}'(t_0) \wedge \ldots \wedge \mathfrak{x}^{(m-1)}(t_0) \wedge \mathfrak{x}_m(h).$$

If h tends to zero, this $(m + 1)$-vector converges to the $(m + 1)$-vector (4.1). In particular, it does not vanish if h is sufficiently small. Thus $L_{m-1}(t_0)$ and $\mathfrak{x}(t_0 + h)$ span an m-space for these h. If h tends to zero, this m-space converges to the m-space spanned by the $(m + 1)$-vector (4.1). This yields our theorem.

In the special case $m = n - 1$ we obtain the *osculating hyperplane* $L_{n-1}(t_0)$. We formulate this case explicitly:

COROLLARY 4.2. *Let C be $(n - 1)$-times differentiable at t_0. Suppose the points $\mathfrak{x}(t_0), \mathfrak{x}'(t_0), \ldots, \mathfrak{x}^{(n-1)}(t_0)$ are linearly independent. Then the osculating hyperplane of C at t_0 exists. It has the equation*

$$(\mathfrak{y}, \mathfrak{x}(t_0), \mathfrak{x}'(t_0), \ldots, \mathfrak{x}^{(n-1)}(t_0)) = 0.$$

We do not prove the following observation.

THEOREM 4.3. *Let C be n-times differentiable in I,*

$$\left.\begin{aligned} &\mathfrak{x}(t) \wedge \mathfrak{x}'(t) \wedge \ldots \wedge \mathfrak{x}^{(n-1)}(t) \neq 0 &(4.5)\\ &\mathfrak{x}(t) \wedge \mathfrak{x}'(t) \wedge \ldots \wedge \mathfrak{x}^{(n-1)}(t) \wedge \mathfrak{x}^{(n)}(t) = 0 &(4.6)\end{aligned}\right\} \text{ for all } t \in I.$$

Then $L_{n-1}(t)$ is constant. Thus C lies in this constant hyperplane.

It should be noted that this theorem becomes false without the assumption (4.5) even if C is of class $C^{(\infty)}$.

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5. Osculating spaces as "subspaces through neighbouring points."

THEOREM 5.1. *Let* $0 < m < n$. *Suppose the curve*

$$C\colon \mathfrak{x} = \mathfrak{x}(t); \qquad t \in I \tag{5.1}$$

is m-times differentiable at t_0 *and satisfies* (4.1); *the parameter values* $t_0, t_1, \ldots, t_m$ *are mutually distinct. Then if* $t_1, \ldots, t_m$ *are sufficiently close to* t_0, *the points*

$$\mathfrak{x}(t_0), \mathfrak{x}(t_1), \ldots, \mathfrak{x}(t_m) \tag{5.2}$$

span an m-space. It converges to $L_m(t_0)$ *if the* t_i *tend to* t_0.

Proof. Let $\mathfrak{a}_{m+1}, \ldots, \mathfrak{a}_n$ be any $n - m$ constant vectors.

By Theorem 3.1,

$$\lim_{t_1,\ldots t_m \to t_0} \frac{(\mathfrak{x}(t_0), \mathfrak{x}(t_1), \ldots, \mathfrak{x}(t_m), \mathfrak{a}_{m+1}, \ldots, \mathfrak{a}_n)}{\prod_{0 \leqslant i < k \leqslant m} (t_k - t_i)} = \frac{(\mathfrak{x}(t_0), \mathfrak{x}'(t_0), \ldots, \mathfrak{x}^{(m)}(t_0), \mathfrak{a}_{m+1}, \ldots, \mathfrak{a}_n)}{1!\,2!\ldots m!}.$$

Since this holds for every choice of $\mathfrak{a}_{m+1}, \ldots, \mathfrak{a}_n$, this implies

$$\lim_{t_1,\ldots, t_m \to t_0} \frac{\mathfrak{x}(t_0) \wedge \mathfrak{x}(t_1) \wedge \ldots \wedge \mathfrak{x}(t_m)}{\prod_{0 \leqslant i < k \leqslant m} (t_k - t_i)} = \frac{\mathfrak{x}(t_0) \wedge \mathfrak{x}'(t_0) \wedge \ldots \wedge \mathfrak{x}^{(m)}(t_0)}{1!\,2!\ldots m!}.$$

By (4.1), the right-hand multivector does not vanish. Hence

$$\mathfrak{x}(t_0) \wedge \mathfrak{x}(t_1) \wedge \ldots \wedge \mathfrak{x}(t_m) \neq 0$$

if the t_i lie sufficiently close to t_0, and the m-space through the points (5.2) converges to the m-space spanned by the $(m + 1)$-vector (4.1), that is, to $L_m(t_0)$ if the t_i converge to t_0; cf. Theorem 4.1.

THEOREM 5.2. *Let* $0 < m < n$. *Suppose the curve* (5.1) *is m-times continuously differentiable at* t_0 *and satisfies* (4.1). *The parameter values* $t_1, t_2, \ldots, t_{m+1}$ *are mutually distinct. Then if* $t_1, \ldots, t_{m+1}$ *lie close enough to* t_0, *the points*

$$\mathfrak{x}(t_1), \ldots, \mathfrak{x}(t_{m+1})$$

span an m-space. It converges to $L_m(t_0)$ *if the* t_i *tend to* t_0.

The proof of this statement is based on Theorem 3.3 rather than 3.1. Otherwise it is parallel to the preceding proof.

5a. A limit case. The question arises whether the results of **5** remain valid if some of the t_i coincide. In our comments we shall only consider Theorem 5.1.

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Let $0 < m < n$. Suppose the curve (5.1) is m-times differentiable at t_0 and satisfies (4.1). The parameter values $t_0, t_1, \ldots, t_r$ are mutually distinct;

$$m_0 \geqslant 0, \quad m_1 \geqslant 0, \ldots, m_r \geqslant 0\,; \quad \sum_0^r (m_i + 1) = m + 1.$$

Suppose the t_i lie sufficiently close to t_0. Then C will be m_i-times differentiable at each t_i and $L_{m_i}(t_i)$ will exist. It is the limit of m_i-spaces through points determined by $m_i + 1$ parameter values $t_{i0} = t_i, t_{i1}, \ldots, t_{im_i}$ converging to t_i. We may assume that all the $m + 1$ parameter values t_{ij} are mutually distinct. Keep the t_i fixed and let the t_{ij} converge to t_i for each i. Any limit space of the m-spaces spanned by the $\mathfrak{x}(t_{ij})$ will contain the $L_{m_i}(t_i)$. This yields:

Remark 5.3. There exist m-spaces containing all the $L_{m_i}(t_i)$ which converge to $L_m(t_0)$ as the t_i tend to t_0.

There remains the question whether the assumption (4.1) is sufficient to ensure that the osculating spaces

$$L_{m_0}(t_0), L_{m_1}(t_1), \ldots, L_{m_r}(t_r)$$

actually span an m-space if the t_i lie near enough to t_0. We have only been able to discuss the case $r = 1$.

Let $k \geqslant 0$, $p \geqslant 0$, $k + p = m + 1$. Without loss of generality let $t_0 = 0$ and put $t_1 = t \neq 0$. If

$$(5.3) \quad \Xi = \mathfrak{x}(0) \wedge \mathfrak{x}'(0) \wedge \ldots \wedge \mathfrak{x}^{(k)}(0) \wedge \mathfrak{x}(t) \wedge \mathfrak{x}'(t) \wedge \ldots \wedge \mathfrak{x}^{(p)}(t) \neq 0,$$

then $L_k(0)$ and $L_p(t)$ span an m-space. If (5.3) holds for all small t, Remark 5.3. will show that this m-space converges to $L_m(0)$ as t tends to zero.

Assume $p \leqslant k + 1$. By Theorem 1.1

$$\mathfrak{x}^{(j)}(t) = \mathfrak{x}^{(j)}(t_0) + \frac{t}{1!}\mathfrak{x}^{(j+1)}(t_0) + \ldots + \frac{t^{m-1-j}}{(m-1-j)!}\mathfrak{x}^{(m-1)}(0)$$
$$+ \frac{t^{m-j}}{(m-j)!}\mathfrak{x}^{j}_{m-j}(t), \qquad \lim_{t\to 0}\mathfrak{x}^{j}_{m-j}(t) = \mathfrak{x}^{(m)}(0).$$

Hence

$$\Xi = \mathfrak{x}(0) \wedge \ldots \wedge \mathfrak{x}^{(k)}(0)$$
$$\wedge\left(\frac{t^{k+1}}{(k+1)!}\mathfrak{x}^{(k+1)}(0) + \ldots + \frac{t^{m-1}}{(m-1)!}\mathfrak{x}^{(m-1)}(0) + \frac{t^m}{m!}\mathfrak{x}^0_m(t)\right)$$
$$\wedge\left(\frac{t^k}{k!}\mathfrak{x}^{(k+1)}(0) + \ldots + \frac{t^{m-2}}{(m-2)!}\mathfrak{x}^{(m-1)}(0) + \frac{t^{m-1}}{(m-1)!}\mathfrak{x}^1_{m-1}(t)\right)$$
$$\wedge \ldots \wedge\left(\frac{t^{k+1-p}}{(k+1-p)!}\mathfrak{x}^{(k+1)}(0) + \ldots + \frac{t^k}{k!}\mathfrak{x}^{(m-1)}(0) + \frac{t^{k+1}}{(k+1)!}\mathfrak{x}^p_{k+1}(t)\right).$$

This yields

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$$\lim_{t\to 0} \frac{\Xi}{t^{(k+1)(p+1)}} = E_{k,p}\,\mathfrak{x}(0) \wedge \mathfrak{x}'(0) \wedge \ldots \wedge \mathfrak{x}^{(m)}(0) \tag{5.4}$$

where

$$E_{k,p} = \begin{vmatrix} \frac{1}{(k+1)!} & \frac{1}{(k+2)!} & \cdots & \frac{1}{(m-1)!} & \frac{1}{m!} \\ \frac{1}{k!} & \frac{1}{(k+1)!} & \cdots & \frac{1}{(m-2)!} & \frac{1}{(m-1)!} \\ \cdots & & & & \\ \frac{1}{(k+1-p)!} & \frac{1}{(k+2-p)!} & \cdots & \frac{1}{k!} & \frac{1}{(k+1)!} \end{vmatrix} = \frac{p!}{m!} E_{k,p-1}.$$

In particular $E_{k,p} \neq 0$ and the right-hand term of (5.4) does not vanish. Thus $\Xi \neq 0$ if t is sufficiently small.

If $p > k + 1$, (5.4) remains valid if $E_{k,p}$ denotes a similar determinant satisfying the same recursion formula. This proves

THEOREM 5.4. *Let $k \geqslant 0$, $p \geqslant 0$, $m = k + p + 1 < n$. Suppose the curve C satisfies the assumptions of Theorem 5.1. Then $L_k(t_0)$ and $L_p(t_1)$ span an m-space if t_1 is sufficiently close to t_0. If t_1 tends to t_0, this m-space converges to $L_m(t_0)$.*

6. Families of hyperplanes. Capital German letters denote hyperplane co-ordinate vectors.

Given a family of hyperplanes

$$\Gamma\colon \mathfrak{X} = \mathfrak{X}(t); \qquad t \in I \tag{6.1}$$

in projective n-space R_n.

Let $t_0 \in I$, $t \neq t_0$. The *characteristic subspaces* $\Lambda_k(t_0)$ of Γ at t_0 are defined dually to the osculating spaces of a curve. Put $\Lambda_{n-1}(t_0) = \mathfrak{X}(t_0)$. Suppose $\Lambda_{n-1}(t_0), \ldots, \Lambda_{n-k}(t_0)$ have been defined and they exist. If the intersection of $\Lambda_{n-k}(t_0)$ with $\mathfrak{X}(t)$ is an $(n - k - 1)$-space for every t close to t_0 and if this $(n - k - 1)$-space converges as t tends to t_0, then the limit space $\Lambda_{n-1-k}(t_0)$ is called the characteristic $(n - 1 - k)$-space of Γ at t_0. We obtain from Theorem 4.1 by a duality

THEOREM 6.1. *Let $0 < m < n$. Suppose Γ is m-times differentiable at t_0 and*

$$\mathfrak{X}(t_0) \wedge \mathfrak{X}'(t_0) \wedge \ldots \wedge \mathfrak{X}^{(m)}(t_0) \neq 0. \tag{6.2}$$

Then Γ has characteristic subspaces of the dimensions $n - 1, n - 2, \ldots, n - 1 - m$ at t_0 and $\Lambda_{n-1-m}(t_0)$ has the equation

$$\mathfrak{Y} \wedge \mathfrak{X}(t_0) \wedge \mathfrak{X}'(t_0) \wedge \ldots \wedge \mathfrak{X}^{(m)}(t_0) = 0$$

[*or in point co-ordinates*

$$\mathfrak{y}\mathfrak{X}(t_0) = \mathfrak{y}\mathfrak{X}'(t_0) = \ldots = \mathfrak{y}\mathfrak{X}^{(m)}(t_0) = 0].$$

Theorems 5.1 and 5.2 can also readily be translated to families of hyperplanes.

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THEOREM 6.2. *Let* $0 < m < n$. *Suppose* Γ *is m-times* [*continuously*] *differentiable at* t_0 *and satisfies* (6.2). *The parameter values* $t_0, t_1, \ldots, t_m$ [$t_1, \ldots, t_m, t_{m+1}$] *are mutually distinct. Then if the* t_i *are sufficiently close to* t_0, *the intersection of the hyperplanes*

$$\mathfrak{X}(t_0), \mathfrak{X}(t_1), \ldots, \mathfrak{X}(t_m) \qquad [\mathfrak{X}(t_1), \ldots, \mathfrak{X}(t_m), \mathfrak{X}(t_{m+1})]$$

is an $(n - 1 - m)$-*space. It converges to* $\Lambda_{n-1-m}(t_0)$ *if the* t_i *tend to* t_0.

7. On the characteristic curve of a family of hyperplanes. If the family Γ of hyperplanes (6.1) is $(n - 1)$-times differentiable in I and if

$$\mathfrak{X}(t) \wedge \mathfrak{X}'(t) \wedge \ldots \wedge \mathfrak{X}^{(n-1)}(t) \neq 0 \qquad \text{for all } t \in I,$$

then Γ has by Theorem 6.1 a characteristic point $\Lambda_0(t)$ at each t. We call

$$C\colon \Lambda_0 = \Lambda_0(t); \qquad t \in I$$

the *characteristic curve* of Γ. Let $\mathfrak{x}(t)$ be a homogeneous co-ordinate vector of the point $\Lambda_0(t)$. Then

$$\mathfrak{x}(t)\mathfrak{X}(t) = \mathfrak{x}(t)\mathfrak{X}'(t) = \ldots = \mathfrak{x}(t)\mathfrak{X}^{(n-1)}(t) = 0 \qquad \text{for all } t \in I. \tag{7.1}$$

THEOREM 7.1. *Let* $\mathfrak{X}(t)$ *be n-times differentiable at* $t_0 \in I$,

$$(\mathfrak{X}(t_0), \mathfrak{X}'(t_0), \ldots, \mathfrak{X}^{(n)}(t_0)) \neq 0.$$

Then the characteristic curve C has osculating spaces $L_k(t_0)$ *of every dimension at* t_0, *and*

$$L_k(t_0) = \Lambda_k(t_0); \qquad k = 0, 1, \ldots, n - 1.$$

Proof. There is a neighbourhood N of t_0 such that $\mathfrak{X}(t)$ is $(n - 1)$-times differentiable in N and that

$$(\mathfrak{X}(t), \mathfrak{X}'(t), \ldots, \mathfrak{X}^{(n-1)}(t), \mathfrak{X}^{(n)}(t_0)) \neq 0 \qquad \text{for all } t \in N. \tag{7.2}$$

This follows from our assumptions and from the fact that the left-hand term of (7.2) is differentiable and therefore continuous at t_0.

By (7.1) and (7.2)

$$\mathfrak{x}(t)\mathfrak{X}^{(n)}(t_0) \neq 0 \qquad \text{for all } t \in N.$$

We can therefore norm $\mathfrak{x}(t)$ such that

$$\mathfrak{x}(t)\mathfrak{X}^{(n)}(t_0) = 1 \qquad \text{for all } t \in N. \tag{7.3}$$

Then the differentiability of (7.2) at t_0 implies that of $\mathfrak{x}(t)$ there. In particular, $\mathfrak{x}(t)$ will be continuous at t_0.

Define the points $\mathfrak{y}_0, \mathfrak{y}_1, \ldots, \mathfrak{y}_n$ through

$$\mathfrak{y}_i\mathfrak{X}^{(n-k)}(t_0) = \begin{cases} 1 & \\ & \text{if} \\ 0 & \end{cases} \begin{matrix} k = i \\ \\ k \neq i \end{matrix} \qquad i, k = 0, 1, \ldots, n. \tag{7.4}$$

Thus

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$$(\mathfrak{y}_0, \mathfrak{y}_1, \ldots, \mathfrak{y}_n)(\mathfrak{X}^{(n)}(t_0), \mathfrak{X}^{(n-1)}(t_0), \ldots, \mathfrak{X}(t_0)) = 1.$$

In particular

$$(\mathfrak{y}_0, \mathfrak{y}_1, \ldots, \mathfrak{y}_n) \neq 0.$$

Hence for each i the points $\mathfrak{y}_0, \mathfrak{y}_1, \ldots, \mathfrak{y}_i$ span an i-space. Since they lie in each of the spaces $\mathfrak{X}(t_0), \ldots, \mathfrak{X}^{(n-i-1)}(t_0)$, they must lie in $\Lambda_i(t_0)$.
This implies

LEMMA 1. *The points* $\mathfrak{y}_0, \mathfrak{y}_1, \ldots, \mathfrak{y}_i$ *span* $\Lambda_i(t_0)$; $i = 0, 1, \ldots, n-1$.

LEMMA 2.

$$\lim_{t\to t_0} \frac{\mathfrak{x}(t)\mathfrak{X}^{(n-k)}(t_0)}{(t-t_0)^i} = \begin{cases} \dfrac{(-1)^i}{i!} & \text{if } k = i \\ 0 & \text{if } k > i \end{cases} \qquad i, k = 0, 1, \ldots, n.$$

Proof. Let $0 \leqslant i \leqslant k \leqslant n$. We have

$$\lim_{t\to t_0} \mathfrak{x}(t)\mathfrak{X}^{(n-k)}(t_0) = \mathfrak{x}(t_0)\mathfrak{X}^{(n-k)}(t_0) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k > 0 \end{cases}.$$

This verifies our statement if $i = 0$. Suppose it is proved up to $i - 1 \geqslant 0$ [thus $k > 0$].

By Theorem 1.1,

$$\mathfrak{X}^{(n-k)}(t) = \mathfrak{X}^{(n-k)}(t_0) + \sum_{h=1}^{i-1} \frac{(t-t_0)^h}{h!}\mathfrak{X}^{(n-k+h)}(t_0) + \frac{(t-t_0)^i}{i!}\mathfrak{X}_i^{n-k}(t);$$

$$\lim_{t\to t_0} \mathfrak{X}_i^{n-k}(t) = \mathfrak{X}^{(n-k+i)}(t_0).$$

Hence

$$\frac{\mathfrak{x}(t)\mathfrak{X}^{(n-k)}(t)}{(t-t_0)^i} = \frac{\mathfrak{x}(t)\mathfrak{X}^{(n-k)}(t_0)}{(t-t_0)^i} + \sum_{h=1}^{i-1} \frac{1}{h!}\frac{\mathfrak{x}(t)\mathfrak{X}^{(n-k+h)}(t_0)}{(t-t_0)^{i-h}} + \frac{1}{i!}\mathfrak{x}(t)\mathfrak{X}_i^{n-k}(t).$$

Here

$$\lim_{t\to t_0} \frac{\mathfrak{x}(t)\mathfrak{X}^{(n-k)}(t)}{(t-t_0)^i} = \lim_{t\to t_0} 0 = 0,$$

$$\lim_{t\to t_0} \mathfrak{x}(t)\mathfrak{X}_i^{n-k}(t) = \mathfrak{x}(t_0)\mathfrak{X}^{(n-k+i)}(t_0) = \begin{cases} 1 & \text{if } k = i \\ 0 & \text{if } k > i \end{cases}.$$

By our induction assumption

$$\lim_{t\to t_0} \frac{\mathfrak{x}(t)\mathfrak{X}^{(n-k+h)}(t_0)}{(t-t_0)^{i-h}} = \begin{cases} \dfrac{(-1)^{i-h}}{(i-h)!} & \text{if } k = i \\ 0 & \text{if } k > i \end{cases} \qquad 1 \leqslant h \leqslant i-1.$$

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Hence

$$L_{i,k} = \lim_{t \to t_0} \frac{\mathfrak{x}(t)\mathfrak{X}^{(n-k)}(t_0)}{(t-t_0)^i}$$

exists and we have $L_{i,k} = 0$ if $k > i$. Finally

$$L_{i,i} = -\sum_{h=1}^{i} \frac{(-1)^{i-h}}{h!(i-h)!} = -\frac{1}{i!}\left(\sum_{h=0}^{i} (-1)^{i-h}\binom{i}{h} - (-1)^i\right) = \frac{(-1)^i}{i!}.$$

This proves Lemma 2. We only need the following observation:

(7.5) $$\lim_{t \to t_0} \frac{\mathfrak{x}(t)\mathfrak{X}^{(n-i)}(t_0)}{(t-t_0)^i} \text{ exists and is not zero}; \quad i = 0, 1, \ldots, n.$$

By making the neighbourhood N of t_0 smaller, we may therefore assume

(7.6) $$\mathfrak{x}(t)\mathfrak{X}^{(n-i)}(t_0) \neq 0 \text{ for all } t \in N,\ t \neq t_0;\ i = 0, 1, \ldots, n.$$

Furthermore (7.5) implies

LEMMA 3. *Let* $0 \leqslant i < k \leqslant n$. *Then*

$$\lim_{t \to t_0} \frac{\mathfrak{x}(t)\mathfrak{X}^{(n-k)}(t_0)}{\mathfrak{x}(t)\mathfrak{X}^{(n-i)}(t_0)} = 0.$$

The point $\mathfrak{x}(t)$ must be a linear combination

$$\mathfrak{x}(t) = \sum_0^n \alpha_k(t)\mathfrak{y}_k$$

of the $n+1$ linearly independent points $\mathfrak{y}_k$. Multiplying this equation by $\mathfrak{X}^{(n-i)}(t_0)$ we determine the $\alpha_k(t)$ and obtain

LEMMA 4.

$$\mathfrak{x}(t) = \sum_0^n \mathfrak{x}(t)\mathfrak{X}^{(n-k)}(t_0) \cdot \mathfrak{y}_k.$$

Trivially $L_0(t_0) = \Lambda_0(t_0)$. Thus Theorem 7.1 holds true for $i = 0$. Suppose it is proved up to $i - 1 \geqslant 0$. Thus $L_{i-1}(t_0) = \Lambda_{i-1}(t_0)$ is spanned by $\mathfrak{y}_0, \mathfrak{y}_1, \ldots, \mathfrak{y}_{i-1}$. By Lemma 4,

(7.7) $$\begin{cases} \mathfrak{x}(t) = \sum_0^{i-1} \mathfrak{x}(t)\mathfrak{X}^{(n-k)}(t_0) \cdot \mathfrak{y}_k + \mathfrak{x}(t)\mathfrak{X}^{(n-i)}(t_0) \cdot \mathfrak{z}_i(t), \\ \mathfrak{z}_i(t) = \sum_{k=i}^{n} \frac{\mathfrak{x}(t)\mathfrak{X}^{(n-k)}(t_0)}{\mathfrak{x}(t)\mathfrak{X}^{(n-i)}(t_0)} \cdot \mathfrak{y}_k. \end{cases}$$

By Lemma 3

(7.8) $$\lim_{t \to t_0} \mathfrak{z}_i(t) = \mathfrak{y}_i.$$

On account of (7.7), the i-space through $L_{i-1}(t)$ and $\mathfrak{x}(t)$ is spanned by the points $\mathfrak{y}_0, \mathfrak{y}_1, \ldots, \mathfrak{y}_{i-1}, \mathfrak{z}_i(t)$; cf. (7.6). By (7.8) it converges to the i-space

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spanned by $\mathfrak{y}_0, \mathfrak{y}_1, \ldots, \mathfrak{y}_{i-1}, \mathfrak{y}_i$, that is, to $\Lambda_i(t_0)$ if t tends to t_0. This proves our theorem.

8. Osculating spheres. Given a curve

$$C\colon P = P(t); \qquad t \in I$$

in conformal n-space Γ_n. Thus C is the continuous image in Γ_n of the interval I.

Let t_0, t_1, t_2 be three mutually distinct parameter values. If the circle through $P(t_0), P(t_1), P(t_2)$ is uniquely determined for all t_1 and t_2 sufficiently close to t_0 and if it converges to the circle $\Gamma_1(t_0)$ if t_1 and t_2 converge independently to t_0, then $\Gamma_1(t_0)$ is called the *osculating circle* or the osculating 1-sphere of C at t_0.

Let $t_0 \in I$ be fixed, $t \neq t_0$. Suppose we have already defined $\Gamma_1(t_0), \Gamma_2(t_0), \ldots, \Gamma_{k-1}(t_0)$ and they exist; $k \geqslant 2$. It can happen that the k-sphere through the $(k-1)$-sphere $\Gamma_{k-1}(t_0)$ and $P(t)$ is unique if t lies sufficiently close to t_0 and that it converges if t tends to t_0. Then the limit k-sphere $\Gamma_k(t_0)$ will be called the *osculating k-sphere* of C at t_0.

We can formulate conditions for the existence of $\Gamma_k(t_0)$ in terms of arbitrary polyspherical co-ordinates. The following approach seems convenient. Let $\xi_1, \ldots, \xi_n$ be the co-ordinates of a point P in euclidean n-space with respect to some normed orthogonal co-ordinate system; $\xi_0 = \sum_1^n \xi_\lambda^2$. We associate with P the homogeneous co-ordinate vector

$$\mathfrak{x} = (x_1, x_2, \ldots, x_n, x_{n+1}, x_{n+2}) = \rho(\xi_1, \xi_2, \ldots, \xi_n, \tfrac{1}{2}(\xi_0 - 1), \tfrac{1}{2}(\xi_0 + 1))$$

where $\rho \neq 0$ is an arbitrary scalar. If

$$\mathfrak{y} = (y_1, \ldots, y_{n+2}),$$

put

$$\mathfrak{x}\mathfrak{y} = \sum_1^{n+1} x_i y_i - x_{n+2} y_{n+2}.$$

Thus $\mathfrak{x}\mathfrak{x} = 0$ and $\mathfrak{x}$ can also be interpreted as the homogeneous co-ordinate vector of a point $\bar{P}$ on the unit sphere $\bar{\Gamma}_n$ if the latter is imbedded into projective R_{n+1}. If we adjoin an ideal point with the co-ordinate vector

$$(0, 0, \ldots, 0, x_{n+1}, x_{n+1})$$

to euclidean n-space, we arrive at conformal n-space Γ_n. The mapping $P \to \bar{P}$ will then be a homeomorphism of Γ_n onto $\bar{\Gamma}_n$.

An $(n-1)$-sphere Γ_{n-1} in Γ_n is given by equations

$$\mathfrak{a}\mathfrak{x} = 0, \ \mathfrak{x}\mathfrak{x} = 0. \tag{8.1}$$

It corresponds to the $(n-1)$-sphere $\bar{\Gamma}_{n-1}$ in which the hyperplane $\mathfrak{a}\mathfrak{x} = 0$ in R_{n+1} intersects $\bar{\Gamma}_n$. Thus it contains real points if and only if $\mathfrak{a}\mathfrak{a} \geqslant 0$. If $\mathfrak{a}\mathfrak{a} = 0$, Γ_{n-1} contains exactly one real point, viz. the point P with the co-ordinate vector $\mathfrak{a}$. We then identify Γ_{n-1} with P.

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Suppose the curve C is given by means of the vector function

$$\mathfrak{x} = \mathfrak{x}(t); \; \mathfrak{x}\mathfrak{x} = 0; \qquad t \in I.$$

Its image in R_{n+1} is a curve

$$\bar{C}\colon \bar{P} = \bar{P}(t); \qquad t \in I.$$

THEOREM 8.1. *Let $\mathfrak{x}(t)$ be twice differentiable at t_0,*

$$\mathfrak{x}(t_0) \wedge \mathfrak{x}'(t_0) \wedge \mathfrak{x}''(t_0) \neq 0.$$

Then C has an osculating circle $\Gamma_1(t_0)$ at t_0. It satisfies the equations

$$\mathfrak{y} \wedge \mathfrak{x}(t_0) \wedge \mathfrak{x}'(t_0) \wedge \mathfrak{x}''(t_0) = 0, \; \mathfrak{y}\mathfrak{y} = 0. \tag{8.2}$$

Thus $\Gamma_1(t_0)$ has a parametric representation

$$\mathfrak{y} = \lambda_0 \mathfrak{x}(t_0) + \lambda_1 \mathfrak{x}'(t_0) + \lambda_2 \mathfrak{x}''(t_0)$$

where the λ_i are subject to the condition $\mathfrak{y}\mathfrak{y} = 0$.

Proof. Let t_0, t_1, t_2 be mutually distinct. Theorem 5.1 implies: If t_1 and t_2 lie sufficiently close to t_0, the three points $\bar{P}(t_i)$ span a plane which converges to the osculating plane $\bar{L}_2$ of $\bar{C}$ at t_0 if t_1 and t_2 converge to t_0. Hence the circle through the $\bar{P}(t_i)$ then converges to the circle

$$\bar{\Gamma}_1 = \bar{L}_2 \cap \bar{\Gamma}_n.$$

Since the first equation of (8.2) represents $\bar{L}_2$ in R_{n+1}, $\bar{\Gamma}_1$ is given by (8.2). The mapping $\bar{\Gamma}_n \to \Gamma_n$ being topological, the image of the circle through the $\bar{P}(t_i)$ converges to the image of $\bar{\Gamma}_1$. This proves our theorem.

The theorems of **4** and **5** are now readily translated.

THEOREM 8.2. *Let $m \geqslant 2$. If $\mathfrak{x}(t)$ is m-times differentiable at t_0 and if*

$$\mathfrak{x}(t_0) \wedge \mathfrak{x}'(t_0) \wedge \ldots \wedge \mathfrak{x}^{(m)}(t_0) \neq 0, \tag{8.3}$$

then C has osculating spheres of every dimension $\leqslant m - 1$ at t_0 and $\Gamma_{m-1}(t_0)$ has the equations

$$\mathfrak{y} \wedge \mathfrak{x}(t_0) \wedge \mathfrak{x}'(t_0) \wedge \ldots \wedge \mathfrak{x}^{(m)}(t_0) = 0; \; \mathfrak{y}\mathfrak{y} = 0.$$

THEOREM 8.3. *Let $m \geqslant 2$. Let $\mathfrak{x}(t)$ be m-times* [*continuously*] *differentiable at t_0 and satisfy* (8.3). *Suppose*

$$t_0, t_1, \ldots, t_m \qquad [t_1, \ldots, t_m, t_{m+1}]$$

are mutually distinct. Then if the t_i lie sufficiently close to t_0, there exists exactly one $(m - 1)$-sphere through the points

$$\mathfrak{x}(t_0), \mathfrak{x}(t_1), \ldots, \mathfrak{x}(t_m) \qquad [\mathfrak{x}(t_1), \ldots, \mathfrak{x}(t_m), \mathfrak{x}(t_{m+1})].$$

It converges to $\Gamma_{m-1}(t_0)$ if the t_i tend to t_0.

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