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THE 4-RANK OF $K_2(O)$

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1. Introduction. Let O_F denote the integers of an algebraic number field F. Classically the Dirichlet Units Theorem gives us the structure of the K-group $K_1(O_F)$. Then recently the structure of the K-group $K_3(O_F)$ was found by Merkurjev and Suslin, [**11**]. But as of now we have only limited information about the structure of the tame kernel $K_2(O_F)$.

For special classes of number fields, the following rank formulas are known. Fix a rational prime number p, let S denote the set of infinite and p-adic places of the number field F, $g_p(F)$ the number of p-adic places of F, and $C(F)$ the S-ideal class group of F. Under the assumption that the group μ_{p^n} of p^n-th roots of unity is contained in F, we obtain from Tate [**14**], the exact sequence

$$(1.1) \qquad 1 \longrightarrow C(F)/C(F)^{p^n} \longrightarrow K_2(O_F)/K_2(O_F)^{p^n} \longrightarrow \coprod \mu_{p^n} \longrightarrow 1$$

where the product is taken over $r_1(F) + g_p(F) - 1$ copies of μ_{p^n}. This yields immediately, for arbitrary number fields F, the well known 2-rank formula

$$(1.2) \qquad 2\text{-rk}\, K_2(O_F) = r_1(F) + g_2(F) - 1 + 2\text{-rk}\, C(F)$$

and, for number fields F containing a primitive fourth root of unity, the 4-rank formula

$$(1.3) \qquad \text{If } \sqrt{-1} \in F, \text{ then } 4\text{-rk } K_2(O_F) = g_2(F) - 1 + 4\text{-rk}\, C(F).$$

Let $M = F(\sqrt{-1})$, consider the norm $N : C(M) \longrightarrow C(F)$ and the natural homomorphism $i_* : C(F) \longrightarrow C(M)$. We denote by ${}_2C(F)$ the subgroup of $C(F)$ of ideal classes of order at most 2. Under the assumption that M contains a primitive 2^n-th root of unity and F is totally real, we know from Kolster, [**8**], for $n \geqq 2$:

$$(1.4) \qquad 2^n\text{-rk}\, K_2(O_F) = g_2(M) - g_2(F) + 2^{n-1}\text{-rk}\ker N/i_*({}_2C(F)).$$

This yields, for totally real number fields F, a 4-rank formula for $K_2(O_F)$. During the preparation of these notes, it was Kolster himself who noticed that the same 4-rank formula can be proved whenever F does not contain a fourth root of unity; that is,

$$(1.5) \qquad \text{If } \sqrt{-1} \notin F, \text{ then } \\ 4\text{-rk}\, K_2(O_F) = g_2(M) - g_2(F) + 2\text{-rk}\ker N/i_*({}_2C(F)).$$

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In this paper we concentrate on the structure of the 2-primary subgroup of $K_2(O_F)$ and present a unified approach for deducing 4-rank formulas for $K_2(O_F)$ for arbitrary number fields F. The intimate connection between the structure of 2-prim $K_2(O_F)$ and classical issues of Algebraic Number Theory involving S-units and S-ideal class groups will be emphasized. Having a variety of applications in mind, we are aiming for various 4-rank formulas in computable terms, see (4.9), (5.2), (5.4), (5.5). This approach yields, as a special case, the above formula (1.3), and also the general form of (1.5) appears as a natural reformulation.

Our main tools are the decomposition, Section 3, and the factorization, Section 4, of the crucial homomorphism χ that takes values in the square class group of $C(F)$.

In the applications, Section 5, extreme cases become explicit. We have made an effort to include examples, Section 6, mainly of imaginary quadratic number fields. The appendix on S-class groups makes our results readily applicable to quadratic number fields.

Based on this approach the deduction of higher rank formulas for $K_2(O_F)$ for arbitrary number fields F is in fact conceivable. We would like to point out recent related papers [**3**], [**7**], [**9**].

2. Preliminaries. For a group X, let ${}_2X = \{a \in X : a^2 = 1\}$. In order to determine the 4-rank of $K_2(O_F)$, we will count the number of elements in ${}_2X \cap X^2$ for $X = K_2(O_F)$. For a number field F, the standard notation will be:

S set of infinite and dyadic places of F
$g_2(F)$ number of dyadic places of F
$\#S = r_1(F) + r_2(F) + g_2(F)$
R_F ring of S-integers of F
U_F group of S-units of F
$C(F)$ S-ideal class group of F.

Let $M = F(\sqrt{-1})$ and consider the following groups of square classes

$$G_F = \{cl(b) \in F^*/F^{*2} : \nu_p(b) \equiv 0 \bmod 2 \text{ for all } p \notin S\}$$

$$H_F = \{cl(b) \in G_F : b \in N_{M/F}(M^*)\}.$$

An element of $K_2(F)$ lies in ${}_2K_2(F)$ if and only if it is a Steinberg symbol of the form $\{-1, b\}$ for some $b \in F^*$, [**14**]. Now, $K_2(O_F)$ is the intersection of the kernels of the tame symbols

$$\tau_p : K_2(F) \longrightarrow (O_F/p)^*$$
$$\{a, b\} \longrightarrow cl\left((-1)^{\nu_p(a)\nu_p(b)} \cdot a^{\nu_p(b)} \cdot b^{-\nu_p(a)}\right)$$

for all finite places p of F, while $K_2(R_F)$ is the intersection of the kernels of the τ_p for all $p \notin S$. For a dyadic place p, the order of $(O_F/p)^*$ is odd, hence

$$\text{2-prim}\, K_2(O_F) = \text{2-prim}\, K_2(R_F) \quad \text{and}$$
$${}_2K_2(O_F) = \{\{-1, b\} : cl(b) \in G_F\}.$$

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The symbol $\{-1, b\}$ is a square in $K_2(F)$ if and only if the element $b \in F^*$ is a norm from M/F, [**12**]. Hence

$$_2K_2(O_F) \cap K_2(F)^2 = \{\{-1, b\} : cl(b) \in H_F\}.$$

Let $cl(b) \in H_F$, thus $\{-1, b\} = y^2$ for some $y \in K_2(F)$ with $\tau_p(y) = \pm 1$ in $(O_F/p)^*$ for all finite places p of F. Choose any fractional R_F-ideal A with

$$\tau_p(y) = (-1)^{\nu_p(A)} \quad \text{for all } p \notin S.$$

Definition 2.1. Let $\chi : H_F \longrightarrow C(F)/C(F)^2$ be the homomorphism given by $cl(b) \longrightarrow cl(A)$.

We check that χ is well-defined: if $\{-1, b\} = y^2 = y_1^2$ with $y, y_1 \in K_2(F)$ and, for all $p \notin S$,

$$\tau_p(y) = (-1)^{\nu_p(A)}, \quad \tau_p(y_1) = (-1)^{\nu_p(A_1)}$$

for some fractional R_F-ideals A, A_1, then $y_1 = y \cdot \{-1, c\}$ for some $c \in F^*$ and

$$\begin{aligned}(-1)^{\nu_p(A_1)} &= \tau_p(y_1) = \tau_p(y) \cdot \tau_p\{-1, c\} \\ &= (-1)^{\nu_p(A)} \cdot (-1)^{\nu_p(c)} = (-1)^{\nu_p(cA)}.\end{aligned}$$

This shows that

$$\nu_p(A_1) \equiv \nu_p(cA) \bmod 2 \text{ for all } p \notin S;$$

thus $cl(A_1) = cl(cA) = cl(A)$ in $C(F)/C(F)^2$. In fact, χ is a homomorphism.

Just from the definition of χ we obtain: $cl(b) \in \ker \chi$ if and only if there exists a $c \in F^*$ with

$$\nu_p(cA) \equiv 0 \bmod 2 \text{ for all } p \notin S,$$

where $\{-1, b\} = y^2$ and $\tau_p(y) = (-1)^{\nu_p(A)}$. This means that there exists a $c \in F^*$ with

$$1 = (-1)^{\nu_p(cA)} = \tau_p(y\{-1, c\})$$

for all finite places p of F; so, $y\{-1, c\} \in K_2(O_F)$. This shows that $cl(b)$ is in the kernel of χ if and only if $\{-1, b\}$ is a square element in $K_2(O_F)$. So,

$$(2.2) \qquad _2K_2(O_F) \cap K_2(O_F)^2 = \{\{-1, b\} : cl(b) \in \ker \chi\}.$$

All we have to do in order to find the 4-rank of $K_2(O_F)$ is to find the 2-rank of $\{\{-1, b\} : cl(b) \in \ker \chi\}$. The kernel of the natural homomorphism

$$\begin{aligned}\alpha : G_F &\longrightarrow {}_2K_2(O_F) \\ cl(b) &\longrightarrow \{-1, b\}\end{aligned}$$

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has 2-rank $r_2(F)+1$, [**14**]. The kernel of α is contained in the kernel of χ, hence

$$4\text{-rk}\, K_2(O_F) = 2\text{-rk}\ker\chi - 2\text{-rk}\ker\alpha,$$

and we have obtained

PROPOSITION 2.3. *For any number field F,*

$$4\text{-rk}\, K_2(O_F) = 2\text{-rk}\ker\chi - r_2(F) - 1.$$

From here on, our purpose will be to make 2-rk ker χ more explicit. This can be treated as a problem from the Algebraic Number Theory, which divides up naturally into an S-unit part and an S-ideal class group part. Consider the epimorphism

$$\chi_0 : G_F \longrightarrow {}_2C(F)$$
$$cl(b) \longrightarrow cl(B),$$

where B is the fractional R_F-ideal satisfying

$$bR_F = B^2.$$

The kernel of χ_0 is U_F/U_F^2, whose 2-rank is $r_1(F)+r_2(F)+g_2(F)$, by the Dirichlet S-Units Theorem. The exact sequence

$$1 \longrightarrow U_F/U_F^2 \longrightarrow G_F \xrightarrow{\chi_0} {}_2C(F) \longrightarrow 1$$

shows

LEMMA 2.4. $2\text{-rk}\, G_F = r_1(F) + r_2(F) + g_2(F) + 2\text{-rk}\, C(F)$.

We restrict χ_0 to H_F and consider the composition

$$\chi_1 : H_F \xrightarrow{\chi_0} C(F) \longrightarrow C(F)/C(F)^2;$$

that is, in terms of $bR_F = B^2$, the resulting homomorphism χ_1 is defined by

Definition 2.5. Let $\chi_1 : H_F \longrightarrow C(F)/C(F)^2$ be the homomorphism given by $cl(b) \longrightarrow cl(B)$.

Already now we can determine the 4-rank of $K_2(O_F)$ under the simplifying assumption that F contains a primitive fourth root of unity. Namely, if $i = \sqrt{-1} \in F$ and $cl(b) \in H_F$, then $\{-1, b\} = \{i, b\}^2$ in $K_2(F)$ and

$$\tau_p\{i, b\} = i^{\nu_p(b)} = (-1)^{\nu_p(b)/2} \quad \text{in } (O_F/p)^*.$$

This shows

$$\tau_p\{i, b\} = (-1)^{\nu_p(A)} \quad \text{for all } p \notin S,$$

where $bR_F = A^2$. So, by definitions (2.1) and (2.5):

$$\chi = \chi_1 \quad \text{if } \sqrt{-1} \in F.$$

Hence, (2.3) simplifies to

$$4\text{-rk}\, K_2(O_F) = 2\text{-rk}\ker\chi_1 - r_2(F) - 1.$$

Moreover, the assumption $\sqrt{-1} \in F$ yields $H_F = G_F$. From the short exact sequence and the commutative triangle

$$\begin{array}{ccccccccc} 1 & \longrightarrow & U_F/U_F^2 & \longrightarrow & H_F & \xrightarrow{\chi_0} & {}_2C(F) & \longrightarrow & 1 \\ & & & & \chi_1 \downarrow & \swarrow & & & \\ & & & & C(F)/C(F)^2 & & & & \end{array}$$

we conclude

$$\begin{aligned} 2\text{-rk}\ker\chi_1 &= 2\text{-rk}\ker\chi_0 + 4\text{-rk}\, C(F) \\ &= r_2(F) + g_2(F) + 4\text{-rk}\, C(F). \end{aligned}$$

Thus we have proved

PROPOSITION 2.6. *If $\sqrt{-1} \in F$, then $H_F = G_F$, $\chi = \chi_1$ and*

$$4\text{-rk}\, K_2(O_F) = g_2(F) - 1 + 4\text{-rk}\, C(F).$$

This is the result (1.3), *quoted above.*

3. Decomposition. In the preceding section we have seen that $\chi = \chi_1$ if $\sqrt{-1} \in F$. However, in general, this simple equality is no longer valid. In this section we will exhibit another homomorphism

$$\chi_2 : H_F \longrightarrow C(F)/C(F)^2$$

such that χ decomposes into the product $\chi = \chi_1 \cdot \chi_2$.

For $cl(b) \in H_F$ put $E = F(\sqrt{b})$, recall $M = F(\sqrt{-1})$. Since $b \in F^*$ is a norm from M/F, it follows that -1 is a norm from E/F. So, we can choose an

$$e \in E \quad \text{with } N_{E/F}(e) = -1.$$

Furthermore, for each place $p \notin S$ of F we choose an extension P of p to E. Clearly, such a p is either inert or splits in E/F. If p splits in E/F, $pR_E = P\bar{P}$, then

$$\nu_P(e) + \nu_{\bar{P}}(e) = \nu_p(N(e)) = \nu_p(-1) = 0;$$

in particular,

$$\nu_P(e) \equiv \nu_{\bar{P}}(e) \bmod 2.$$

Definition 3.1. Let

$$\chi_2 : H_F \longrightarrow C(F)/C(F)^2$$

be the homomorphism given by

$$cl(b) \longrightarrow cl\left(\prod_{p \notin S} p^{\nu_P(e)}\right).$$

There is no problem in checking that χ_2 is well-defined. We observe that $\chi_2(cl(b))$ is trivial if -1 is the norm of an S-unit e in $E = F(\sqrt{b})$. Clearly, χ_2 is the trivial map if $\sqrt{-1} \in F$. The fact that χ_2 is a homomorphism will follow from the Decomposition Theorem:

THEOREM 3.2. *For any number field F, we have*

$$\chi = \chi_1 \cdot \chi_2.$$

Proof. The identity

$$\chi(cl(b)) = \chi_1(cl(b)) \cdot \chi_2(cl(b))$$

in $C(F)/C(F)^2$ is clear if $cl(b) \in H_F$ is trivial. So, for $cl(b) \in H_F$ we may assume that

$$b = a^2 + 1$$

for some $a \in F^*$ since $b \in F^*$ is a norm from M/F. By (2.5),

$$\chi_1(cl(b)) = cl\left(\prod_{p \notin S} p^{\nu_p(b)/2}\right).$$

The element $e = a + \sqrt{b} \in F(\sqrt{b}) = E$ satisfies

$$N_{E/F}(e) = a^2 - b \cdot 1^2 = -1.$$

Hence, by (3.1) with an extension $P|p$,

$$\chi_2(cl(b)) = cl\left(\prod_{p \notin S} p^{\nu_P(a+\sqrt{b})}\right).$$

The proof of this decomposition formula amounts to showing

$$\chi(cl(b)) = cl\left(\prod_{p\notin S} p^{\nu_p(b)/2+\nu_p(a+\sqrt{b})}\right).$$

Now,

$$\{-1,b\} = \{-1,b\}\{1-b,b\} = \{b-1,b\} = \{a^2,b\} = \{a,b\}^2$$

in $K_2(F)$ and, by (2.1), it is only left to prove that for each $p\notin S$ we have

$$\tau_p\{a,b\} = (-1)^{\nu_p(b)/2+\nu_p(a+\sqrt{b})} \quad \text{in } (O_F/p)^*.$$

This will follow directly from the next two lemmas (3.3) and (3.4). We notice that the above formula shows $\tau_p\{a,b\}=1$ for p dyadic.

LEMMA 3.3. *If $\nu_p(b)\geqq 0$ then $\tau_p\{a,b\}=(-1)^{\nu_p(b)/2}$:*
If $\nu_p(b)<0$, then $\tau_p\{a,b\}=1$.

Proof. If $\nu_p(a^2+1)>0$, then a is a p-adic unit and

$$\tau_p\{a,b\} = (-1)^\circ\cdot a^{\nu_p(b)}\cdot b^\circ = (a^2)^{\nu_p(b)/2} = (-1)^{\nu_p(b)/2} \quad \text{in } (O_F/p)^*.$$

If $\nu_p(a^2+1)=0$, then $\nu_p(a)\geqq 0$. If a is also a p-adic unit, we have

$$\tau_p\{a,b\} = (-1)^\circ\cdot a^\circ\cdot b^\circ = 1 = (-1)^{\nu_p(b)/2} \quad \text{in } (O_F/p)^*.$$

If $\nu_p(a)>0$, we have as well that

$$\tau_p\{a,b\} = (-1)^\circ\cdot a^\circ\cdot (a^2+1)^{-\nu_p(a)} = 1 = (-1)^{\nu_p(b)/2} \quad \text{in } (O_F/p)^*.$$

If $\nu_p(a^2+1)<0$, then $\nu_p(a)<0$. More precisely,

$$a^{-2}(1+a^2) = 1+a^{-2} \equiv 1 \bmod p$$

implies

$$2\nu_p(a) = \nu_p(a^2+1).$$

We have

$$\begin{aligned}\tau_p\{a,b\} &= (-1)^2\cdot a^{\nu_p(a^2+1)}\cdot (a^2+1)^{-\nu_p(a)}\\ &= (a^2/1+a^2)^{\nu_p(a)} = 1 \quad \text{in } (O_F/p)^*.\end{aligned}$$

Essentially, the computations in the proof of (3.3) have been performed before in [**6**]. Let P denote an extension of the non-dyadic finite place p of F to $E=F(\sqrt{b})$.

LEMMA 3.4. *If* $\nu_p(b) \geqq 0$, *then* $\nu_P(a+\sqrt{b}) = 0$;
If $\nu_p(b) < 0$, *then* $\nu_P(a+\sqrt{b}) = \pm\nu_p(b)/2$.

Proof. If $\nu_p(a^2+1) \geqq 0$, then $\nu_p(a) \geqq 0$. The minimal polynomial of $a+\sqrt{b}$ over F is $t^2 - 2at - 1$ and we conclude that, for $P|p$, $a+\sqrt{b}$ is a P-adic unit; that is,

$$\nu_P(a+\sqrt{b}) = 0.$$

If $\nu_p(a^2+1) < 0$, then $2\nu_p(a) = \nu_p(a^2+1)$ as we have noted in (3.3). If p were inert in E/F, we would have $\nu_P(a+\sqrt{b}) = 0$, contradicting $\nu_p(a) < 0$. Thus p splits in E/F; that is, b is a local square in the completion of F at p. Putting $a = \epsilon\pi^{-s}$ with ϵ a p-adic unit, $s > 0$, we now have $b = \delta^2\pi^{-2s}$ with δ a p-adic unit. From $b = a^2+1$ we conclude that

$$\delta^2 = \epsilon^2 + \pi^{2s};$$

by replacing δ with $-\delta$, if necessary, we therefore may assume that $\delta \equiv \epsilon \bmod \pi$. Using $p \notin S$, we see that $\delta+\epsilon$ is a p-adic unit and, because

$$\pi^{2s} = \delta^2 - \epsilon^2 = (\delta+\epsilon)(\delta-\epsilon),$$

we conclude that $\nu_p(\delta-\epsilon) = 2s$. We have

$$a+\sqrt{b} = \pi^{-s}(\epsilon+\delta) \quad \text{or} \quad a+\sqrt{b} = \pi^{-s}(\epsilon-\delta),$$

so

$$\nu_P(a+\sqrt{b}) = -s = \nu_p(b)/2 \quad \text{or } \nu_P(a+\sqrt{b}) = -s+2s = s = -\nu_p(b)/2.$$

Thus, in either case,

$$\tau_p\{a,b\} = (-1)^{\nu_p(b)/2+\nu_P(a+\sqrt{b})}$$

and we have proved the Decomposition Theorem (3.2).

In order to make the decomposition $\chi = \chi_1 \cdot \chi_2$ more effective, we shall give another description of χ_2. For $cl(b) \in H_F$ we can choose an

$$m \in M \quad \text{with } N_{M/F}(m) = b.$$

For each place $p \notin S$ of F we choose an extension P of p to M.

PROPOSITION 3.5. *The homomorphism* χ_2 *is also given by*

$$\chi_2 : H_F \longrightarrow C(F)/C(F)^2$$
$$cl(b) \longrightarrow cl\left(\prod_{p\notin S} p^{\nu_P(m)}\right).$$

Proof. Choose $cl(b) \in H_F$; clearly the class of $\prod_{p \notin S} p^{\nu_P(m)}$ in $C(F)/C(F)^2$ is well-defined. Hence, by (3.1), it suffices to show that for a particular choice of

$$m \in M = F(\sqrt{-1}) \quad \text{with } N_{M/F}(m) = b,$$
$$e \in E = F(\sqrt{b}) \quad \text{with } N_{E/F}(e) = -1,$$

and a pair P_1, P_2 of extensions of any place $p \notin S$ to M and E, respectively, we have

$$\nu_{P_1}(m) \equiv \nu_{P_2}(e) \bmod 2.$$

This is clear if $cl(b)$ is trivial or $cl(b) = cl(-1)$ in H_F. So, we need only concern ourselves with $cl(b) \neq F^{*2}, -F^{*2}$. Consider then the composite number field

$$M \cdot E = F(\sqrt{-1}, \sqrt{b});$$

the extension $M \cdot E/F$ is relative abelian of degree 4 over F with elementary abelian Galois group. As before, we may choose $b \in F^*$ such that

$$b = a^2 + 1 \quad \text{for some } a \in F^*.$$

Thus $m = 1 + ai \in M$ satisfies $N_{M/F}(m) = b$, and $e = a + \sqrt{b} \in E$ satisfies $N_{E/F}(e) = -1$. Everything follows now from the identity

$$(1 + ai)(1 - i)^2 = (a + \sqrt{b})\left(1 + (a - \sqrt{b})i\right)^2.$$

Hence there exists a non-zero $x \in M \cdot E$ with

$$m = ex^2 \quad \text{in } M \cdot E.$$

Any $p \notin S$ is unramified in $M \cdot E/F$. Fix an extension P of p to $M \cdot E$. Then P is an extension to $M \cdot E$ of an extension P_1 of p to M and of an extension P_2 of p to E. These places P_1 and P_2 satisfy

$$\nu_P(m) = \nu_{P_1}(m) \quad \text{and} \quad \nu_P(e) = \nu_{P_2}(e).$$

However, we know already that

$$\nu_P(m) \equiv \nu_P(e) \bmod 2,$$

thus

$$\nu_{P_1}(m) \equiv \nu_{P_2}(e) \bmod 2.$$

Using the description (3.5) for the homomorphism χ_2 will make the decomposition $\chi = \chi_1 \cdot \chi_2$ most useful in determining the 2-rank of the kernel of χ and hence the 4-rank of $K_2(O_F)$. For number fields F with

$$\sqrt{-1} \notin F$$

we will now embed χ_2 in an appropriate exact sequence.

By analogy with G_F we have the square class group G_M for $M = F(\sqrt{-1})$. Consider the homomorphism $N : G_M \longrightarrow H_F$ induced by the norm $M^* \longrightarrow F^*$ and the homomorphism $j : G_F \longrightarrow G_M$ induced by the inclusion $F^* \longrightarrow M^*$. The image of j is equal to the kernel of N, the kernel of j is the cyclic group C_2 of order 2 generated by $cl(-1) \in G_F$. So far,

$$1 \longrightarrow C_2 \longrightarrow G_F \longrightarrow G_M \xrightarrow{N} H_F$$

is exact. We are going to extend this sequence.

LEMMA 3.6. *The image of $N : G_M \longrightarrow H_F$ is equal to the kernel of*

$$\chi_2 : H_F \longrightarrow C(F)/C(F)^2.$$

Proof. The containment $\operatorname{im} N \subseteq \ker \chi_2$ is immediate. For the reversed containment, suppose $cl(b) \in \ker \chi_2$; that is, there is an $m \in M$ with $N_{M/F}(m) = b$ and

$$cl\left(\prod_{p \notin S} p^{\nu_P(m)}\right) = 1 \quad \text{in } C(F)/C(F)^2.$$

Hence there is a $y \in F^*$ such that

$$\nu_p(y) \equiv \nu_P(m) \bmod 2$$

for every place $p \notin S$ of F with extension $P|p$ to M. From

$$N_{M/F}(ym) = y^2 N_{M/F}(m) \quad \text{and}$$
$$\nu_P(ym) = \nu_p(y) + \nu_p(m) \equiv 0 \bmod 2$$

we now conclude that $cl(ym) \in G_M$ satisfies

$$N(cl(ym)) = N(cl(m)) = cl(b) \quad \text{in } H_F.$$

This shows $\ker \chi_2 \subseteq \operatorname{im} N$.

Let us turn to the S-ideal class group $C(M)$ of M and consider the homomorphism

$$i : C(F)/C(F)^2 \longrightarrow C(M)/C(M)^2$$

that is induced by the canonical homomorphism $i_* : C(F) \longrightarrow C(M)$.

LEMMA 3.7. *The image of* $\chi_2 : H_F \longrightarrow C(F)/C(F)^2$ *is equal to the kernel of*

$$i : (CF)/C(F)^2 \longrightarrow C(M)/C(M)^2.$$

Proof. First we prove the containment $\operatorname{im}\chi_2 \subseteq \ker i$. An element $\chi_2(cl(b))$ is a class in $C(F)/C(F)^2$ of a fractional R_F-ideal of the form

$$\prod_{p \notin S} p^{\nu_P(m)}$$

where $m \in M$, $N_{M/F}(m) = b$ with $cl(b) \in H_F$ and $P|p$. If p is inert in M/F, then

$$p^{\nu_P(m)} \cdot R_M = P^{\nu_P(m)};$$

if p splits in M/F, $pR_M = P\bar{P}$, then

$$p^{\nu_P(m)} \cdot R_M = P^{\nu_P(m)}\bar{P}^{\nu_P(m)} = P^{\nu_P(m)}\bar{P}^{\nu_{\bar{P}}(m)}\bar{P}^{s}$$

with

$$s = \nu_P(m) - \nu_{\bar{P}}(m) \equiv \nu_P(m) + \nu_{\bar{P}}(m) = \nu_p(N(m)) = \nu_p(b) \equiv 0 \bmod 2.$$

So, s is even, and the class $i(\chi_2(cl(b)))$ in $C(M)/C(M)^2$ is the class of the fractional R_M-ideal mR_M, hence trivial. Thus $\operatorname{im}\chi_2 \subseteq \ker i$.

Next we prove the containment $\ker i \subseteq \operatorname{im}\chi_2$. Consider a fractional R_F-ideal A with $cl(A) \in \ker i$; that is,

$$AR_M = mB^2$$

for some $m \in M$ and some fractional R_M-ideal B. Put $b = N_{M/F}(m)$. Then $\nu_p(b) \equiv 0 \bmod 2$ for all $p \notin S$, so $cl(b) \in H_F$ and, in $C(M)/C(M)^2$,

$$\chi_2(cl(b)) = cl\left(\prod_{p \notin S} p^{\nu_P(m)}\right) = cl\left(\prod_{p \notin S} p^{\nu_p(A)}\right) = cl(A)$$

since $\nu_P(m) \equiv \nu_P(AR_M) \equiv \nu_p(A) \bmod 2$. Thus $\ker i \subseteq \operatorname{im}\chi_2$.

These two lemmas add two more terms to the above exact sequence and place the homomorphism χ_2 in proper context. We have obtained

PROPOSITION 3.8. *For a number field F with $\sqrt{-1} \notin F$, we have an exact sequence*

$$1 \longrightarrow C_2 \longrightarrow G_F \longrightarrow G_M \xrightarrow{N} H_F \xrightarrow{\chi_2} C(F)/C(F)^2 \longrightarrow C(M)/C(M)^2.$$

COROLLARY 3.9. *If $\sqrt{-1} \notin F$, then*

$$2\text{-rk} \ker \chi_2 = r_2(F) + 1 + g_2(M) - g_2(F) + 2\text{-rk}\, C(M) - 2\text{-rk}\, C(F).$$

Proof. By (3.8) we have an exact sequence

$$1 \longrightarrow C_2 \longrightarrow G_F \longrightarrow G_M \longrightarrow \ker \chi_2 \longrightarrow 1.$$

Now, by (2.4),

$$\begin{aligned} 2\text{-rk}\, G_F &= r_1(F) + r_2(F) + g_2(F) + 2\text{-rk}\, C(F) \\ 2\text{-rk}\, G_M &= r_1(F) + 2r_2(F) + g_2(M) + 2\text{-rk}\, C(M). \end{aligned}$$

The assertion follows by considering the alternating sum of 2-ranks.

4. Factorization. We continue to assume that F is a number field with $\sqrt{-1} \notin F$. In view of (2.3), our objective is to find the 2-rank of the kernel of

$$\chi : H_F \longrightarrow C(F)/C(F)^2$$

for all such number fields F.

The first part of this section consists of unit considerations. We are motivated by the following observation. Let U_M be the group of S-units of $M = F(\sqrt{-1})$ and, for $m \in U_M$, put $b = N_{M/F}(m)$. Then clearly $cl(b) \in H_F$. Moreover, we notice:

$$cl(b) \in \ker \chi_1, \text{ by (3.1), since } b \in U_F$$

$$cl(b) \in \ker \chi_2, \text{ by (3.5), since } b \in N_{M/F}(U_M).$$

Hence, $cl(b) \in \ker \chi$, by (3.2), and we have seen that

$$N(U_M/U_M^2) \subseteq \ker \chi,$$

where

$$N : U_M/U_M^2 \longrightarrow U_F/U_F^2$$

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is the homomorphism induced by the norm $M^* \longrightarrow F^*$.

Now we are going to exhibit a natural group H_0 with

$$N(U_M/U_M^2) \subseteq H_0 \subseteq \ker\chi$$

satisfying

$$2\text{-rk}\, H_0 = g_2(M) - g_2(F) + r_2(F) + 1.$$

This will yield

$$2\text{-rk} \ker\chi \geqq g_2(M) - g_2(F) + r_2(F) + 1$$

and hence, by (2.3),

$$4\text{-rk}\, K_2(O_F) \geqq g_2(M) - g_2(F);$$

that is, for all number fields F, a lower bound for the 4-rank of $K_2(O_F)$ is given by the number of dyadic primes of F that split in M/F.

Definition 4.1. Let $cl(b) \in H_F$; then $cl(b)$ lies in H_0 if and only if there is an $m \in M$ and a fractional R_F-ideal B satisfying

$$N_{M/F}(m) = b \text{ and } mR_m = BR_M.$$

It is clear that H_0 is a subgroup of H_F that contains $N(U_M/U_M^2)$; in fact, $H_0 \subset \ker\chi$. This containment will also follow directly from the Factorization Theorem (4.8).

Recall, from section 2, the epimorphism

$$\chi_0 : G_F \longrightarrow {}_2C(F).$$

Suppose $cl(b) \in H_0$ with $N_{M/F}(m) = b$ and $mR_M = BR_M$; taking norms yields $bR_F = B^2$, thus, by definition,

$$\chi_0(cl(b)) = cl(B) \text{ in } C(F)$$

for this R_F-ideal B, which lies in the kernel of

$$i_* : C(F) \longrightarrow C(M).$$

So, the restriction of χ_0 to H_0 yields a homomorphism

$$\chi_0 : H_0 \longrightarrow \ker i_*$$

which is clearly surjective. We put

$$N_0 = \ker(N : U_M/U_M^2 \longrightarrow U_F/U_F^2).$$

LEMMA 4.2. *There is an exact sequence*

$$1 \longrightarrow N_0 \longrightarrow U_M/U_M^2 \xrightarrow{N} H_0 \xrightarrow{\chi_0} \ker i_* \longrightarrow 1.$$

Proof. All that is left to show is the exactness at H_0. The containment

$$N(U_M/U_M^2) \subseteq \ker \chi_0$$

is immediate.

Now suppose $cl(b) \in H_0$ lies in $\ker \chi_0$. Then we have an $m \in M$ with $N_{M/F}(m) = b$ and a principal fractional R_F-ideal B with $mR_M = BR_M$; so,

$$B = aR_F \quad \text{for some } a \in F^*.$$

But this means $m = a \cdot u$ for some S-unit $u \in U_M$. We conclude

$$cl(b) = cl(N_{M/F}(m)) = cl(a^2 N_{M/F}(u)) = cl(N_{M/F}(u));$$

that is, $cl(b)$ is the class of the norm of an S-unit in U_M. Thus

$$\ker \chi_0 \subseteq N(U_M/U_M^2).$$

The 2-ranks of N_0 and $\ker i_*$ are not easily accessible. The idea is now to produce a second exact sequence involving N_0 and $\ker i_*$ such that 2-rk H_0 drops out explicitly.

Let $u \in U_M$ with $cl(u) \in N_0$; then $N_{M/F}(u)$ is a square in F^*, actually in U_F. Thus u and $\bar{u}$ differ only by a square in M^*. Therefore, for some $b \in F^*$ and some $m \in M^*$, we can write

$$b = um^2 \quad \text{in } M^*.$$

In particular, $cl(b) \in G_F$. Then consider

$$\mu : N_0 \longrightarrow {}_2C(F)$$
$$cl(u) \longrightarrow \chi_0(cl(b))$$

and check that μ is well-defined. In fact, μ is a homomorphism with image

$$\mu(N_0) = \ker i_*.$$

The kernel of the natural homomorphism

$$\nu : U_F/U_F^2 \longrightarrow U_M/U_M^2$$

is cyclic of order 2, generated by the class of -1, and it is immediate that

$$\nu(U_F/U_F^2) \subseteq N_0.$$

So far, we have

$$1 \longrightarrow C_2 \longrightarrow U_F/U_F^2 \xrightarrow{\nu} N_0 \quad \text{and} \quad N_0 \xrightarrow{\mu} \ker i_* \longrightarrow 1.$$

LEMMA 4.3. *There is an exact sequence*

$$1 \longrightarrow C_2 \longrightarrow U_F/U_F^2 \xrightarrow{\nu} N_0 \xrightarrow{\mu} \ker i_* \longrightarrow 1.$$

Proof. All that is left to show is the exactness at N_0. The containment

$$\nu(U_F/U_F^2) \subseteq \ker \mu$$

is immediate.

Now suppose $cl(u) \in N_0$ lies in $\ker \mu$. Then we can write $b = um^2$ for some $b \in F^*$, $bR_M = B^2$ with a principal R_F-ideal B. Hence $b = vn^2$ for some $v \in U_F$ and $v = u(mn^{-1})^2$. So $mn^{-1} \in U_M$ and $cl(u)$ in N_0 is in the image of ν. Thus

$$\ker \mu \subseteq \nu(U_F/U_F^2).$$

The combination of these two lemmas implies immediately:

PROPOSITION 4.4. *For a number field F with $\sqrt{-1} \notin F$, we have*

$$2\text{-rk}\, H_0 = g_2(M) - g_2(F) + r_2(F) + 1$$

and hence

$$4\text{-rk}\, K_2(O_F) \geqq g_2(M) - g_2(F).$$

Proof. By (4.2) and (4.3) we know

$$2\text{-rk} \ker i_* - 2\text{-rk}\, N_0 = 2\text{-rk}\, H_0 - 2\text{-rk}\, U_M/U_M^2 \quad \text{and}$$
$$2\text{-rk} \ker i_* - 2\text{-rk}\, N_0 = 1 - 2\text{-rk}\, U_F/U_F^2;$$

hence

$$\begin{aligned} 2\text{-rk}\, H_0 &= 1 + 2\text{-rk}\, U_M/U_M^2 - 2\text{-rk}\, U_F/I_F^2 \\ &= 1 + (r_1(F) + 2r_2(F) + g_2(M)) - (r_1(F) + r_2(F) + g_2(F)) \\ &= g_2(M) - g_2(F) + r_2(F) + 1. \end{aligned}$$

Since $H_0 \subseteq \ker \chi$, this shows, in view of (2.3),

$$4\text{-rk}\, K_2(O_F) \geqq g_2(M) - g_2(F).$$

We are going to relate the subgroup H_0 of $\ker \chi$ of the kernel of the epimorphism

$$\begin{aligned} \alpha : G_F &\longrightarrow {}_2K_2(O_F) \\ cl(b) &\longrightarrow \{-1, b\}. \end{aligned}$$

COROLLARY 4.5. *If F is a number field with $\sqrt{-1} \notin F$ for which 2-prim $K_2(O_F)$ is elementary abelian, then*

$$H_0 = \ker \alpha.$$

Proof. If 2-prim $K_2(O_F)$ is elementary abelian, then $2\text{-rk} \ker \chi = r_2(F) + 1$, by (2.3). Since $\ker \alpha \subseteq \ker \chi$ and $2\text{-rk} \ker \alpha = r_2(F) + 1$ we obtain

$$\ker \alpha = \ker \chi.$$

Moreover, by (4.4), we conclude that $g_2(M) = g_2(F)$ and hence $2\text{-rk}\, H_0 = r_2(F) + 1$. Since $H_0 \subseteq \ker \chi$ we obtain

$$H_0 = \ker \chi;$$

so, the kernel of α is equal to H_0.

This is the one case in which we have the opportunity of determining effectively the Tate kernel, $\ker \alpha$.

The second part of this section consists of class group considerations. The Factorization Theorem (4.8) will identify the difference between $4\text{-rk}\, K_2(O_F)$ and $g_2(M) - g_2(F)$ as the 2-rank of the kernel of a natural norm homomorphism defined on class groups.

We denote the subgroup of elements of order at most 2 of the quotient of $C(M)$ modulo the image of $i_* : C(F) \longrightarrow C(M)$ by $\Lambda(F)$; that is,

$$\Lambda(F) = {}_2(C(M)/i_*C(F)).$$

Suppose $cl(b) \in H_F$ and $N_{M/F}(m) = b$. For each $p \notin S$ choose an extension $P|p$ to M and define a fractional R_F-ideal B by $\nu_p(B) = \nu_P(m)$ for $p \notin S$. Then mB^{-1} has even order at every prime ideal of R_M. Thus, there exists a fractional R_M-ideal C such that

$$mR_M = BC^2.$$

Definition 4.6. Let $\lambda : H_F \longrightarrow \Lambda(F)$ be the homomorphism given by $cl(b) \longrightarrow cl(C)$.

It may be routinely verified that λ is well-defined, hence a homomorphism. Just by comparing $mR_M = BR_M$ with $mR_M = BC^2$ in the definitions (4.1) and (4.6), we notice that

$$\ker \lambda = H_0.$$

Moreover, λ is surjective. Namely, let $cl(C) \in \Lambda(F)$; then BC^2 is principal for some fractional R_F-ideal B, $BC^2 = mR_M$ with $m \in M$, say. Put $b = N_{M/F}(m)$; so,

$$bR_F = (B \cdot N_{M/F}(C))^2.$$

This shows that $cl(b) \in H_F$ and $\lambda(cl(b)) = cl(C)$. Hence

$$\operatorname{im} \lambda = \Lambda(F),$$

and we have proved

LEMMA 4.7. *There is a short exact sequence*

$$1 \longrightarrow H_0 \longrightarrow H_F \xrightarrow{\lambda} \Lambda(F) \longrightarrow 1.$$

The missing link is now provided by the homomorphism

$$n_{M/F} : \Lambda(F) \longrightarrow C(F)/C(F)^2$$

that is induced by the norm

$$N : C(M) \longrightarrow C(F).$$

As a consequence of the decomposition (3.2) we prove the following factorization for χ.

THEOREM 4.8. *For a number field F with $\sqrt{-1} \notin F$, there is a commutative diagram*

$$\begin{array}{ccc} H_F & \xrightarrow{\lambda} & \Lambda(F) \\ & {\scriptstyle\chi}\searrow \quad \swarrow{\scriptstyle n_{M/F}} & \\ & C(F)/C(F)^2 & \end{array}$$

Proof. Suppose $cl(b) \in H_F$, $m \in M$ with $N_{M/F}(m) = b$. Write $mR_M = BC^2$ with fractional R_F, R_M-ideals B, C, respectively. From

$$bR_F = (BN_{M/F}(C))^2$$

we see immediately, by (2.5), that

$$\chi_1(cl(b)) = cl(B)cl(N_{M/F}(C)) \quad \text{in } C(F)/C(F)^2.$$

Now, by (3.5),

$$\chi_2(cl(b)) = cl\left(\prod_{p \notin S} p^{\nu_P(m)}\right)$$

where, for $P|p$, we have

$$\nu_P(m) = \nu_P(BR_M) + 2\nu_P(C) = \nu_p(B) + 2\nu_P(C) \equiv \nu_p(B) \bmod 2.$$

Therefore,

$$\chi_2(cl(b)) = cl(B) \quad \text{in } C(F)/C(F)^2.$$

Then, by (3.2),

$$\begin{aligned} \chi(cl(b)) &= \chi_1(cl(b)) \cdot \chi_2(cl(b)) = cl(B)^2 \cdot cl(N_{M/F}(C)) \\ &= cl(N_{M/F}(C)) = n_{M/F} \circ \lambda(cl(b)) \quad \text{in } C(F)/C(F)^2 \end{aligned}$$

since $\lambda(cl(b)) = cl(C)$ in $\Lambda(F)$, by (4.6). The factorization formula is established.

Once again we see that $H_0 = \ker \lambda$ is a subgroup of $\ker \chi$. The consequence we are interested in is the 4-rank formula:

COROLLARY 4.9. *For a number field F with $\sqrt{-1} \notin F$, we have*

$$4\text{-rk}\, K_2(O_F) = g_2(M) - g_2(F) + 2\text{-rk} \ker n_{M/F}.$$

Proof. From the short exact sequence (4.7) and the commutative triangle (4.8) we have the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & H_0 & \longrightarrow & H_F & \xrightarrow{\lambda} & \Lambda(F) & \longrightarrow & 1 \\ & & & & \chi\downarrow & \swarrow n_{M/F} & & & \\ & & & & C(F)/C(F)^2 & & & & \end{array}$$

and conclude

$$2\text{-rk}\ker\chi = 2\text{-rk}\, H_0 + 2\text{-rk}\ker n_{M/F}.$$

In (4.4) we have computed the 2-rank of H_0, hence

$$2\text{-rk}\ker\chi = g_2(M) - g_2(F) + r_2(F) + 1 + 2\text{-rk}\ker n_{M/F},$$

which, in view of (2.3), yields our claim.

For applications we refer to the next section. Let us notice that for

$$N : C(M) \longrightarrow C(F) \quad \text{and} \quad n_{M/F} : \Lambda(F) \longrightarrow C(F)/C(F)^2$$

we have a natural isomorphism

$$\ker n_{M/F} = {}_2(\ker N/i_*({}_2C(F)));$$

in particular,

$$2\text{-rk}\ker n_{M/F} = 2\text{-rk}\ker N/i_*({}_2C(F)),$$

which yields the reformulation:

COROLLARY 4.10. *For a number field F with $\sqrt{-1} \notin F$, we have*

$$4\text{-rk}\, K_2(O_F) = g_2(M) - g_2(F) + 2\text{-rk}\ker N/i_*({}_2C(F)).$$

This is the 4-rank formula as stated in (1.5).

5. Applications.

Elementary abelian 2-prim $K_2(O_F)$. Let us apply the 4-rank formulas (2.6) and (4.10) in order to obtain a characterization of all number fields F for which the 2-primary subgroup of $K_2(O_F)$ is elementary abelian. We will denote by $h(M/F)$ the relative S-class number of M/F; that is,

$$h(M/F) = \#\ker(N : C(M) \longrightarrow C(F)).$$

If $\sqrt{-1} \in F$, then $4\text{-rk}\, K_2(O_F) = 0$ if and only if $g_2(F) = 1$ and $4\text{-rk}\, C(F) = 0$, by (2.6).

If $\sqrt{-1} \notin F$ then $4\text{-rk}\, K_2(O_F) = 0$ if and only if $g_2(M) = g_2(F)$ and the 2-primary subgroup of the kernel of $N : C(M) \longrightarrow C(F)$ is equal to $i_*({}_2C(F))$, by (4.10). The kernel of $i_* : C(F) \longrightarrow C(M)$ is contained in ${}_2C(F)$, hence

$$2^{2\text{-rk}C(F)} = \# i_*({}_2C(F)) \cdot \# \ker i_*.$$

So, the condition that 2-prim $\ker N = i_*({}_2C(F))$ means that $2^{2\text{-rk}C(F)}$ is the exact 2-power dividing $h(M/F) \cdot \# \ker i_*$. We have obtained

PROPOSITION 5.1. *For a numbered field F with $\sqrt{-1} \in F$, we have:* 2-prim $K_2(O_F)$ *is elementary abelian if and only if F has only one dyadic prime and* 2-prim $C(F)$ *is elementary abelian.*

For a number field F with $\sqrt{-1} \notin F$, we have: 2-prim $K_2(O_F)$ *is elementary abelian if and only if no dyadic prime splits in M/F and* $2^{2\text{-rk}C(F)} \| h(M/F) \cdot \# \ker i_*$.

How explicit is this divisibility condition? The relative S-class number $h(M/F)$ can be expressed in terms of S-class numbers

$$h(F) = \# C(F) \quad \text{and} \quad h(M) = \# C(M)$$

in the following way:

$$h(M/F) = \begin{cases} h(M)/h(F) & \text{if there is a place in } S \text{ that} \\ & \quad \text{does not split in } M/F; \\ 2h(M)/h(F) & \text{if all places in } S \text{ split in } M/F. \end{cases}$$

Namely, the norm $N : C(M) \longrightarrow C(F)$ is surjective unless $r_1(F) = 0$ and all dyadic primes of F split in M/F, however the index of $N(C(M))$ in $C(F)$ is 2 in the exceptional case. In particular, in (5.1) we can replace the condition $2^{2\text{-rk}C(F)} \| h(M/F) \cdot \# \ker i_*$ by

$$2^{2\text{-rk}C(F)} \| (h(M)/h(F)) \cdot \# \ker i_*.$$

For totally real number fields F this simplifies further, since $\ker i_*$ is trivial if $r_2(F) = 0$ and $g_2(M) = g_2(F)$, [**8**]. A determination of all real quadratic number fields F with 2-prim $K_2(O_F)$ elementary abelian has been given in [**1**].

However, if F is not totally real, then $\ker i_*$ might be non-trivial even if $g_2(M) = g_2(F)$. This feature makes the application of (5.1) more delicate. We have been informed about research in progress concerning the determination of all imaginary quadratic number fields F with 2-prim $K_2(O_F)$ elementary abelian. For examples we refer to Section 6.

Extreme cases. Using the decomposition $\chi = \chi_1 \cdot \chi_2$, we would like to produce more explicit 4-rank formulas for $K_2(O_F)$ in the extreme cases when $\chi = \chi_1$ or $\chi = \chi_2$.

PROPOSITION 5.2. *If $H_F = G_F$, then χ_2 is trivial if and only if either $\sqrt{-1} \in F$ or*

$$1 + g_2(M) + 2\text{-rk}\, C(M) = 2(g_2(F) + 2\text{-rk}\, C(F)).$$

In that case

$$4\text{-rk}\, K_2(O_F) = g_2(F) - 1 + 4\text{-rk}\, C(F).$$

Proof. If $\sqrt{-1} \in F$, then everything is clear, by (2.6). We now consider the case $\sqrt{-1} \notin F$. Since by assumption $H_F = G_F$ it follows that -1 is a norm from M/F and hence F is totally complex. Then, by (2.4),

$$2\text{-rk}\, H_F = 2\text{-rk}\, G_F = r_2(F) + g_2(F) + 2\text{-rk}\, C(F)$$

and, by (3.9),

$$2\text{-rk}\ker \chi_2 = r_2(F) + 1 + g_2(M) - g_2(F) + 2\text{-rk}\, C(M) - 2\text{-rk}\, C(F).$$

We equate

$$2\text{-rk}\, H_F = 2\text{-rk}\ker \chi_2$$

and the first assertion follows.

Next, if we assume χ_2 is trivial, then we have $\chi = \chi_1$, by (3.2), and since $G_F = H_F$ we have an exact sequence together with a commutative triangle

$$\begin{array}{ccccccccc} 1 & \longrightarrow & U_F/U_F^2 & \longrightarrow & H_F & \xrightarrow{\chi_0} & {}_2C(F) & \longrightarrow & 1 \\ & & & & \chi_1 \downarrow & \swarrow & & & \\ & & & & C(F)/C(F)^2 & & & & \end{array}$$

Hence

$$2\text{-rk}\ker \chi = 2\text{-rk}\ker \chi_1 = 2\text{-rk}\, U_F/U_F^2 + 4\text{-rk}\, C(F).$$

Since

$$2\text{-rk}\, U_F/U_F^2 = (r_2(F) + 1) + g_2(F) - 1,$$

application of (2.3) finishes the proof.

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The reader will notice that (5.2) generalizes the old result (2.6), for the assumptions in (5.2) apply to a wider class of number fields F than just the ones with $\sqrt{-1} \in F$.

COROLLARY 5.3. *Suppose F is a totally complex number field with $\sqrt{-1} \notin F$. If $g_2(F) = 1$, then $H_F = G_F$ and χ_2 is trivial if and only if either*

i) $$g_2(M) = 1 \quad \text{and} \quad 2\text{-rk}\, C(M) = 2(2\text{-rk}\, C(F))$$

or

ii) $$g_2(M) = 2 \quad \text{and} \quad 1 + 2\text{-rk}\, C(M) = 2(2\text{-rk}\, C(F)).$$

Proof. If $r_1(F) = 0$ and $g_2(F) = 1$, then $H_F = G_F$; namely: if $cl(b) \in G_F$, then $\nu_p(b) \equiv 0 \bmod 2$ and hence $(-1, b)_p = +1$ for every finite non-dyadic place p of F. By assumption, F has no real infinite places, and so by reciprocity

$$(-1, b)_p = +1$$

also for the unique dyadic place p of F. Hence b is a norm from M/F and $cl(b) \in H_F$. Now apply (5.2).

We shall remind the reader of this in a later example (6.6) and the concluding exercise (7.3). For the other extreme we have

PROPOSITION 5.4. *If χ_1 is trivial and $\sqrt{-1} \notin F$, then*

$$4\text{-rk}\, K_2(O_F) = g_2(M) - g_2(F) + 2\text{-rk}\, C(M) - 2\text{-rk}\, C(F).$$

Proof. By (3.2) and (3.9) we have

$$\begin{aligned} 2\text{-rk} \ker \chi &= 2\text{-rk} \ker \chi_2 \\ &= r_2(F) + 1 + g_2(M) - g_2(F) + 2\text{-rk}\, C(M) - 2\text{-rk}\, C(F). \end{aligned}$$

Then (2.3) finishes the proof.

Clearly

$$\chi_1 : H_F \longrightarrow {}_2C(F) \longrightarrow C(F)/C(F)^2$$

is trivial if $4\text{-rk}\, C(F) = 2\text{-rk}\, C(F)$.

COROLLARY 5.5. *Let F be a number field with $4\text{-rk}\, C(F) = 2\text{-rk}\, C(F)$ and $\sqrt{-1} \notin F$. Then*

$$4\text{-rk}\, K_2(O_F) = g_2(M) - g_2(F) + 2\text{-rk}\, C(M) - 2\text{-rk}\, C(F).$$

We point out a special case

COROLLARY 5.6. *Let F be a number field with odd S-class number $h(F)$ and $\sqrt{-1} \notin F$. Then*

$$4\text{-rk}\, K_2(O_F) = g_2(M) - g_2(F) + 2\text{-rk}\, C(M).$$

In particular then $2\text{-prim}\, K_2(O_F)$ *is elementary abelian if and only if no dyadic prime splits in M/F and the S-class number $h(M)$ is also odd.*

Examples illustrating these assertions are quite common. For the convenience of the reader we will provide in the appendix the explicit determination of $2\text{-rk}\, C(F)$ and $2\text{-rk}\, C(M)$ in case of all quadratic number fields F.

Furthermore, computations of $\ker \chi$ involving χ_1 and χ_2 can be carried out in many examples which do not fall into either of the two extremes; an illustration is the proof of (6.4).

6. Examples. Due to the lack of 4-rank formulas for number fields F that are neither totally real nor contain $\sqrt{-1}$, so far information about 2-prim $K_2(O_F)$ for imaginary quadratic number fields F has been limited. For several imaginary quadratic number fields F of small discriminant, in absolute value, the whole group $K_2(O_F)$ has been computed in [**13**]. We put

$$F = \mathbf{Q}(\sqrt{d}) \quad \text{with } d < 0, \text{ squarefree}$$

$$h(F) = \#C(F) \quad S\text{-class number of } F.$$

If $d \neq -1$, then $M = F(\sqrt{-1})$ is an abelian extension of $\mathbf{Q}$ with degree 4 and Galois group $C_2 \times C_2$. The 2-ranks of the S-class groups $C(F)$ and $C(M)$ have been listed in the appendix.

The wild kernel wild (O_F) (Hilbert Kernel) is a subgroup of the tame kernel $K_2(O_F)$ whose 2-rank was determined in [**2**] for all quadratic fields F. This allows us to describe the quadratic number fields F whose wild kernel has a trivial 2-primary subgroup; that is, for which $\#\text{wild}(O_F)$ is odd. The complete list of such imaginary quadratic number fields F was given in [**5**]:

Let $F = \mathbf{Q}(\sqrt{d})$ be imaginary quadratic. Then $\#\text{wild}(O_F)$ is odd if and only if

$$(6.1)\qquad \begin{array}{ll} d = -1, -2, -p, -2p & \textit{with a prime } p \equiv \pm 3 \bmod 8 \textit{ or} \\ d = -pq & \textit{with primes } p \equiv 3 \bmod 8, q \equiv 5 \bmod 8 \textit{ or} \\ d = -p & \textit{with a prime } p \equiv 7 \bmod 8. \end{array}$$

This leads to the following simple characterization of all imaginary quadratic number fields with a wild kernel of odd order.

PROPOSITION 6.2. *Let $F = \mathbf{Q}(\sqrt{d})$ be imaginary quadratic. Then:*

$\#\text{wild}(O_F)$ *is odd if and only if*

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$h(F)$ is odd *if and only if*

2-prim $K_2(O_F)$ *is elementary abelian of rank* $g_2(F) - 1$.

Proof. For $F = \mathbf{Q}(\sqrt{-1})$ the tame kernel $K_2(O_F)$ is trivial, hence all three properties hold for F. We will assume that $d < -1$.

We check (7.1) and find out that $F = \mathbf{Q}(\sqrt{d})$ occurs in the list (6.1) if and only if 2-rk $C(F) = 0$; hence the first two properties are equivalent.

The condition that 2-rk $K_2(O_F) = g_2(F) - 1$ means that $h(F)$ is odd, by (1.2). Thus (5.6) applies and yields:

2-prim $K_2(O_F)$ is elementary abelian of rank $g_2(F) - 1$ if and only if $g_2(M) = g_2(F)$ and $h(M)$ is odd;

that is, if and only if $d \not\equiv 7 \bmod 8$ and $h(M)$ is odd. Now compare (7.1) and (7.2) and notice that

$$2\text{-rk}\, C(F) = 0 \text{ if and only if } d \not\equiv 7 \bmod 8 \text{ and } 2\text{-rk}\, C(M) = 0.$$

Hence the last two properties are equivalent.

Let us investigate 2-prim $K_2(O_F)$ for the fields $F = \mathbf{Q}(\sqrt{-p})$, where p denotes a rational prime number.

For $p \not\equiv 1 \bmod 8$ we conclude that 2-rk $C(F) = 0$, by (7.1), hence the structure of 2-prim $K_2(O_F)$ is known by (6.2).

For $p \equiv 1 \bmod 8$ we conclude that 2-rk $C(F) = 1$, by (7.1). Since $g_2(F) = 1$, $g_2(M) = 2$ we obtain 2-rk $K_2(O_F) = 1$, by (1.2), and 4-rk $K_2(O_F) \geqq 1$, by (4.4). This yields

Example (6.3) Let $F = \mathbf{Q}(\sqrt{-p})$ with a prime p. Then

2-prim $K_2(O_F) = \{1\}$	if $p = 2$ or $p \equiv 3, 5 \bmod 8$
2-prim $K_2(O_F) = C_2$	if $p \equiv 7 \bmod 8$
2-prim $K_2(O_F)$ is cyclic of order divisible by 4	if $p \equiv 1 \bmod 8$.

The imaginary quadratic number field F with smallest discriminant, in absolute value, for which 2-prim $K_2(O_F)$ was not determined in [**13**], is $F = \mathbf{Q}(\sqrt{-35})$. In this regard, we deduce

Example 6.4. Let $F = \mathbf{Q}(\sqrt{-pq})$ with primes $p \equiv 7 \bmod 8$ and $q \equiv 5 \bmod 8$. Then:

$$2\text{-prim}\, K_2(O_F) = C_2, \text{ generated by } \{-1, -1\}.$$

Proof. From $g_2(F) = 1$ and, by (7.1), 2-rk $C(F) = 1$ we see in view of (1.2) that

$$2\text{-rk}\, K_2(O_F) = 1.$$

Now, 2-rk $G_F = 3$, by (2.4), and hence the classes of $-1, 2, q$ form a basis for G_F. Clearly the classes of 2 and q belong to H_F; namely

$$2 = N_{M/F}(1+i) \quad \text{and} \quad q = N_{M/F}(a+bi),$$

where $q = a^2 + b^2$ for some $a, b \in \mathbf{Z}$. However, also -1 is a norm from M/F since the level of F is 2, compare [**10**]. So, $G_F = H_F$,

$$2\text{-rk}\, H_F = 3,$$

and the classes of $-1, 2, q$ form a basis for H_F.

From (3.9) we conclude that

$$2\text{-rk}\ker\chi_2 = 2,$$

since $g_2(M) = 1$ and 2-rk $C(M) = 1$, by (7.2). As noted above, 2 is the norm of the S-unit $1+i$ from M/F; so, $cl(2) \in H_F$ is a norm from G_M and hence, by (3.5),

$$\chi_2(cl(2)) = 1 \quad \text{in } C(F)/C(F)^2.$$

To see that $\chi_2(cl(q))$ is also trivial we refer to our comment after the original definition (3.1) of χ_2. It is enough to show that there exists an $e \in F(\sqrt{q}) = \mathbf{Q}(\sqrt{-pq}, \sqrt{q})$ with $N_{E/F}(e) = -1$. We can make the choice $e = \epsilon$, the fundamental unit of $\mathbf{Q}(\sqrt{q})$. Then

$$N_{E/F}(\epsilon) = N_{\mathbf{Q}(\sqrt{q})/\mathbf{Q}}(\epsilon) = -1$$

in view of $q \equiv 1 \bmod 4$. Hence

$$\chi_2(cl(q)) = 1 \quad \text{in } C(F)/C(F)^2.$$

This implies that

$$\chi_2(cl(-1)) \neq 1 \quad \text{in } C(F)/C(F)^2.$$

Since, see (2.5), $\chi_1(cl(-1))$ is trivial the decomposition $\chi = \chi_1 \cdot \chi_2$, (3.2), tells us that

$$\chi(cl(-1)) \neq 1 \quad \text{in } C(F)/C(F)^2.$$

Hence, $cl(-1) \notin \ker\chi$; so, by (2.2), the Steinberg symbol $\{-1, -1\}$ is not a square in $K_2(O_F)$. Thus we have exhibited, in the cyclic group 2-prim $K_2(O_F)$, an element of order 2 that is not a square; that is, 2-prim $K_2(O_F)$ is cyclic of order 2, and $\{-1, -1\}$ is the generator.

In this example it can be shown that the kernel of

$$i_* : C(F) \longrightarrow C(M)$$

is non-trivial even though 2-prim $K_2(O_F)$ is elementary abelian. In a similar way, we can determine all fields $F = \mathbf{Q}(\sqrt{-pq})$ with primes p, q, for which the 2-primary subgroup of $K_2(O_F)$ is elementary abelian.

We state the following example without proof

Example 6.5. Let $E = \mathbf{Q}(\sqrt{pqr})$, with primes $p, q, r \equiv 3 \bmod 8$ satisfying $(\frac{p}{q}) = (\frac{q}{r}) = (\frac{r}{p}) = +1$. Then 2-prim $K_2(O_E)$ is elementary abelian of rank 4, while 2-prim $K_2(O_F)$ is elementary abelian of rank 2.

To illustrate (6.5) the choices $p = 3$, $q = 11$, $r = 19$ may be used. Our final example will appeal to both (5.3), in which case χ_2 is trivial, and (5.5), in which case χ_1 is trivial.

Example 6.6. Let $F = \mathbf{Q}(\sqrt{-2p})$ with a prime $p \equiv 9 \bmod 16$. Then

$$2\text{-prim}\, K_2(O_F) = C_2.$$

Proof. We note that $g_2(F) = g_2(M) = 1$ and, by (7.1), 2-rk $C(F) = 1$. This tells us, by (1.2), that 2-rk $K_2(O_F) = 1$.

First suppose that p cannot be written as $x^2 + 32y^2$ with $x, y \in \mathbf{Z}$. Then 2-rk $C(M) = 1$, by (7.2), since p is in A^-. Now from (24.6) in [**4**] it will follow that 8 divides the ordinary class number of F, and hence the order of the S- class group $C(F)$ is divisible by 4. Thus 2-rk $C(F)$ = 4-rk $C(F)$ and (5.5) applies. We conclude that

$$4\text{-rk}\, K_2(O_F) = 0.$$

Next suppose that p can be written as $x^2 + 32y^2$ for some $x, y \in \mathbf{Z}$. Then 2-rk $C(M) = 2$, by (7.2), since p is in A^+. We deduce from (5.3) that $H_F = G_F$ and χ_2 is trivial. Thus, by (5.2),

$$4\text{-rk}\, K_2(O_F) = 4\text{-rk}\, C(F).$$

Now it will follow from (24.6) in [**4**] that 4 is the exact 2-power dividing the ordinary class number of F.

It may be seen as follows that the dyadic prime of F is not principal. If it were then it would have a generator which is in $O_F = \mathbf{Z}[\sqrt{-2p}]$ and has norm 2. This implies that $2 = a^2 + 2pb^2$ has a solution in rational integers, which it obviously does not.

We conclude that the order of the S-class group $C(F)$ is congruent to $2 \bmod 4$; hence 4-rk $C(F) = 0$. Again we obtain that

$$4\text{-rk}\, K_2(O_F) = 0.$$

Thus, in any case, 2-prim $K_2(O_F)$ is elementary abelian of rank 1.

To illustrate (6.6) the primes $73 \in A^-$ and $41 \in A^+$ may be chosen. In a similar way, we can determine all fields $F = \mathbf{Q}(\sqrt{-2p})$ with a prime p for which the 2-primary subgroup of $K_2(O_F)$ is elementary abelian.

7. Appendix. The objective is to make the determination of the 4- rank of $K_2(O_F)$ explicit for the quadratic number fields F. We put

$$F = \mathbf{Q}(\sqrt{d}), \quad M = \mathbf{Q}(\sqrt{d}, \sqrt{-1})$$

with $d \in \mathbf{Z}$, $|d| > 1$, square free. As before S is the set of infinite and dyadic primes, $C(F)$ and $C(M)$ are the S-ideal class groups of F and M, respectively. Let

$$\begin{aligned} 2^s &= \text{\# of elements in } \{1, -1, 2, -1\} \text{ that are norms from } F/\mathbf{Q} \\ t &= \text{\# of odd prime divisors of } d \\ t_1 &= \text{\#of prime divisors } p \text{ of } d \text{ with } p \equiv 1 \bmod 4. \end{aligned}$$

Lemma 7.1. *The 2-rank of the S-class group $C(F)$ is given by*

$$2\text{-rk}\, C(F) = \begin{cases} s+t-1 & \text{if } d \not\equiv 1 \bmod 8,\ d < 0 \\ s+t-2 & \text{if } d \equiv 1 \bmod 8,\ d < 0 \\ s+t-2 & \text{if } d \not\equiv 1 \bmod 8,\ d > 0 \\ s+t-3 & \text{if } d \equiv 1 \bmod 8,\ d > 0. \end{cases}$$

Proof. This determination can be performed in terms of genus theory, dating back to Gauss. Explicitly, the above list has been given in [**1**].

For rational primes $p \equiv 1 \bmod 8$, we let

$$\begin{aligned} &p \in A^+ \quad \text{if and only if} \quad p = x^2 + 32y^2 \quad \text{for some} \quad x, y \in \mathbf{Z}, \\ &p \in A^- \quad \text{if and only if} \quad p \neq x^2 + 32y^2 \quad \text{for all} \quad x, y \in \mathbf{Z}. \end{aligned}$$

LEMMA 7.2. *The 2-rank of the S-class group $C(M)$ is given by*

$$\begin{cases} t_1+t-1 & \text{if } d \equiv \pm 1 \bmod 8 \text{ and } p \equiv 7 \bmod 8 \text{ or } p \in A^+ \\ & \quad \text{for all primes } p \text{ dividing } d \\ t_1+t-2 & \text{if } d \equiv \pm 1 \bmod 8 \text{ and no prime } p \equiv 5 \bmod 8 \text{ divides } d, \\ & \quad \text{but either a prime } p \equiv 3 \bmod 8 \text{ or } p \in A^- \\ & \quad \text{divides } d \\ t_1+t-3 & \text{if } d \equiv \pm 1 \bmod 8 \text{ and there is a prime } p \equiv 5 \bmod 8 \\ & \quad \text{dividing } d \\ t_1+t-1 & \text{if } d \equiv \pm 3 \bmod 8 \text{ and no prime } p \equiv 5 \bmod 8 \text{ divides } d \\ t_1+t-2 & \text{if } d \equiv \pm 3 \bmod 8 \text{ and there is a prime } p \equiv 5 \bmod 8 \\ & \quad \text{dividing } d \\ t_1+t & \text{if } d \equiv 0 \bmod 2 \text{ and } p \equiv 7 \bmod 8 \text{ or } p \in A^+ \\ & \quad \text{for all odd primes } p \text{ dividing } d \\ t_1+t-1 & \text{if } d \equiv 0 \bmod 2 \text{ and no prime } p \equiv 5 \bmod 8 \text{ divides } d \text{ but} \\ & \quad \text{either a prime } p \equiv 3 \bmod 8 \text{ or } p \in A^- \text{ divides } d \\ t_1+t-2 & \text{if } d \equiv 0 \bmod 2 \text{ and there is a prime } p \equiv 5 \bmod 8 \\ & \quad \text{dividing } d. \end{cases}$$

This lemma can be established by appeal to the S-version of the exact hexagon associated with $M/\mathbf{Q}(\sqrt{-1})$. The reader is referred to [**4**], particularly Section 6.

We would like to finish by suggesting an exercise that makes use of the above two lists. Consider the imaginary quadratic fields $F = \mathbf{Q}(\sqrt{d})$ with

$$d < -1 \text{ square free}, \quad d \not\equiv 1 \bmod 8.$$

Then (5.3) applies to these fields F.

Exercise 7.3. Using (7.1), (7.2), and (5.3) determine those d for which χ_2 is trivial.

For such d it will follow by (5.2) that

$$4\text{-rk}\, K_2(O_F) = 4\text{-rk}\, C(F).$$

In particular, χ_2 is trivial for $d = -p_1 p_2 \cdot \ldots \cdot p_t$ with $t \geqq 1$ primes $p_1, p_2, \ldots, p_t$ in A^+. For those d we may add, by (4.4), in view of $g_2(F) = 1$, $g_2(M) = 2$ that

$$4\text{-rk}\, K_2(O_F) = 4\text{-rk}\, C(F) \geqq 1.$$

References

1. B. Brauckmann, *Die 2-Sylowgruppe des zahmen Kerns total-reeller Zahlkörper*, Thesis, Münster (1987).
2. J. Browkin and A. Schinzel, *On Sylow 2-subgroups of $K_2(O_F)$ for quadratic number fields F*, J. reine angew. Math *331* (1982), 104–113.
3. A. Candiotti and K. Kramer, *On the 2-Sylow subgroup of the Hilbert kernel of K_2 of number fields*, preprint (1987).
4. P. E. Conner and J. Hurrelbrink, *Class number parity*, Series Pure Math. *8* (World Scientific Publ. Co., Singapore, 1988).
5. J. Hurrelbrink, *Class numbers, units, and K_2*, to appear in Proc. Alg. K-Th. Conf. Lake Louise, Canada (1987), NATO ASI Series.
6. J. Hurrelbrink and M. Kolster, *On the 2-primary part of the Birch-Tate conjecture for cyclotomic fields*, Contemp. Math. *55* (1986), 519–528.
7. F. Keune, *On the structure of K_2 of the ring of integers in a number field*, preprint (1987).
8. M. Kolster, *The structure of the 2-Sylow subgroup of $K_2(O)$, I*, Comment. Math. Helvetici *61* (1986), 376–388.
9. ——— *The structure of the 2-Sylow subgroup of $K_2(O)$, II*, K-Theory *1* (1987), 467–479.
10. T. Y. Lam, *The algebraic theory of quadratic forms* (Reading, Mass, 1973).
11. A. S. Merkurjev and A. A. Suslin, *On the K_3 of fields*, preprint (1987).
12. J. Milnor, *Introduction to algebraic K-theory*, Annals Math. Stud. *72* (Princeton, NJ, 1971).
13. J. Tate, *Appendix to: The Milnor ring of a global field*, SLNM *342* (1973), 429–446.
14. ——— *Relations between K_2 and Galois cohomology*, Inventiones math. *36* (1976), 257–274.

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