Foldable fans, cscK surfaces and local K-moduli

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ABSTRACT

We study the moduli space of constant scalar curvature Kähler (cscK) surfaces around toric surfaces. To this end, we introduce the class of foldable surfaces: smooth toric surfaces whose lattice automorphism group contains a non-trivial cyclic subgroup. We classify such surfaces and show that they all admit a cscK metric. We then study the moduli space of polarised cscK surfaces around a point given by a foldable surface, and show that it is locally modelled on a finite quotient of a toric affine variety with terminal singularities.

1. Introduction

The construction of moduli spaces of varieties is a central problem in complex geometry. Pioneered by Riemann, the case of moduli spaces of curves is now fairly well understood. In higher dimensions, however, the situation becomes much more delicate. Indeed, iterated blow-ups (see e.g. [Kol13, Example 4.4]) or the presence of non-discrete automorphisms induce non-separatedness in moduli considerations. It is then remarkable that, despite their very simple combinatorial description, toric surfaces are subject to these two issues, and typically correspond to pathological points in the moduli space of surfaces. In this paper, we will show that when one restricts to constant scalar curvature Kähler (cscK) surfaces, the moduli space enjoys a very nice structure near its toric points.

Our motivation to restrict to cscK surfaces comes from the Yau–Tian–Donaldson conjecture (YTD conjecture, see [Yau93, Tia97, Don02]), which predicts that the existence of a cscK metric on a given polarised Kähler manifold should be equivalent to (a uniform version of) K-polystability. In the Fano case, the proof of this conjecture led to the recent construction of the moduli space of K-polystable Fano varieties by Odaka [Oda15] and Li, Wang and Xu [LWX18, LWX19]. See [Tia97, Ber16, CDS15, Tia15] for the YTD conjecture and also [Xu21, Part II] and the references therein for a historical survey on the related moduli space. Moreover, from Odaka's work [Oda12, Oda13], in the canonically polarised case K-stability is equivalent to having semi-log canonical singularities, a class that was successfully introduced to construct the so-called KSBA moduli space (see [Kol13] for a survey of the moduli of varieties of general

type). While a general algebraic construction of a moduli space for K-polystable varieties seems out of reach at the moment, on the differential geometric side Dervan and Naumann recently built a separated coarse moduli space for compact polarised cscK manifolds [DN20], extending Fujiki and Schumacher's construction that dealt with the case of discrete automorphism groups [FS90] (see also Inoue's work on the moduli space of Fano manifolds with Kähler–Ricci solitons [Ino19]).

However, it seems to the author that the Fano case and its log or pair analogues are the only sources of examples of those moduli spaces, when non–discrete automorphism groups are allowed (see also the discussion in [DN20, Example 4.16], which may lead to further examples via moduli of polystable bundles). Restricting to surfaces, moduli spaces of Kähler–Einstein del Pezzo surfaces were explicitly constructed in [OSS16]. Our focus will then be on the moduli space of polarised cscK surfaces, and its geometry around points corresponding to the toric points. Given that they satisfy the YTD conjecture, by Donaldson's work [Don02], toric surfaces attract a lot of attention, and they appear as natural candidates on which to test the general machinery (compare also with [KP21, Pet22]). As the full classification of cscK toric surfaces is yet to be carried out, we will consider a subclass whose fans carry further symmetries, which we introduce now.

DEFINITION 1.1. Let N be a rank two lattice and Σ a complete smooth fan in $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$. The fan Σ is called *foldable* if its lattice automorphism group $\operatorname{Aut}(N, \Sigma)$ contains a non-trivial cyclic subgroup. The associated toric surfaces will be called *foldable surfaces*.

In order to understand this class of surfaces amongst the toric ones, we show that all crystallographic groups arise as lattice automorphism groups of two dimensional smooth fans (see Section 3.1).

PROPOSITION 1.2. *Let N be a rank two lattice and Σ a complete smooth fan in $N_{\mathbb{R}}$. Then $\operatorname{Aut}(N, \Sigma)$ is isomorphic to one of the groups in the following set*

$$\{C_1, C_2, C_3, C_4, C_6, D_1, D_2, D_3, D_4, D_6\}.$$

Moreover, any group in the above list is isomorphic to the lattice automorphism group of some complete two dimensional smooth fan.

In Proposition 1.2, we denote the cyclic group of order p by C_p and the dihedral group of order $2p$ by D_p. We make a distinction between C_2 and D_1 by assuming that C_2 acts via $-\operatorname{Id}$ on N while D_1 acts through a reflection (see Section 3.1). Then, from Proposition 1.2, a complete and smooth two dimensional fan Σ in $N_{\mathbb{R}}$ is foldable if and only if $\operatorname{Aut}(N, \Sigma)$ is not isomorphic to C_1 or D_1. The associated class of toric surfaces then provides a wide class of examples of cscK surfaces (see Proposition 3.14).

PROPOSITION 1.3. *Let X be a foldable surface. Then X admits a cscK metric in some Kähler class.*

This result relies on the classification of foldable surfaces (Section 3.2) and an application of Arezzo, Pacard, Singer, and Székelyhidi's blow-up theorem for extremal Kähler metrics [APS11, Szé15]. Similar arguments were used in [Tip14] to show that any toric surface admits an iterated toric blow-up with a cscK metric.

REMARK 1.4. To our knowledge, all known examples of toric cscK surfaces are foldable (see e.g. [Tip14] for a family of examples with unbounded Picard rank, and [WZ11] for a complete

classification of polarised cscK toric surfaces up to Picard rank 4). It would be interesting to produce examples of non-foldable toric cscK surfaces.

Our main result shows that the moduli space of polarised cscK surfaces, which is known to be a complex space [FS90, DN20], is quite well behaved around a foldable surface, as it is locally modelled on a finite quotient of a toric terminal singularity (see Section 4.3 for a definition of this class of singularities, which originated in the MMP).

THEOREM 1.5. *Let* $(X, [\omega])$ *be a polarised cscK foldable surface with fan* Σ *in* $N_{\mathbb{R}}$, *and let* $G = \mathrm{Aut}(N, \Sigma)$. *Denote by* $[X] \in \mathscr{M}$ *the corresponding point in the moduli space* $\mathscr{M}$ *of polarised cscK surfaces. Then there are:*

(i) *a Gorenstein toric affine* G-*variety* W *with at worst terminal singularities, and*

(ii) *open neighbourhoods* $\mathscr{W}$ *and* $\mathscr{U}$ *of, respectively, (the image of) the torus fixed point* $x \in \mathscr{W} \subset W/G$ *and* $[X] \in \mathscr{U} \subset \mathscr{M}$,

such that $(\mathscr{U}, [X])$ *and* $(\mathscr{W}, x)$ *are isomorphic as pointed complex spaces.*

This result sheds some light on the singularities that may appear on the moduli space of polarised cscK surfaces. Note that by [BGLJ24, Theorem 4], the good moduli space of smooth K-polystable Fano manifolds of fixed dimension and volume has klt type singularities (see Section 4.3 for definitions). Also, a combination of Braun, Greb, Langlois and Moraga's work on GIT quotients of klt type singularities [BGLJ24, Theorem 1], together with the construction of Dervan and Naumann's moduli space for cscK polarised manifolds [DN20, Section 3], directly implies that the moduli space of polarised cscK surfaces has klt type singularities around its toric points (see Proposition 2.3). Hence Theorem 1.5 provides a refinement of Proposition 2.3, showing that, up to a finite covering, the toric structure is locally preserved on the moduli space, and that the singularities are terminal, at a foldable point. On the other hand, one might not expect such a nice structure at toric boundary points of a (still hypothetical) compactification $\overline{\mathscr{M}}$. Indeed, the local structure of the K-moduli space of Fano varieties was studied around singular toric varieties in [KP21, Pet22], where much wilder phenomena were observed, mainly due to the existence of obstructed deformations.

REMARK 1.6. It would be interesting to understand whether or not the finite quotient W/G in Theorem 1.5 is still terminal. Note, however, that it is a non-trivial problem to discover whether a finite quotient of a terminal singularity is terminal (see [KM98, Remark, p. 161]).

REMARK 1.7. Because of cohomology vanishings for toric surfaces, the germ of the moduli space of polarised cscK surfaces at a toric point does not depend on the chosen cscK polarisation. See, however, [ST22] for local wall-crossing type phenomena when the polarisation is pushed away from the cscK locus of the Kähler cone.

REMARK 1.8. Theorem 1.5 provides another instance in which canonical Kähler metrics, or K-polystability, become useful in moduli problems. We would like to mention two other approaches to producing moduli spaces for (or around) toric varieties. In [Mee19], the notion of an analytic stack is introduced to produce moduli spaces of integrable complex structures modulo diffeomorphisms. Those spaces carry more information, as they classify a wider class of varieties, but are typically non-separated. In another direction, turning from the classical setting to the non-commutative one, quantum toric varieties admit moduli spaces which are orbifolds; see [KLMV21].

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We proceed now to an overview of the proof of Theorem 1.5. If $(X, [\omega])$ is a cscK foldable surface, relying on [Szé10, DN20] and an observation in [RT14], the deformation theory of $(X, [\omega])$ is entirely encoded by the set of unobstructed polystable points in $H^1(X, TX)$ under the action of $\mathrm{Aut}(X)$, the latter being reductive by Matsushima and Lichnerowicz's theorem [Mat57, Lic58]. Nill's study of toric varieties with reductive automorphism groups [Nil06] then shows that either X is isomorphic to $\mathbb{CP}^2$ or $\mathbb{CP}^1 \times \mathbb{CP}^1$, in which case it is rigid, or $\mathrm{Aut}^0(X) \simeq T$, where T is the torus of X. Then, by Ilten's work [Ilt11], $H^2(X, TX) = 0$, so that, locally, the moduli space we are after is given by (a neighbourhood of the origin in) the GIT quotient $H^1(X, TX)//\mathrm{Aut}(X)$, where $\mathrm{Aut}(X) \simeq T \rtimes G$. Using [Ilt11] again, the deformation theory of toric surfaces is explicitly described in terms of their fan, and the affine toric variety $W := H^1(X, TX)//T$ in Theorem 1.5 can be explicitly computed. The proof of Theorem 1.5 then goes as follows. By a classification result (see Lemma 4.3), X can be obtained by successive C_p-equivariant blow-ups of some 'minimal' models in a list of six toric surfaces. We then use the combinatorial description of toric terminal singularities (see e.g. [CLS11, § 11.4]) to prove the result for surfaces in that list. Then, using convex geometry, we show that the desired properties for the local moduli space are preserved after the C_p-equivariant blow-ups, and hold around $[X]$.

REMARK 1.9. In Section 4.4, we produce an example of a toric surface X such that $\mathrm{Aut}(X) \simeq T$ and the quotient $H^1(X, TX)//T$ is not $\mathbb{Q}$-Gorenstein. We do not know whether this surface carries a cscK metric. If not, the expectation is that the theory should be extended to smooth pairs (X, D) where $D \subset X$ is a simple normal crossing divisor, considering singular or Poincaré type cscK metrics that are non-singular away from D, such as in [AAS21].

The paper is organised as follows. In Section 2, we settle the necessary background material on cscK manifolds. Section 3 is devoted to the classification of foldable surfaces and the proofs of Proposition 1.2 and Proposition 1.3. In Section 4, we carry over the construction of the local moduli spaces, and prove Theorem 1.5. Finally, in Section 5.1, we provide maps between the local moduli spaces and speculate about their Weil–Petersson geometry, while in Section 5.2 we discuss the higher dimensional case.

1.1 Notation

We will use the notation from [CLS11]. For a toric variety X, $\mathrm{Aut}(X)$ stands for its automorphism group, and $\mathrm{Aut}^0(X) \subset \mathrm{Aut}(X)$ the connected component of the identity. We denote by T the torus of X, and by N the lattice of its one-parameter subgroups, with dual lattice M and pairing $\langle m, u \rangle$, for $(m, u) \in M \times N$. The fan of X will be denoted by Σ (or Σ_X) and the letters τ, σ will be used for cones in Σ. For $0 \leqslant j \leqslant \dim(X)$, $\Sigma(j)$ is the set of j-dimensional cones in Σ. For $\mathbb{K} \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$, we let $N_{\mathbb{K}} := N \otimes_{\mathbb{Z}} \mathbb{K}$, and similarly $M_{\mathbb{K}} = M \otimes_{\mathbb{Z}} \mathbb{K}$. Finally, X_{Σ} may be used to refer to the toric variety associated to Σ.

2. Background on cscK metrics and their moduli

Let (X, Ω) be a compact Kähler manifold with Kähler class Ω. We will give a very brief overview of extremal metrics and their deformations, and refer the reader to [Gau, Szé14] for a more comprehensive treatment.

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2.1 Extremal Kähler metrics

Extremal Kähler metrics were introduced by Calabi [Cal21a] and provide canonical representatives of Kähler classes. They are defined as the critical points of the so-called Calabi functional, which assigns to each Kähler metric in Ω the L^2-norm of its scalar curvature. They include, as special cases of interest, cscK metrics and thus Kähler–Einstein metrics. The following obstruction to the existence of a cscK metric is due to Matsushima and Lichnerowicz [Mat57, Lic58].

THEOREM 2.1 [Mat57, Lic58]. *Assume that X carries a cscK metric. Then the automorphism group of X is reductive.*

Later on, Futaki discovered another obstruction to the existence of a cscK metric on (X, Ω): the vanishing of the so-called Futaki invariant [Fut83, Fut88]. Moreover, from [Cal21b], an extremal Kähler metric in the Kähler class Ω is cscK precisely when the Futaki invariant Fut_Ω vanishes. Together with Arezzo, Pacard, Singer, and Székelyhidi's results on blow-ups of extremal Kähler metrics, we deduce the following existence result for toric surfaces.

THEOREM 2.2 [APS11, Szé15]. *Let (X, Ω) be a smooth compact polarised toric surface with cscK metric $\omega \in \Omega$. Let $Z \subset X$ be a finite set of torus fixed points, and $G \subset \mathrm{Aut}(X)$ a finite subgroup such that:*

(i) *the set Z is G-invariant;*

(ii) *the class Ω is G-invariant; and*

(iii) *the adjoint action of G on $\mathrm{Lie}(\mathrm{Aut}(X))$ has no fixed point but zero.*

Then, there is $\varepsilon_0 > 0$ such that $(\mathrm{Bl}_Z(X), \Omega_\varepsilon)$ carries a cscK metric in the class $\Omega_\varepsilon := \pi^*\Omega - \varepsilon c_1(E)$ for $\varepsilon \in (0, \varepsilon_0)$, where $\pi : \mathrm{Bl}_Z(X) \to X$ stands for the blow-down map and $E = \sum_{z \in Z} E_z$ is the exceptional locus of π.

Proof. From [APS11, Szé15], there is ε_0 such that $(\mathrm{Bl}_Z(X), \Omega_\varepsilon)$ admits an extremal Kähler metric for $\varepsilon \in (0, \varepsilon_0)$. We only need to show the vanishing of the Futaki invariant $\mathrm{Fut}_{\Omega_\varepsilon}$. This will follow from its equivariance, as already used in [ST24, Proposition 2.1]. Define $\tilde{X} = \mathrm{Bl}_Z(X)$. By (i) and (ii), G lifts to a subgroup of $\mathrm{Aut}(\tilde{X})$ such that Ω_ε is G-invariant. Then, by equivariance of $\mathrm{Fut}_{\Omega_\varepsilon} \in (\mathrm{Lie}(\mathrm{Aut}(\tilde{X})))^*$ (see e.g. [Fut88, Chapter 3] or [LS94, § 3.1]), we have, for all $g \in G$ and $v \in (\mathrm{Lie}(\mathrm{Aut}(\tilde{X})))$,

$$\mathrm{Fut}_{\Omega_\varepsilon}(\mathrm{Ad}_g(v)) = \mathrm{Fut}_{\Omega_\varepsilon}(v).$$

Let $\tilde{w} \in \mathrm{Lie}(\mathrm{Aut}(\tilde{X}))$. Then $\tilde{w}$ is the lift of an element $w \in \mathrm{Lie}(\mathrm{Aut}(X))$ that vanishes on Z. Moreover,

$$\tilde{w}_0 := \sum_{g \in G} \mathrm{Ad}_g(\tilde{w})$$

is a fixed point under the adjoint action of G on $\mathrm{Lie}(\mathrm{Aut}(\tilde{X}))$. As Z is G-invariant, we deduce that it is the lift of a fixed point $w_0 \in \mathrm{Lie}(\mathrm{Aut}(X))$ under the adjoint action of G. From (iii), we deduce that $\tilde{w}_0 = 0$. Hence, by equivariance of $\mathrm{Fut}_{\Omega_\varepsilon}$,

$$0 = \mathrm{Fut}_{\Omega_\varepsilon}(\tilde{w}_0) = \sum_{g \in G} \mathrm{Fut}_{\Omega_\varepsilon}(\mathrm{Ad}_g(\tilde{w})) = |G|\, \mathrm{Fut}_{\Omega_\varepsilon}(\tilde{w})$$

where $|G|$ is the cardinal of G. This concludes the proof. $\square$

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2.2 Local moduli of polarised cscK manifolds

A moduli space $\mathscr{M}$ for polarised cscK manifolds was constructed in [DN20], generalising the results in [FS90] by taking care of non-discrete automorphisms by means of [Szé10, Ino19]. This space is Hausdorff and is endowed with a complex space structure. It is a coarse moduli space in the following sense: its points are in bijective correspondence with isomorphism classes of polarised cscK manifolds and, for any complex analytic family $(\mathscr{X}, \mathscr{L}) \to \mathscr{S}$ of polarised cscK manifolds over some reduced analytic space $\mathscr{S}$ ($\mathscr{L}$ stands for the relative polarisation of $\mathscr{X} \to \mathscr{S}$), there is an induced map $\mathscr{S} \to \mathscr{M}$. In what follows, we will restrict to the connected components corresponding to the moduli space of polarised cscK surfaces, still denoted by $\mathscr{M}$. Roughly, $\mathscr{M}$ is constructed by patching together local moduli spaces $(\mathscr{M}_X)_{X \in \mathfrak{S}}$ parameterised by $\mathfrak{S}$, the set of polarised cscK surfaces, each $\mathscr{M}_X$ being obtained as some open set in an analytic GIT quotient $W_X /\!/ G$. Our case of interest is when X is a cscK toric surface. The first cohomology group of the tangent sheaf, namely $H^1(X, TX)$, plays a central role in our study. Indeed, it naturally arises as the space of infinitesimal deformations for X, and will play the role of the tangent space at X to the base of a semi-universal family of deformations of X (see e.g. [Kod05, Chapter 4]). The tangent bundle TX is naturally $\mathrm{Aut}(X)$-equivariant, meaning that the $\mathrm{Aut}(X)$-action lifts to an action on TX in such a way that the projection map $TX \to X$ is equivariant (the lift is given by the differential of the action). Using pullbacks, we see that the bundles $\Lambda^{p,q}TX^*$ are also equivariant. Then, $\Lambda^{p,q}TX^* \otimes TX$ is equivariant, from which we deduce that $\mathrm{Aut}(X)$ acts naturally on TX-valued (p,q)-forms. More explicitly, for any $\mathrm{Aut}(X)$-equivariant bundle $E \to X$, we deduce an action of $\mathrm{Aut}(X)$ on the sections of E by setting, for $s \in \Gamma(E)$, $g \in \mathrm{Aut}(X)$ and $x \in X$:

$$(g \cdot s)(x) := g(s(g^{-1}x)).$$

From holomorphicity of the action, we deduce that the $\mathrm{Aut}(X)$-action on TX-valued $(0,p)$-forms commutes with the Dolbeault operator, and hence induces a representation of $\mathrm{Aut}(X)$ on

$$H^1(X, TX) \simeq \frac{\ker\left(\overline{\partial} : \Omega^{0,1}(TX) \to \Omega^{0,2}(TX)\right)}{\mathrm{Im}\left(\overline{\partial} : \Omega^{0,0}(TX) \to \Omega^{0,1}(TX)\right)}.$$

In a more formal way, we could say that the Dolbeault resolution of the holomorphic tangent bundle is actually a resolution by injectives in the category of $\mathrm{Aut}(X)$-equivariant sheaves, and thus that the right derived functors of the global sections functor Γ, namely the cohomology groups $H^\bullet(X, TX)$, inherit an $\mathrm{Aut}(X)$-structure. We then extract from [DN20] the following proposition.

PROPOSITION 2.3. *Let (X, Ω) be a polarised cscK toric surface. Then the local moduli space $\mathscr{M}_X$ is given by an open neighbourhood of $\pi(0)$ in the GIT quotient*

$$\pi : H^1(X, TX) \to H^1(X, TX)/\!/\mathrm{Aut}(X).$$

In particular, its singularities are of klt type.

The above GIT quotient in this affine setting is given by the good categorical quotient that we now recall. Let R be the ring of regular functions on $H^1(X, TX)$. The $\mathrm{Aut}(X)$-action on $H^1(X, TX)$ induces an $\mathrm{Aut}(X)$-module structure on R, given by

$$g \cdot f := f(g \cdot)$$

for any $g \in \mathrm{Aut}(X)$ and $f \in R$. We then introduce $R^{\mathrm{Aut}(X)}$ as the set of $\mathrm{Aut}(X)$-invariant elements. By a classical result of Hilbert and Nagata, $R^{\mathrm{Aut}(X)}$ is finitely generated as soon as

$\mathrm{Aut}(X)$ is reductive. In that case, the GIT quotient is given by

$$H^1(X, TX)//\mathrm{Aut}(X) := \mathrm{Spec}(R^{\mathrm{Aut}(X)}),$$

with quotient map

$$\pi : \mathrm{Spec}(R) \to \mathrm{Spec}(R^{\mathrm{Aut}(X)})$$

corresponding to the inclusion of the finitely generated algebras

$$R^{\mathrm{Aut}(X)} \to R.$$

We will then say that a point in $H^1(X, TX)$ is polystable if its $\mathrm{Aut}(X)$-orbit is closed and that it is stable if, in addition, its stabiliser is discrete. We refer the reader to [CLS11, § 5.0] for a short account of good categorical quotients and to [CLS11, § 14.1] for the special case of toric GIT quotients. Finally, we refer to Section 4.3 for the definition of klt type singularities.

Proof of Proposition 2.3. The tangent space at X, denoted $\tilde{H}^1(TX)$, of the base of a semi-universal family of complex deformations of X compatible with the polarisation Ω is constructed in [CS14, Lemma 6.1] (see also [Szé10]). In general, the construction of $\tilde{H}^1(TX)$ goes as follows. First, the usual Kuranishi complex for deformations of complex structures on X is

$$\cdots \to \Omega^{0,k}(TX) \xrightarrow{\overline{\partial}} \Omega^{0,k+1}(TX) \to \cdots,$$

where the extension of $\overline{\partial}$ to $(0, p)$-forms is given in local coordinates, for $\beta = \sum_j \beta_j \otimes \frac{\partial}{\partial z^j}$, by

$$\overline{\partial}\beta = \sum_j \overline{\partial}\beta_j \otimes \frac{\partial}{\partial z^j}.$$

We are interested in deformations that are compatible with some cscK metric $\omega \in \Omega$. Following [FS90], we define maps

$$\iota_\bullet^k : \Omega^{0,k}(TX) \to \Omega^{0,k+1}$$
$$\beta \quad \mapsto \quad \iota_\beta \omega$$

where $\iota_\beta \omega$ is obtained by the composition of first contraction and then alternation operators. For $k \geqslant 1$, we then set

$$\Omega_\omega^{0,k}(TX) := \ker \iota_\bullet^k,$$

while we define

$$\Omega_\omega^{0,0}(TX) := \mathscr{C}^\infty(X, \mathbb{C}).$$

Together with the restriction of $\overline{\partial}$, and the map $\overline{\partial}^0 := \mathscr{D}$ defined, for $f \in \mathscr{C}^\infty(X, \mathbb{C})$, by

$$\mathscr{D}(f) := \overline{\partial}(\nabla_\omega^{1,0} f),$$

where $\nabla_\omega^{1,0} f$ is given by $\overline{\partial} f = \omega(\nabla_\omega^{1,0} f, \cdot)$, we obtain another elliptic complex $(\Omega_\omega^{0,\bullet}(TX), \overline{\partial}^\bullet)$. We denote by $\tilde{H}^\bullet(TX)$ the associated cohomology groups. Then, from this complex, following Kuranishi's techniques, one can build a semi-universal family $(\mathscr{X}, \mathscr{L}) \to Z$ of polarised deformations of (X, Ω), where $Z \subset B \subset \tilde{H}^1(TX)$ is an analytic subspace (corresponding to integrable infinitesimal deformations) of some open ball B centred at zero (again, we refer to [CS14, Lemma 6.1] for the detailed construction). As in [Doa22], setting $K := \mathrm{Aut}(X, \omega)$, the group of holomorphic isometries of (X, ω), this construction can actually be made K-equivariantly, and even locally $K^{\mathbb{C}}$-equivariantly on Z, for the natural $K^{\mathbb{C}}$-action on $\tilde{H}^1(TX)$.

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From [DN20, § 3], $\mathscr{M}_X$ is given by an open neighbourhood of 0 in the analytic GIT quotient $W_X /\!/ G$, where $G = K^{\mathbb{C}}$, and where $W_X \subset \tilde{H}^1(TX)$ is the smallest G-invariant Stein space containing Z. As noted in [RT14, Lemma 2.10], as X is toric, by Bott, Steenbrink and Danilov's vanishing of $h^{0,1}(X)$ and $h^{0,2}(X)$ (see [CLS11, Theorem 9.3.2]), the vector space $\tilde{H}^1(TX)$ is G-equivariantly isomorphic to $H^1(X, TX)$. Also, for a toric surface, $\mathrm{Aut}(X)$ is a linear algebraic group, so that $G \simeq \mathrm{Aut}(X)$, as the Lie algebra of $\mathrm{Aut}(X)$ has no parallel vector field (see [Gau, Chapter 2]). Finally, $H^2(X, TX) = 0$ by [Ilt11, Corollary 1.5], so that any small element of $H^1(X, TX)$ is integrable, and $Z = B \subset H^1(X, TX)$. Hence $W_X = H^1(X, TX)$, and putting all this together, we obtain the first part of Proposition 2.3. The statement about klt type singularities follows directly from [BGLJ24, Theorem 1], which asserts that the GIT quotient of a klt type singularity is of klt type, and which implies in particular that the GIT quotient of $H^1(X, TX)$ (which is smooth) by $\mathrm{Aut}(X)$ is of klt type. $\qquad\square$

3. Foldable toric surfaces

Let N be a rank two lattice and Σ be a fan in $N_{\mathbb{R}}$ with associated toric surface X. We will assume Σ to be *smooth*, that is, that each cone $\sigma \in \Sigma$ is generated by elements in N that form part of a $\mathbb{Z}$-basis, and to be *complete*, i.e.

$$\bigcup_{\sigma \in \Sigma} \sigma = N_{\mathbb{R}}.$$

Hence, X is a smooth and compact toric surface.

3.1 Lattice automorphisms and fans

We will be interested in automorphism groups of smooth toric surfaces. Denote by $T := N \otimes_{\mathbb{Z}} \mathbb{C}^*$ the torus of X, and by $\mathrm{Aut}(N, \Sigma)$ the group of *lattice automorphisms* of Σ. Recall that $\mathrm{Aut}(N, \Sigma)$ is the subgroup of $\mathrm{GL}_{\mathbb{Z}}(N)$ consisting of elements $g \in \mathrm{GL}_{\mathbb{Z}}(N)$ such that the induced isomorphisms $g \in \mathrm{GL}_{\mathbb{R}}(N_{\mathbb{R}})$ send Σ bijectively to Σ. By a result of Demazure [Dem70, Proposition 11], the automorphism group $\mathrm{Aut}(X)$ of X is a linear algebraic group isomorphic to

$$\mathrm{Aut}^0(X) \rtimes G,$$

where G is a quotient of $\mathrm{Aut}(N, \Sigma)$. We will later on be interested in the GIT quotient of $H^1(X, TX)$ by $\mathrm{Aut}(X)$, assuming X to admit a cscK metric. A combination of Matsushima and Lichnerowicz's obstruction together with Nill's work on toric varieties with reductive automorphism group [Nil06] implies the following proposition.

PROPOSITION 3.1. *Assume that X is a toric surface that admits a cscK metric. Denote by Σ its fan and by N its lattice of one-parameter subgroups. Then either*

$$X \in \{\mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1\},$$

or

$$\mathrm{Aut}(X) \simeq T \rtimes \mathrm{Aut}(N, \Sigma).$$

Proof. By Matsushima and Lichnerowicz's theorem [Mat57, Lic58], $\mathrm{Aut}(X)$ is reductive. The result then follows from Nill's result [Nil06, Theorem 1.8] together with Demazure's structure theorem [Dem70, Proposition 11, p. 581], which we now recall. Demazure's root system for (N, Σ), which we denote here by $\mathscr{R}$ (it corresponds to $\mathrm{Rac}(\Sigma)$ in Demazure's notation [Dem70]),

is the set

$$\mathscr{R} := \{m \in M \mid \exists \rho \in \Sigma(1) \text{ with } \langle m, u_\rho \rangle = 1 \text{ but } \forall \rho' \in \Sigma(1), \ \rho' \neq \rho, \ \langle m, u_{\rho'} \rangle \leqslant 0\},$$

where we use the notation $u_\rho \in N$ for the primitive generator of the ray ρ. Demazure's structure theorem then asserts that the identity component $\operatorname{Aut}(X)^0$ of $\operatorname{Aut}(X)$ is generated by T and some unipotent elements $\{U_m, m \in \mathscr{R}\}$. In particular,

$$\dim(\operatorname{Aut}(X)) = \dim(T) + |\mathscr{R}|.$$

Then, the quotient $\operatorname{Aut}(X)/\operatorname{Aut}(X)^0$ is finite and isomorphic to a quotient of $\operatorname{Aut}(N, \Sigma)$ by a subgroup $W(N, \Sigma) \subset \operatorname{Aut}(N, \Sigma)$ that is the Weyl group of a maximal reductive subgroup of $\operatorname{Aut}(X)$ with root system $\mathscr{R} \cap -\mathscr{R}$. Then Nill's theorem [Nil06, Theorem 1.8] gives an upper bound for the dimension of the automorphism group of a smooth complete toric variety, when it is reductive. In the case of a smooth toric surface X with reductive automorphism group, the outcome is that either X is a product of projective spaces or $\dim(\operatorname{Aut}(X)) = 2$. In the latter case, from the previous discussion on Demazure's result, this implies that the set of Demazure's roots $\mathscr{R}$ for (N, Σ) is empty. Hence, $\operatorname{Aut}(X)^0 = T$ and $\operatorname{Aut}(X)/\operatorname{Aut}(X)^0 \simeq \operatorname{Aut}(N, \Sigma)$, which concludes the proof. $\qquad\square$

As $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ are rigid, we will now focus on the case

$$\operatorname{Aut}(X) \simeq T \rtimes \operatorname{Aut}(N, \Sigma),$$

and characterise the possible finite groups that arise as lattice automorphism groups of rank two complete fans.

LEMMA 3.2. *Let $g \in \operatorname{Aut}(N, \Sigma)$, and denote by $m \in \mathbb{N}$ the order of g. Then*

$$m \in \{1, 2, 3, 4, 6\}.$$

Moreover, the complex linear extension of g to $N_{\mathbb{C}}$ is conjugated (in $\operatorname{GL}(N_{\mathbb{C}}) \simeq \operatorname{GL}_2(\mathbb{C})$) to the following:

(i) *if $m = 1$, then $g \sim \operatorname{Id}$,*

(ii) *if $m = 2$ and $\det(g) = 1$, then $g \sim -\operatorname{Id}$,*

(iii) *if $m = 2$ and $\det(g) = -1$, then $g \sim \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$,*

(iv) *if $m = 3$, then $g \sim \begin{bmatrix} j & 0 \\ 0 & j^2 \end{bmatrix}$,*

(v) *if $m = 4$, then $g \sim \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, and*

(vi) *if $m = 6$, then $g \sim \begin{bmatrix} -j & 0 \\ 0 & -j^2 \end{bmatrix}$,*

where we write $j = e^{i\frac{2\pi}{3}}$ and denote by $\sim$ the equivalence relation given by conjugation, and where we use an isomorphism $N_{\mathbb{C}} \simeq \mathbb{C}^2$.

The proof is an elementary exercise in linear algebra. We include it for convenience of the reader.

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Proof. Fix an isomorphism $N \simeq \mathbb{Z}^2$ and identify g with an element of $\mathrm{GL}_2(\mathbb{Z})$. The characteristic polynomial of

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then reads

$$\chi(Y) = Y^2 - (a+d)Y + (ad - bc).$$

On the other hand, Jordan's normal form for the $\mathbb{C}$-linear extension of g implies that it is conjugated (in $\mathrm{M}_2(\mathbb{C})$) to

$$\begin{bmatrix} \alpha & \delta \\ 0 & \beta \end{bmatrix}$$

for $(\alpha, \beta) \in \mathbb{C}^2$ and $\delta \in \{0, 1\}$. As g is both invertible and of finite order, we deduce that $\delta = 0$ and g is conjugated to

$$\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}.$$

Moreover, $\alpha^m = \beta^m = 1$, so that α and β are m-th roots of unity in $\mathbb{C}$. On the other hand, by conjugation invariance, we also have

$$\det(g) = \alpha\beta = ad - bc \in \{-1, +1\}$$

as $g \in \mathrm{GL}_2(\mathbb{Z})$ and

$$\mathrm{trace}(g) = \alpha + \beta = a + d \in \mathbb{Z}.$$

If α and β belong to $\{-1, +1\}$, that is if $m = 2$, then we are done. If not, at least one of α or β is not real, and thus α and β must be complex conjugated roots of $\chi(Y)$. Hence

$$\mathrm{trace}(g) = \alpha + \overline{\alpha} \in \mathbb{Z},$$

but also

$$|\alpha + \overline{\alpha}| < 2$$

so that

$$\alpha + \overline{\alpha} \in \{-1, 0, 1\}.$$

Hence

$$\alpha \in \{\pm i, \pm j, \pm j^2\},$$

from which the result follows easily. □

Recall that we denote by C_p the cyclic group of order p and by D_p the dihedral group of order p. We then have the following proposition.

PROPOSITION 3.3. *The group* $\mathrm{Aut}(N, \Sigma)$ *is isomorphic to one of the groups in the following set*

$$\{C_1, C_2, C_3, C_4, C_6, D_1, D_2, D_3, D_4, D_6\}.$$

Proof. As $\mathrm{Aut}(N, \Sigma)$ is finite, we can fix an $\mathrm{Aut}(N, \Sigma)$-invariant euclidean metric on $N_{\mathbb{R}}$ (again considering the $\mathbb{R}$-linear extension of $\mathrm{Aut}(N, \Sigma)$). Then, by the classification of finite subgroups of the group of orthogonal transformations of the plane, we deduce that $\mathrm{Aut}(N, \Sigma)$ is isomorphic
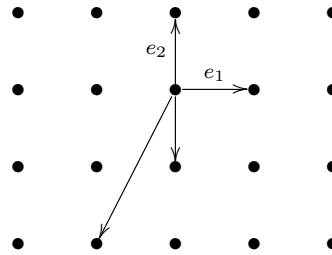
Figure 1. Fan Σ'_1 of $\mathbb{F}_2$.
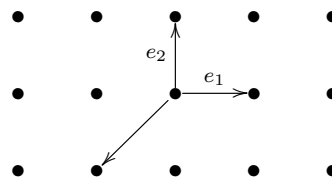
Figure 2. Fan Σ'_3 of $\mathbb{P}_2$.

to C_p or D_p, for some $p \in \mathbb{N}^*$. As those groups admit elements of order p, by Lemma 3.2, the result follows. □

REMARK 3.4. This proposition simply recovers the well-known classification of crystallographic groups in dimension 2.

We will now show that all the groups listed above arise as lattice automorphism groups of rank two complete smooth fans. Note that C_2 and D_1 will be distinguished by the fact that their generator is $-\mathrm{Id}$ in the first case and a reflection in the second case. Consider then the following fans in $\mathbb{Z}^2$, where we denote by $\{e_1, e_2\}$ the standard basis of $\mathbb{Z}^2$.

Example 3.5. Let $\mathbb{F}_n$ be the nth Hirzebruch surface, that is, the total space of the fibration

$$\mathbb{P}(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(n)) \to \mathbb{CP}^1.$$

Note that $\mathbb{F}_0 = \mathbb{CP}^1 \times \mathbb{CP}^1$. Then, up to isomorphism, the fan $\Sigma_{\mathbb{F}_n}$ of $\mathbb{F}_n$ is described by

$$\Sigma_{\mathbb{F}_n}(1) = \{\mathbb{R}_+ \cdot (0,-1),\ \mathbb{R}_+ \cdot (1,0),\ \mathbb{R}_+ \cdot (0,1),\ \mathbb{R}_+ \cdot (-1,-n)\}.$$

As an example, $\mathbb{F}_2$ admits the fan description of Figure 1, which we will denote by Σ'_1. In this figure, and those that follow, the dots represent the lattice points, and we only represent the ray generators of the complete two dimensional fan.

Example 3.6. In Figures 2 and 3, respectively, the fans $\Sigma_{\mathbb{P}^2} = \Sigma'_3$ and $\Sigma_{\mathbb{P}^1 \times \mathbb{P}^1} = \Sigma'_4$ of $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ are depicted (the numbering of the fans is motivated by Proposition 3.9 below).

Example 3.7. Recall that blowing up a smooth toric surface along a torus fixed point produces a new smooth toric surface. By the orbit cone correspondence, such a fixed point corresponds to a two dimensional cone

$$\sigma = \mathbb{R}_+ \cdot e_i + \mathbb{R}_+ \cdot e_{i+1}$$

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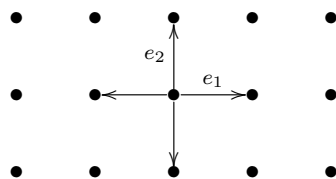
FIGURE 3. Fan Σ_4' of $\mathbb{CP}^1 \times \mathbb{CP}^1$.
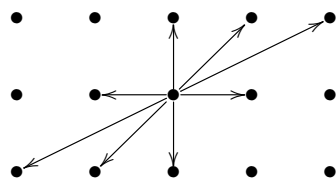
FIGURE 4. Fan Σ_2': iterated blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$.
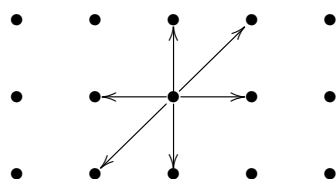
FIGURE 5. Fan Σ_6': blow-up of $\mathbb{P}^2$ along its three fixed points.

in the fan of the blown-up surface, and the fan of the resulting surface is obtained by adding the ray generated by $e_i + e_{i+1}$ to the set of rays of the initial surface (see [CLS11, Chapter 3, §3]). By iterated blow-ups, we obtain the following fans. Figure 4 represents an iterated blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$. The following Figure 5 is a two points blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ and, at the same time, a three points blow-up of $\mathbb{CP}^2$.

Example 3.8. Here are further examples to complete our classification of lattice automorphism groups of rank two complete smooth fans (Figures 6, 7, 8, 9 and 10). The first is a single blow-up of $\mathbb{F}_2$, which cancels the symmetry of $\Sigma_1' = \Sigma_{\mathbb{F}_2}$ (Figure 6). The next (Figure 7) is obtained from Σ_2' by blowing up in a C_2-equivariant way (recall that $C_2 = \langle -\mathrm{Id} \rangle$). The third example is obtained from the fan of $\mathbb{P}^2$ by three successive blow-ups of three distinct fixed points, in a symmetric way (Figure 8). The next example is obtained by two successive blow-ups of $\mathbb{P}^1 \times \mathbb{P}^1$ at four fixed points (Figure 9). The last is obtained from Σ_6' by two successive blow-ups at six fixed points (Figure 10).
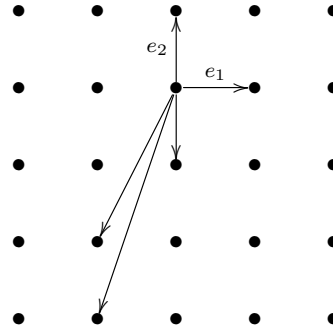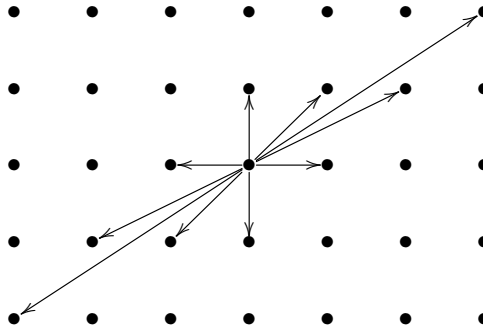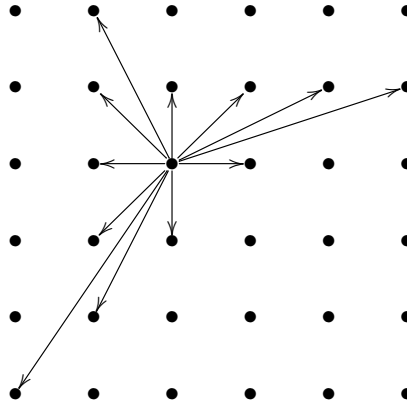
PROPOSITION 3.9. *Let $p \in \{1, 2, 3, 4, 6\}$. Then we have*

$$\mathrm{Aut}(\mathbb{Z}^2, \Sigma_p) \simeq C_p$$

and

$$\mathrm{Aut}(\mathbb{Z}^2, \Sigma_p') \simeq D_p.$$

Proof. The proof is straightforward, although a bit tedious, so we will only give a sketch of it. Fix an orientation of $\mathbb{R}^2$, and pick a two dimensional cone σ of Σ_p (the proof is the same for Σ_p').

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FIGURE 6. Fan Σ_1: one-point blow-up of $\mathbb{F}_2$.

FIGURE 7. Fan Σ_2.

FIGURE 8. Fan Σ_3.

Let $g \in \operatorname{Aut}(\mathbb{Z}^2, \Sigma_p)$. Then $g \cdot \sigma \in \Sigma_p(2)$. Moreover, g maps the two facets of σ to the two facets of $g \cdot \sigma$. As the fan Σ_p is explicitly described, for each possible $\tau = g \cdot \sigma \in \Sigma_p(2)$, we deduce the possible matrix coefficients for g,

$$g = \begin{bmatrix} a_\tau & b_\tau \\ c_\tau & d_\tau \end{bmatrix} \in \operatorname{GL}_2(\mathbb{Z}).$$

For any possible τ, one can then explicitly compute the images of the rays of Σ_p by g. They should all belong to $\Sigma_p(1)$, and this enables us to select the allowed cones τ for g to be a lattice automorphism of Σ_p. The result then follows easily. □

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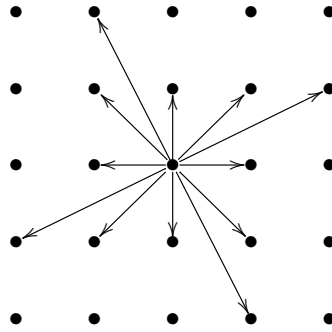
FIGURE 9. Fan Σ_4.
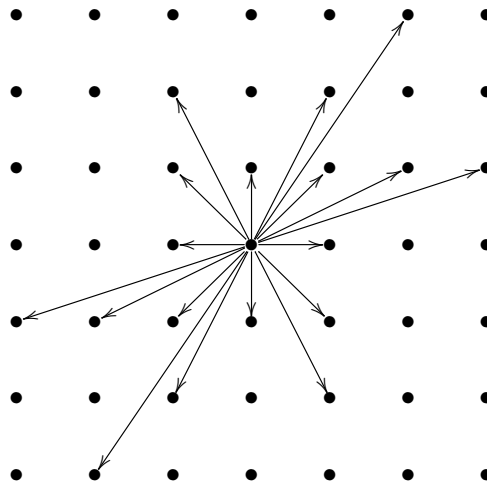
FIGURE 10. Fan Σ_6.

REMARK 3.10. One can show similarly that for any Hirzebruch surface $\mathbb{F}_n$ with $n \geqslant 1$ and with fan $\Sigma_{\mathbb{F}_n}$, one has $\operatorname{Aut}(N, \Sigma_{\mathbb{F}_n}) \simeq D_1$.

3.2 Classification of foldable surfaces

We first recall the following definition.

DEFINITION 3.11. We say that a smooth and complete two dimensional fan is *foldable* if its lattice automorphism group contains a non-trivial cyclic group. A *foldable surface* is a toric surface whose fan is foldable.

We now proceed to a partial classification of foldable toric surfaces. Notice that if $G \subset \operatorname{Aut}(N, \Sigma)$, then we can find a G-action on X that is faithful as soon as $\operatorname{Aut}(X) \simeq T \rtimes \operatorname{Aut}(N, \Sigma)$.

DEFINITION 3.12. A G-equivariant blow-up of X is a blow-up of X along a G-orbit of torus fixed points. We say that a smooth toric surface $\tilde{X}$ is obtained from X by successive G-equivariant blow-ups if there is a sequence of G-equivariant blow-ups:

$$\tilde{X} = X_k \to X_{k-1} \to \cdots \to X_1 \to X_0 = X.$$

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Denote by $\mathrm{Bl}_{p_1,p_2,p_3}(\mathbb{P}^2)$ the toric surface associated to the fan of Figure 5, that is, the blow-up of $\mathbb{P}^2$ at its three torus fixed points.

PROPOSITION 3.13. *Let X be a foldable surface with fan Σ. Assume that $\mathrm{Aut}(N,\Sigma)$ contains a subgroup isomorphic to C_p, $p \geqslant 2$.*

(i) *If $p \in \{2,4\}$, then X is obtained from $\mathbb{P}^1 \times \mathbb{P}^1$ by successive C_p-equivariant blow-ups.*

(ii) *If $p = 3$, then X is obtained from $\mathbb{P}^2$ by successive C_p-equivariant blow-ups.*

(iii) *If $p = 6$, then X is obtained from $\mathrm{Bl}_{p_1,p_2,p_3}(\mathbb{P}^2)$ by successive C_p-equivariant blow-ups.*

Proof. Through its representation

$$C_p \to \mathrm{GL}_\mathbb{R}(N_\mathbb{R}),$$

the subgroup C_p of $\mathrm{Aut}(N,\Sigma)$ acts on $N_\mathbb{R}$ by rotations (once an invariant euclidean metric is fixed), and hence has no fixed points. We deduce that the action of C_p on $\Sigma(1)$ is free, and then p divides $|\Sigma(1)|$, the number of rays of Σ.

On the other hand, from the classification of toric surfaces (see e.g. [CLS11, Chapter 10]), we must have $|\Sigma(1)| \geqslant 3$. Moreover, if $|\Sigma(1)| = 3$, then $X \simeq \mathbb{P}^2$, while if $|\Sigma(1)| = 4$, then $X \simeq \mathbb{F}_n$ for some $n \in \mathbb{N}$. By Remark 3.10, we see that $X \simeq \mathbb{P}^1 \times \mathbb{P}^1$ in that case.

So we may assume that $|\Sigma(1)| \geqslant 5$, and that p divides $|\Sigma(1)|$. By the classification of toric surfaces, we know that X is obtained by successive blow-ups from $\mathbb{F}_n$, $n \geqslant 0$, or from $\mathbb{P}^2$. Then, there is a ray ρ_i in $\Sigma(1)$ generated by the sum of two primitive elements

$$u_i = u_{i-1} + u_{i+1}$$

generating adjacent rays ρ_{i-1} and ρ_{i+1} in $\Sigma(1)$ (that is both $\rho_i + \rho_{i-1}$ and $\rho_i + \rho_{i+1}$ belong to $\sigma(2)$). Note that

$$\det(u_{i+1}, u_{i-1}) = \det(u_i - u_{i-1}, u_{i-1}) = \det(u_i, u_{i-1}) = 1$$

so the contraction to a point of the divisor associated to the ray ρ_i is smooth. By linearity, for any $g \in C_p$ we have

$$g \cdot u_i = g \cdot u_{i-1} + g \cdot u_{i+1},$$

and hence we can contract to points the C_p-orbit of the torus invariant divisor D_{ρ_i} associated to ρ_i and obtain a smooth toric surface whose fan still admits C_p as a subgroup of its lattice symmetry group. By an inductive argument on $|\Sigma(1)|$, we obtain the result. $\qquad\square$

From the classification, we obtain the following proposition.

PROPOSITION 3.14. *Let X be a foldable surface. Then X admits a cscK metric.*

Proof. This is simply an induction on the number of C_p-equivariant blow-ups in Proposition 3.13, using Theorem 2.2 at each step, and the fact that $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^2$ and $\mathrm{Bl}_{p_1,p_2,p_3}(\mathbb{P}^2)$ admit cscK metrics in C_p-equivariant classes (for $\mathrm{Bl}_{p_1,p_2,p_3}(\mathbb{P}^2)$, this follows again from Theorem 2.2 applied to $\mathbb{P}^2$). $\qquad\square$

4. The local cscK moduli around a foldable surface

Let X be a smooth and complete toric surface with fan Σ and one-parameter subgroups lattice N. We will denote by $(u_i)_{0 \leqslant i \leqslant r}$ the primitive elements of N that generate the rays

$$\rho_i = \mathbb{R}_+ \cdot u_i \in \Sigma(1),$$

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labelled in counterclockwise order, so that, for each i, (u_i, u_{i+1}) is a positively oriented $\mathbb{Z}$-basis of N, and $\rho_i + \rho_{i+1} \in \Sigma(2)$, with the convention $u_0 = u_r$.

4.1 Deformation theory of toric surfaces

We recall here some results from [Ilt11] regarding the deformation theory of toric surfaces. As X is toric, dualising the generalised Euler sequence yields (see e.g. [CLS11, Theorem 8.1.6])

$$0 \to \mathrm{Pic}(X)^* \otimes_{\mathbb{Z}} \mathscr{O}_X \to \bigoplus_{i=1}^{r} \mathscr{O}_X(D_{\rho_i}) \to TX \to 0,$$

where $\mathrm{Pic}(X)$ stands for the Picard group of X and D_{ρ_i} is the torus invariant divisor associated to the ray $\rho_i \in \Sigma(1)$. This sequence is actually a short exact sequence of $\mathrm{Aut}(X)$-equivariant bundles, and induces a long exact sequence of $\mathrm{Aut}(X)$-modules in cohomology. From a classical vanishing theorem (see e.g. [CLS11, Theorem 9.3.2]), we have $H^1(X, \mathscr{O}_X) = 0$ and $H^2(X, \mathscr{O}_X) = 0$, and this long exact sequence induces an isomorphism of $\mathrm{Aut}(X)$-modules:

$$\bigoplus_{i=1}^{r} H^1(X, \mathscr{O}_X(D_{\rho_i})) \simeq H^1(X, TX). \tag{1}$$

In particular, the action of the torus T on X induces a representation of T on

$$V := H^1(X, TX)$$

compatible with the above isomorphism. The complete reducibility theorem for torus actions (see [CLS11, Proposition 1.1.2] and the references therein) provides a weight space decomposition that we will denote by

$$V = \bigoplus_{m \in M} V_m.$$

More precisely, V splits into T-invariant subspaces $V_m \subset V$ where the T-action on V_m is given by

$$t \cdot x = \chi^m(t)\, x$$

for $t \in T$, $x \in V_m$, and where $\chi^m : T \to \mathbb{C}^*$ is the character associated to the weight $m \in M$. Using the isomorphism (1), Ilten obtained, for each weight $m \in M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ ([Ilt11, Corollary 1.5]),

$$\dim(V_m) = \sharp \left\{ \rho_i \in \Sigma(1) \ \middle| \ \begin{array}{l} \langle m, u_i \rangle = -1 \\ \langle m, u_{i\pm 1} \rangle < 0 \end{array} \right\}. \tag{2}$$

where $\sharp$ stands for the cardinal.

Example 4.1. We will provide explicit examples of weight space decompositions for $V = H^1(X, TX)$, when X is the smooth toric surface associated to a foldable fan. First, $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^2$ and $\mathrm{Bl}_{p_1,p_2,p_3}(\mathbb{P}^2)$ are rigid, as can be checked directly using Formula (2). In those cases, $V = 0$. We denote by Y_2 the foldable toric surface associated to the fan Σ_2' of Figure 4. We also introduce the foldable toric surfaces Y_4 and Y_3 associated to the fans in Figures 11 and 12.

One can compute directly that, for $i \in \{2, 3, 4\}$,

$$\mathrm{Aut}(N, \Sigma_{Y_i}) \simeq D_i.$$
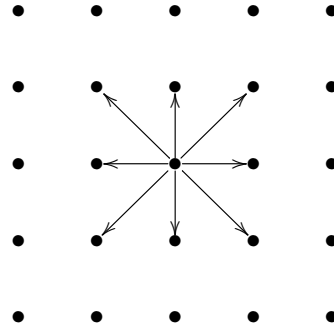
FIGURE 11. Fan of Y_4.
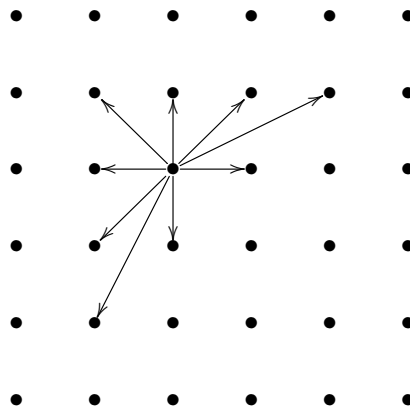
FIGURE 12. Fan of Y_3.

Set $V_i = H^1(Y_i, TY_i)$. Denote by (e_1, e_2) the standard basis of $N = \mathbb{Z}^2$ and by (e_1^*, e_2^*) the dual basis of $M = (\mathbb{Z}^2)^* \simeq \mathbb{Z}^2$. Testing the conditions in Formula (2) for each ray generator gives

$$V_2 = V_{+e_2^*}^1 \oplus V_{-e_2^*}^1 \oplus V_{-e_1^*+e_2^*}^1 \oplus V_{e_1^*-e_2^*}^1,$$

$$V_3 = V_{-e_2^*}^2 \oplus V_{e_1^*}^2 \oplus V_{-e_1^*+e_2^*}^2,$$

$$V_4 = V_{+e_1^*}^1 \oplus V_{-e_1^*}^1 \oplus V_{+e_2^*}^1 \oplus V_{-e_2^*}^1, \tag{3}$$

where V_m^j stands for a j-dimensional weight m representation of T.

REMARK 4.2. One can check, by unravelling the isomorphisms used in [Ilt11, § 1], that the C_3-action on Y_3 induces an action on the coordinates χ_j^m of V_3, with two orbits given by

$$\left\{ \chi_j^{-e_2^*}, \chi_j^{e_1^*}, \chi_j^{-e_1^*+e_2^*} \right\}, \quad j \in \{1, 2\}$$

where χ_j^m, for $1 \leqslant j \leqslant \dim(V_m)$, stand for the generators of the weight m space V_m. Similarly, the D_2 (respectively D_4) action on Y_2 (respectively Y_4) induces a transitive action on the coordinates of V_2 (respectively V_4).

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4.2 The toric GIT quotient

We will assume from now on that X carries a cscK metric, so, by Proposition 3.1, we have

$$\mathrm{Aut}(X) \simeq T \rtimes \mathrm{Aut}(N, \Sigma).$$

We will set

$$G := \mathrm{Aut}(N, \Sigma).$$

From Proposition 2.3, the local moduli space $\mathscr{M}_X$ of polarised cscK surfaces around $[X] \in \mathscr{M}$ is given by a neighbourhood of the origin in the GIT quotient of V by $T \rtimes G$. Assume now that X is foldable. To prove Theorem 1.5, it is then enough to show that the GIT (or categorical) quotient

$$W := V /\!/ T$$

is toric, Gorenstein and terminal. We will end this section by showing that W is indeed an affine toric variety. We first need the following lemma, where by $C_p \subset G$ we mean that there is an injection of C_p in G.

LEMMA 4.3. *Let X be a foldable toric surface. Then one of the following holds:*

(i) *If X is rigid, then $X \in \{\mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1, \mathrm{Bl}_{p_1,p_2,p_3}(\mathbb{P}^2)\}$.*

(ii) *If X is not rigid and $C_2 \subset G$, $X \in \{Y_2, Y_4\}$ or X is obtained from Y_2 or Y_4 by successive C_2-equivariant blow-ups.*

(iii) *If X is not rigid and $C_3 \subset G$, $X = Y_3$ or X is obtained from Y_3 by successive C_3-equivariant blow-ups.*

Recall that Y_2, Y_3 and Y_4 were defined in Example 4.1.

Proof. Note that, by Example 4.1, $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathrm{Bl}_{p_1,p_2,p_3}(\mathbb{P}^2)$ are rigid. If X is not one of those three foldable surfaces, then, from Proposition 3.13, it either belongs to $\{Y_2, Y_3, Y_4\}$ or is obtained from Y_j by successive C_p-equivariant blow-ups, where $p = 2$ for $j \in \{2, 4\}$ and $p = 3$ for $j = 3$. From [Ilt11, Corollary 1.6], $H^1(Y_j, TY_j)$ injects in $H^1(X, TX)$. Hence, from the computations of Example 4.1, we see that X is not rigid, and the result follows. $\square$

PROPOSITION 4.4. *The GIT quotient W inherits the structure of a normal toric affine variety.*

Although the proof is straightforward, we should warn the reader that our setting is slightly different from the standard quotient construction of toric varieties (see e.g. [CLS11, Chapter 5]), as the weight spaces V_m may have dimension greater than 1, and the quotient map $\tilde{\sigma} \to \sigma$ (see the proof below) may send several rays to the same ray. This detailed proof will also settle the necessary notation for the following section.

Proof. If $V = 0$, then there is nothing to prove. We then assume that X is not rigid. From Lemma 4.3, X is obtained from Y_j by successive C_p-equivariant blow-ups, where $j \in \{2, 3, 4\}$, for suitable $p \in \{2, 3\}$.

Denote by $d = \dim(V)$ and by $\tilde{N} \simeq \mathbb{Z}^d$ the lattice of one-parameter subgroups of $\tilde{T} \simeq (\mathbb{C}^*)^d$ acting on $V \simeq \mathbb{C}^d$ by multiplication on each coordinate:

$$\forall (t, x) \in \tilde{T} \times V, \ t \cdot x = (t_i x_i)_{1 \leqslant i \leqslant d}.$$

The T-module structure of V is then equivalent to the injection $T \to \tilde{T}$, or the lattice monomorphism

$$B : N \to \tilde{N}$$

defined by

$$B(u) = (\langle m_i, u \rangle)_{1 \leqslant i \leqslant d}$$

where the weights m_i run through all the weights in the weight space decomposition of V (with multiplicities when $\dim(V_m) \geqslant 2$). The fact that B is injective comes again from the injection $V_j \subset V$ (cf. [Ilt11, Corollary 1.6]) and the explicit computation of V_j in Example 4.1. We then consider the quotient

$$N' := \tilde{N}/N,$$

where we identify N with $B(N)$ by abuse of notation. We claim that N' is again a lattice, that is, that N is saturated in $\tilde{N}$. If $X = Y_j$, for $j \in \{2, 3, 4\}$, then this can be checked directly using the description of V_j in Example 4.1. If X is a blow-up of Y_j, then $V_j \subset V$, so that B can be written $B = (B_j, B_+)$, where

$$B_j : N \to \tilde{N}_j$$

is the injection corresponding to the weight space decomposition of V_j, with

$$\tilde{N}_j \simeq \mathbb{Z}^{d_j},$$

for $d_j = \dim(V_j)$. Then, the fact that $B_j(N)$ is saturated in $\tilde{N}_j \subset \tilde{N}$ implies that $B(N)$ is saturated in $\tilde{N}$. Thus, we have a short exact sequence of lattices

$$0 \longrightarrow N \xrightarrow{B} \tilde{N} \xrightarrow{A} N' \longrightarrow 0 \tag{4}$$

where

$$A : \tilde{N} \to \tilde{N}/N$$

denotes the quotient map.

Introduce $(\tilde{e}_i)_{1 \leqslant i \leqslant d}$, the basis of V dual to the coordinates χ^{m_i} of the weight spaces V_{m_i}. This corresponds to a $\mathbb{Z}$-basis of $\tilde{N}$, still denoted $(\tilde{e}_i)_{1 \leqslant i \leqslant d}$. The cone

$$\tilde{\sigma} = \sum_{i=1}^{d} \mathbb{R}_+ \cdot \tilde{e}_i \subset \tilde{N}_{\mathbb{R}}$$

satisfies

$$\mathrm{Spec}(\mathbb{C}[\tilde{\sigma}^{\vee} \cap \tilde{M}]) = V,$$

where

$$\tilde{M} = \mathrm{Hom}_{\mathbb{Z}}(\tilde{N}, \mathbb{Z}).$$

Consider the cone of $N'_{\mathbb{R}}$

$$\sigma = A(\tilde{\sigma}) = \sum_{i=1}^{d} \mathbb{R}_+ \cdot A(\tilde{e}_i)$$

(we will still denote by A and B their $\mathbb{R}$-linear extensions). Our goal is then to show that W is isomorphic to the affine toric variety defined by σ. We first need to prove that σ has the required

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properties to define such a variety. The cone σ is clearly rational, convex and polyhedral. We now prove that σ is strictly convex, that is

$$-\sigma \cap \sigma = \{0\}.$$

So let $v \in -\sigma \cap \sigma$. Then there is $(u_+, u_-) \in (\tilde{\sigma})^2$ such that $A(u_\pm) = \pm v$. We deduce that

$$u_+ + u_- \in \ker(A) \cap \tilde{\sigma} = \operatorname{Im}(B) \cap \tilde{\sigma}.$$

We will then show that

$$\operatorname{Im}(B) \cap \tilde{\sigma} = \{0\}, \tag{5}$$

which implies $u_+ = u_- = 0$ by strict convexity of $\tilde{\sigma}$. Notice that it is enough to show (5) for $X = Y_j$. Indeed, using the decomposition $B = (B_j, B_+)$ as before, if

$$u = B(x) \in \tilde{\sigma},$$

then

$$u_j = B_j(x) \in \tilde{\sigma}_j,$$

where $\tilde{\sigma}_j \subset (\tilde{N}_j)_{\mathbb{R}}$ is the cone corresponding to V_j. If

$$\operatorname{Im}(B_j) \cap \tilde{\sigma}_j = \{0\}, \tag{6}$$

then by injectivity of B_j we deduce that $x = 0$ and then $u = 0$. It remains to prove (6), which follows again from the explicit description of the weights of the T action on V_j. For example, if $j = 2$, and if

$$B_2(x_1, x_2) = (x_2, -x_2, -x_1 + x_2, x_1 - x_2) \in \tilde{\sigma}_2,$$

then by definition of $\tilde{\sigma}_2$ we deduce

$$\begin{cases} x_2 & \geqslant 0, \\ -x_2 & \geqslant 0, \\ -x_1 + x_2 & \geqslant 0, \\ x_1 - x_2 & \geqslant 0, \end{cases}$$

and thus $x_1 = x_2 = 0$. The cases $j \in \{3, 4\}$ are similar. We have just proved that σ is a strongly convex rational polyhedral cone, and it thus defines an affine toric variety.

We now claim that

$$W \simeq \operatorname{Spec}(\mathbb{C}[\sigma^\vee \cap M'])$$

with

$$M' = \operatorname{Hom}_{\mathbb{Z}}(N', \mathbb{Z}).$$

It is equivalent to show that

$$\mathbb{C}[\sigma^\vee \cap M'] \simeq \mathbb{C}[\chi^{m_1}, \ldots, \chi^{m_d}]^T.$$

The latter equality follows easily from the definitions. Indeed, considering the dual sequence to (4),

$$0 \longrightarrow M' \xrightarrow{A^*} \tilde{M} \xrightarrow{B^*} M \longrightarrow 0 \tag{7}$$

for $m' \in M'$, if $\tilde{m} = A^*(m')$, then

$$\chi^{m'} \in \sigma^\vee \iff \forall i \in [\![1, d]\!],\ \langle m', A(\tilde{e}_i) \rangle \geqslant 0$$

$$\iff \forall i \in [\![1, d]\!],\ \langle A^*(m'), \tilde{e}_i \rangle \geqslant 0$$

$$\iff \begin{cases} B^*(\tilde{m}) = 0 \\ \forall i \in [\![1, d]\!],\ \langle \tilde{m}, \tilde{e}_i \rangle \geqslant 0 \end{cases}$$

$$\iff \chi^{\tilde{m}} \in \mathbb{C}[\chi^{m_1}, \dots, \chi^{m_d}]^T,$$

where in the last equivalence we used the facts that, from the definition of the basis $(\tilde{e}_i)_{1 \leqslant i \leqslant d}$,

$$\chi^{\tilde{m}} \in \mathbb{C}[\chi^{m_1}, \dots, \chi^{m_d}] \iff \forall i \in [\![1, d]\!],\ \langle \tilde{m}, \tilde{e}_i \rangle \geqslant 0,$$

and the T-module structure on $\mathbb{C}[\chi^{m_1}, \dots, \chi^{m_d}]$ reads, for any one-parameter subgroup $\lambda_u \colon \mathbb{C}^* \to T$ generated by some $u \in N$,

$$\lambda_u(t) \cdot \chi^{\tilde{m}} = t^{\langle \tilde{m}, B(u) \rangle} \chi^{\tilde{m}}.$$

This proves the claim, and the result follows. $\qquad\square$

4.3 Singularities

We keep the notation from the previous section (see, in particular, the proof of Proposition 4.4). Our goal here is to conclude the proof of Theorem 1.5, by showing that W is Gorenstein and terminal. Let us first recall the definitions of these notions, and their combinatorial characterisation in the toric case (we refer the reader to [KM98, Introduction and Chapter 5] for a more general treatment of these notions).

DEFINITION 4.5. Let Z be a normal toric variety.[1] Then Z is Gorenstein if the canonical divisor K_Z is Cartier ($\mathbb{Q}$-Gorenstein if K_Z is $\mathbb{Q}$-Cartier). In that case, Z has terminal singularities if there is a resolution of singularities $\pi \colon \tilde{Z} \to Z$ such that if we set

$$K_{\tilde{Z}} = \pi^* K_Z + \sum_i a_i E_i,$$

where the divisors E_i are distinct and irreducible, then for all i, we have $a_i > 0$.

One can check that the above definition does not depend on the choice of resolution. Terminal singularities play an important role in the minimal model programme, being the singularities of minimal models. Their logarithmic version has also turned out to be very useful. Recall that a *log pair* is a normal variety Z together with an effective $\mathbb{Q}$-divisor D with coefficients in $[0, 1] \cap \mathbb{Q}$. A *log resolution* for a log pair (Z, D) is a resolution of singularities $\pi \colon \tilde{Z} \to Z$ such that the exceptional locus $\mathrm{Exc}(\pi)$ of π is a divisor and such that $\pi^{-1}(\mathrm{Supp}(D)) \cup \mathrm{Exc}(\pi)$ is a simple normal crossing divisor.

DEFINITION 4.6. Let (Z, D) be a log pair such that $K_Z + D$ is $\mathbb{Q}$-Cartier (the pair is then called $\mathbb{Q}$-Gorenstein). In that case, (Z, D) has klt singularities if there is a log resolution $\pi \colon \tilde{Z} \to Z$

---

[1]Being normal is not required in the most general definition of Gorenstein singularities. However, we will only deal with normal toric varieties, which are rational [CLS11, Theorem 11.4.2], and hence Cohen–Macaulay, which is usually required to define Gorenstein singularities.

such that if we set

$$K_{\tilde{Z}} = \pi^*(K_Z + D) + \sum_i a_i E_i,$$

where the divisors E_i are distinct and irreducible, then for all i, we have $a_i > -1$. Following [BGLJ24], we say that a normal variety Z is of klt type if there exists an effective $\mathbb{Q}$-divisor D such that (Z, D) is a $\mathbb{Q}$-Gorenstein log pair with klt singularities.

At this stage, from [BGLJ24, Theorem 1], we know that W is of klt type. Also, by [CLS11, Corollary 11.4.25], if W is Gorenstein, then it has log terminal singularities, meaning that the pair $(W, 0)$ is klt. We will give a direct proof that it is actually Gorenstein with terminal singularities, using the following characterisation from [CLS11, Proposition 11.4.12] (where we use the notation $\sigma(1)$ to denote the set of rays in a cone σ and u_ρ to denote the primitive generator of a ray ρ in $\sigma(1)$).

PROPOSITION 4.7. *Consider $W = \mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap M'])$, the affine toric variety associated to the rational strictly convex polyhedral cone $\sigma \subset N'_{\mathbb{R}}$ (with lattice N'). Then*

(i) *W is Gorenstein if and only if there exists $m \in M'$ such that for all $\rho \in \sigma(1)$, $\langle m, u_\rho \rangle = 1$; and*

(ii) *in that case, W has terminal singularities if and only if the only lattice points in*

$$\Pi_\sigma := \left\{ \sum_{\rho \in \sigma(1)} \lambda_\rho u_\rho \;\middle|\; \sum_{\rho \in \sigma(1)} \lambda_\rho \leqslant 1, \; 0 \leqslant \lambda_\rho \leqslant 1 \right\} \subset N'_{\mathbb{R}}$$

are given by its vertices.

From the discussion in Section 4.2, the following proposition concludes the proof of Theorem 1.5.

PROPOSITION 4.8. *The affine toric variety W is Gorenstein and has terminal singularities.*

Proof. We will exclude the rigid cases, and, as in the proof of Proposition 4.4, assume that X is obtained from Y_j by successive C_p-equivariant blow-ups, where $j \in \{2, 3, 4\}$, for suitable $p \in \{2, 3\}$.

We use the characterisation in Proposition 4.7, and first need to describe the rays of σ and their primitive generators. Note that, by construction, the rays of σ belong to the set $\{\mathbb{R}_+ \cdot A(\tilde{e}_i), \; 1 \leqslant i \leqslant d\}$. Let $\rho_i = \mathbb{R}_+ \cdot A(\tilde{e}_i)$ be such a ray. We claim that $A(\tilde{e}_i)$ is primitive in N'. The argument is similar to the one used to prove strict convexity in the proof of Proposition 4.4. Suppose, for a contradiction, that there is $a \in \mathbb{N}$, $a \geqslant 2$, and $\tilde{e} \in \tilde{\sigma} \cap \tilde{N}$ such that $A(\tilde{e}_i) = aA(\tilde{e})$. Note that there is $x \in N$ such that

$$B(x) = (B_j(x), B_+(x)) = \tilde{e}_i - a\tilde{e}. \tag{8}$$

By injectivity of B_j, if $B_j(x) = 0$, then $x = 0$ hence $\tilde{e}_i = a\tilde{e}$. This is absurd as $\tilde{e}_i$ is primitive. So we may assume $B_j(x) \neq 0$. Similarly, using Equation (6), we may assume as well that $B_j(x) \notin -\tilde{\sigma}_j$ and $B_j(x) \notin \tilde{\sigma}_j$. Hence $B_j(x)$ must have at least one positive and one negative coordinate in the basis $(\tilde{e}_k)_{1 \leqslant k \leqslant d_j}$. As $\tilde{e} \in \tilde{\sigma} \cap \tilde{N}$, its coordinates in the basis $(\tilde{e}_k)_{1 \leqslant k \leqslant d}$ are non-negative integers,

and, as $a \geqslant 2$, we can write

$$a\tilde{e} = \sum_{k=1}^{d} a_k \tilde{e}_k$$

with $a_k = 0$ or $a_k \geqslant 2$. Then, from Equation (8), $B_j(x)$ has exactly one coordinate equal to 1, while its other non-zero coordinates are all less than or equal to -2. A case-by-case analysis using the description of V_j in Equation (3) (cf. Example 4.1) shows that this is impossible. Indeed, for V_2, the map B_2 is

$$B_2(x_1, x_2) = (x_2, -x_2, -x_1 + x_2, x_1 - x_2),$$

with $x = (x_1, x_2) \in \mathbb{R}^2 = N_{\mathbb{R}}$, so if one coordinate of $B_2(x)$ equals 1, there is another coordinate that equals -1. A similar argument leads to the same conclusion for B_4. For B_3, we can simply use the fact that the weight spaces are two dimensional, and we have

$$B_3(x) = (-x_2, -x_2, x_1, x_1, -x_1 + x_2, -x_1 + x_2),$$

so the value 1 would appear at least twice in the coordinates of $B_3(x)$, if it ever does.

We will then prove that there is $m' \in M'$ such that

$$\forall i \in [\![1, d]\!], \ \langle m', A(\tilde{e}_i) \rangle = 1,$$

which implies that W is Gorenstein. So let $m' \in M'$. Set

$$\tilde{m} = A^*(m') \in \ker(B^*).$$

Then,

$$\forall i \in [\![1, d]\!], \ \langle m', A(\tilde{e}_i) \rangle = 1 \iff \forall i \in [\![1, d]\!], \ \langle \tilde{m}, \tilde{e}_i \rangle = 1$$

$$\iff \quad \tilde{m} = (1, \ldots, 1)$$

where we used the basis $(\tilde{e}_i)$ to produce the coordinates of $\tilde{m}$. Hence, it is equivalent to show that

$$(1, \ldots, 1) \in \ker(B^*),$$

which, by definition of B, is equivalent to

$$\sum_{i=1}^{d} m_i = 0 \in M$$

where the weights m_i describe the T-action on V. This is where we use the fact that X is foldable. Recall that $G = \operatorname{Aut}(N, \Sigma)$. The G-action on N naturally induces a G-action on M by duality, in such a way that the duality pairing is G-invariant. Explicitly, for $m \in M = \operatorname{Hom}(N, \mathbb{Z})$, and $g \in G$, we have

$$g \cdot m := m(g^{-1} \cdot)$$

and thus

$$\langle g \cdot m, g \cdot u \rangle = \langle m, u \rangle$$

for any $u \in N$. By the characterisation of the weights (m_i) in Equation (2), we see that this G-action preserves the set of weights appearing in the decomposition $V = \bigoplus_{m \in M} V_m$. As G contains a non-trivial cyclic group, there is an element $g \in \operatorname{Aut}(N)$ of order p with no fixed point

but 0. This element generates a cyclic group that acts freely on $\{m_i,\ 1 \leqslant i \leqslant d\}$ (freeness comes from the fact that the action is free on the whole of $M_{\mathbb{R}}$). Hence

$$\sum_{i=1}^{d} m_i = \sum_{i'} \sum_{k=0}^{p-1} g^k \cdot m_i'$$

where we picked a single element m_i' in each orbit under this action. As $g \neq \mathrm{Id}$, we have

$$\mathrm{Id} + g + \cdots + g^{p-1} = 0$$

so that for each orbit

$$\sum_{k=0}^{p-1} g^k \cdot m_i' = 0.$$

We then deduce the existence of the required m', and that W is Gorenstein.

We proceed to the proof of the fact that W has terminal singularities. Let

$$\Pi_\sigma := \left\{ \sum_{\rho \in \sigma(1)} \lambda_\rho u_\rho \ \middle|\ \sum_{\rho \in \sigma(1)} \lambda_\rho \leqslant 1,\ 0 \leqslant \lambda_\rho \leqslant 1 \right\}.$$

As described above, we may fix a subset of

$$\{A(\tilde{e}_i),\ 1 \leqslant i \leqslant d\}$$

as a set of generators for the rays in $\sigma(1)$. Let $u' \in \Pi_\sigma \cap N'$ be given by

$$u' = \sum_{i=1}^{d} \lambda_i A(\tilde{e}_i),$$

with

$$0 \leqslant \lambda_i \leqslant 1,$$

and

$$\lambda_1 + \cdots + \lambda_d \leqslant 1,$$

assuming $\lambda_i = 0$ when $A(\tilde{e}_i)$ is not in our fixed chosen set of ray generators. Assume that there is i_0 with $\lambda_{i_0} \neq 0$. We need to show that $\lambda_{i_0} = 1$ and $\lambda_i = 0$ for $i \neq i_0$. By the assumptions on the λ_i, it is enough to show that $\lambda_i \in \mathbb{Z}$ for all i. As $u' \in N'$, there exists $\tilde{u} \in \tilde{N}$ such that

$$A(\tilde{u}) = A\left(\sum_{i=1}^{d} \lambda_i \tilde{e}_i \right),$$

and then there is $x \in N_{\mathbb{R}}$ such that

$$\tilde{u} - \sum_{i=1}^{d} \lambda_i \tilde{e}_i = B(x) = (B_j(x), B_+(x)),$$

where we recall that we still denote by A and B their $\mathbb{R}$-linear extensions to $\tilde{N}_{\mathbb{R}}$ and $N_{\mathbb{R}}$. By injectivity of B_j again, we only need to consider the case when $B_j(x) \neq 0$ and $1 \leqslant i_0 \leqslant d_j$. Indeed, if $B_j(x) = 0$ then by injectivity $x = 0$, and $\tilde{u}$ being in $\tilde{N}$ forces the λ_i to be integral, whereas if $\lambda_i = 0$ for all $i \leqslant d_j$, then $B_j(x) \in \tilde{N}$ (recall that B_j maps to the first d_j-coordinates) and so by saturation of $B_j(N)$ we have $x \in N$ and thus $B(x) \in \tilde{N}$ which, together with $\tilde{u} \in \tilde{N}$, implies that

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the λ_i are integral. Hence, we have reduced the problem to the case when $X = Y_j$. We will again use the explicit descriptions of the spaces V_j from Example 4.1. For V_3, we find the system

$$\begin{cases} \tilde{u}_1 - \lambda_1 = -x_2, \\ \tilde{u}_2 - \lambda_2 = -x_2, \\ \tilde{u}_3 - \lambda_3 = x_1, \\ \tilde{u}_4 - \lambda_4 = x_1, \\ \tilde{u}_5 - \lambda_5 = -x_1 + x_2, \\ \tilde{u}_6 - \lambda_6 = -x_1 + x_2, \end{cases}$$

where $x = (x_1, x_2) \in \mathbb{R}^2 = N_{\mathbb{R}}$ and $\tilde{u} = (\tilde{u}_1, \dots, \tilde{u}_6) \in \mathbb{Z}^6 \simeq \tilde{N}$ (using the basis $(\tilde{e}_i)_{1 \leqslant i \leqslant 6}$), from which we obtain

$$(\lambda_1 + \lambda_3 + \lambda_5, \lambda_1 + \lambda_3 + \lambda_6, \lambda_1 + \lambda_4 + \lambda_6, \lambda_1 + \lambda_4 + \lambda_5) \in \mathbb{N}^4$$

and

$$(\lambda_2 + \lambda_3 + \lambda_5, \lambda_2 + \lambda_3 + \lambda_6, \lambda_2 + \lambda_4 + \lambda_6, \lambda_2 + \lambda_4 + \lambda_5) \in \mathbb{N}^4,$$

where we used the fact that the $\tilde{u}_i$ are integers and $\lambda_i \geqslant 0$ for all i. Together with

$$0 \leqslant \lambda_1 + \cdots + \lambda_6 \leqslant 1, \tag{9}$$

we deduce that actually

$$(\lambda_1 + \lambda_3 + \lambda_5, \lambda_1 + \lambda_3 + \lambda_6, \lambda_1 + \lambda_4 + \lambda_6, \lambda_1 + \lambda_4 + \lambda_5) \in \{0, 1\}^4$$

and

$$(\lambda_2 + \lambda_3 + \lambda_5, \lambda_2 + \lambda_3 + \lambda_6, \lambda_2 + \lambda_4 + \lambda_6, \lambda_2 + \lambda_4 + \lambda_5) \in \{0, 1\}^4.$$

Let us then assume that $i_0 = 1$, to fix our ideas, so $\lambda_1 > 0$, the other cases being similar. We must then have $\lambda_1 + \lambda_3 + \lambda_5 = 1$, and summing with $\lambda_2 + \lambda_4 + \lambda_6$, we see with (9) that $\lambda_2 + \lambda_4 + \lambda_6 = 0$. Hence $\lambda_2 = \lambda_4 = \lambda_6 = 0$. Then, $\lambda_1 + \lambda_4 + \lambda_6 = \lambda_1 > 0$ so $\lambda_1 = 1$. Considering $\lambda_1 + \lambda_3 + \lambda_6 = 1$ and $\lambda_1 + \lambda_4 + \lambda_5 = 1$ yields $\lambda_3 = \lambda_5 = 0$, and the result follows in that case. The cases of V_2 and V_4 are similar, so we will only treat V_4. In that case, in the coordinates $\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_4) \in \mathbb{R}^4 \simeq \tilde{N}_{\mathbb{R}}$ given by $(\tilde{e}_1, \dots, \tilde{e}_4)$, the map A is given by

$$A(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4) = (\tilde{x}_1 + \tilde{x}_2, \tilde{x}_3 + \tilde{x}_4).$$

Hence

$$\begin{cases} A(\tilde{e}_1) = A(\tilde{e}_2), \\ A(\tilde{e}_3) = A(\tilde{e}_4), \end{cases}$$

so we can pick the ray generators $(A(\tilde{e}_1), A(\tilde{e}_3))$ for the two dimensional cone σ. The description of V_4 then implies

$$\begin{cases} \tilde{u}_1 - \lambda_1 = x_1, \\ \tilde{u}_2 = -x_1, \\ \tilde{u}_3 - \lambda_3 = x_2, \\ \tilde{u}_4 = -x_2, \end{cases}$$

hence $x \in N$ and the result follows. $\qquad \square$

4.4 Examples

We first provide two examples of local moduli spaces and then an example of a quotient

$$H^1(X, TX)//T$$

that is not $\mathbb{Q}$-Gorenstein, for a smooth toric surface X.

4.4.1 *A smooth example: Y_4* We consider the toric variety W_4 which is the quotient of V_4 by T (recall the definition of Y_4 in Example 4.1, and that $V_4 = H^1(Y_4, TY_4)$). As seen in the proof of Proposition 4.8, the generators of σ in that case can be taken to be $A(\tilde{e}_1)$ and $A(\tilde{e}_3)$, so that σ is isomorphic to the cone

$$\mathbb{R}_+ \cdot (1, 0) + \mathbb{R}_+ \cdot (0, 1) \subset \mathbb{R}^2.$$

Then,

$$W_4 \simeq \mathbb{C}^2,$$

and the G-action on V_4 (see Remark 4.2) descends to a D_1-action on $W_4 \simeq \mathbb{C}^2$ generated by a reflection

$$(x, y) \mapsto (y, x).$$

Hence, we conclude that the local moduli space is modelled on

$$W_4//G \simeq \mathbb{C}^2.$$

4.4.2 *A singular example: Y_3* Let us now consider W_3, given by the GIT quotient of V_3 by T. We can pick isomorphisms $N \simeq \mathbb{Z}^2$, $\tilde{N} \simeq \mathbb{Z}^6$ and

$$N' = \tilde{N}/N \simeq \mathbb{Z}^4$$

such that the map $B : N \to \tilde{N}$ is given by

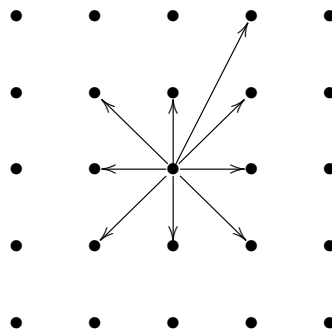$$B = \begin{bmatrix} 0 & -1 \\ 0 & -1 \\ 1 & 0 \\ 1 & 0 \\ -1 & 1 \\ -1 & 1 \end{bmatrix}$$

and the map $A : \tilde{N} \to N'$ by

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

Hence, W_3 is the toric variety associated to the cone

$$\sigma = \sum_{i=1}^{6} \mathbb{R}_+ \cdot e_i' \subset \mathbb{R}^4$$

where the $(e_i')_{1 \leqslant i \leqslant 6}$ are given by the columns of the matrix A. This toric affine variety is singular. It is not even simplicial, as the $(e_i')_{1 \leqslant i \leqslant 6}$, which are the primitive generators of the six rays

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FIGURE 13. Fan of X.

in $\sigma(1)$, do not form a $\mathbb{Q}$-basis for $N'_{\mathbb{Q}}$. The G-action descends to a D_3-action on W_3. Indeed, as explained in the proof of Proposition 4.8, $G \simeq D_3$ acts on the set of weights m_i that appear in the weight space decomposition of $V_3 \simeq \mathbb{C}^6$. This action results in permutations of the χ^{m_i}, and thus in permutations of the elements of the basis $(\tilde{e}_i)_{1 \leqslant i \leqslant 6}$ (we use the notations from the previous proofs of Propositions 4.4 and 4.8). Then, this D_3-action descends to $N' \simeq \mathbb{Z}^4$, by permutations of the e'_i. Explicitly, we compute that the associated representation

$$D_3 \to \mathrm{GL}_4(\mathbb{Z})$$

is generated by

$$\begin{bmatrix} 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

where the first matrix represents an element of order 3 and the second one a reflection. The final quotient $W_3 // D_3$ provides a singular example of a local moduli space.

4.5 A non–Gorenstein example

Here we produce an example of a toric surface X with $\mathrm{Aut}(X) \simeq T$ and $H^1(X, TX) // T$ a non $\mathbb{Q}$-Gorenstein toric variety (see Figure 13). For this, simply blow up Y_4 in a single point to produce the fan within Figure 13.

As X is blown up from Y_4, we have $\mathrm{Aut}^0(X) \simeq T$. We can also check that we actually have $\mathrm{Aut}(X) \simeq T$. Moreover, the arguments in Section 4.2 go through, and we find that, if X were cscK, its local moduli space would be modelled on the GIT quotient of $H^1(X, TX)$ by T. A direct computation again, using the method in Section 4.1, provides the weight space decomposition

$$H^1(X, TX) = V^1_{+e_1^*} \oplus V^2_{-e_1^*} \oplus V^1_{+e_2^*} \oplus V^1_{-e_2^*} \oplus V^1_{e_1^* - e_2^*}.$$

Given that the sum of the weights that appear in this decomposition does not vanish, the quotient $H^1(X, TX) // T$ is not $\mathbb{Q}$-Gorenstein (see the proof of Proposition 4.7).

5. Discussion and perspectives

5.1 Relations between local moduli and Weil–Petersson metrics

In this section we will discuss the fact that the local moduli spaces we considered are related by toric fibrations. Assume that $\pi : X \to X_0$ is a G-equivariant blow-up between two foldable toric

surfaces. We keep the notation from the previous sections, using the subscript 0 to refer to the spaces associated to X_0. From [Ilt11, Corollary 1.6], the corresponding space V_0 injects in V. It is a straightforward exercise to check that we have the following commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & N & \xrightarrow{B} & \tilde{N} & \xrightarrow{A} & N' & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N & \xrightarrow{B_0} & \tilde{N}_0 & \xrightarrow{A_0} & N'_0 & \longrightarrow & 0
\end{array}
$$

where the first vertical arrow is the identity and the last two are surjective. By construction, one sees that the surjective map $N' \to N'_0$ is compatible with σ and σ_0, and induces a toric locally trivial fibration $W \to W_0$ whose fibre is itself an affine toric variety. The whole construction is G-equivariant, and provides maps between the associated local moduli spaces (up to shrinking the neighbourhoods we consider).

The cscK metric on X lives in the class $\pi^*[\omega_0] - \varepsilon E_i$, where ω_0 is cscK on X_0, ε is small and the E_i stand for the exceptional divisors of the blow-up. It would be interesting to understand the behaviour of the associated Weil–Petersson metrics $(\Omega_\varepsilon^{WP})_{0<\varepsilon<\varepsilon_0}$ on the local moduli spaces $\mathscr{W}_\varepsilon \subset W/G$ as constructed in [DN20] when ε goes to zero. It seems natural to expect that the volume of the fibres of the fibration $W/G \to W_0/G$ would go to zero, so that $\mathscr{W}_\varepsilon$ would converge in a Gromov–Hausdorff sense to $\mathscr{W}_0$.

5.2 Higher dimensional case

Many features that hold for toric surfaces fail in a higher dimensions. First, even when it is reductive, the identity component of the automorphism group of a non-rigid toric variety will not a priori be isomorphic to the torus (see [Nil06]). Then, from dimension 3, toric varieties may be obstructed (see [IT20]). Finally, the toric MMP (which produces the classification of toric surfaces) in general produces singular varieties. Our main ingredients for proving Theorem 1.5 therefore do not generalise, a priori, in higher dimensions.

Nevertheless, it would be interesting to study what survives the new difficulties. We expect the right extension of the notion of foldable fans in higher dimension to be fans whose lattice automorphism group admits a subgroup that acts with no fixed point. This raises several questions:

 (i) Do all crystallographic groups arise as lattice automorphism groups of smooth complete fans?

 (ii) Do all foldable toric varieties admit a cscK metric?

(iii) Are foldable toric varieties unobstructed?

(iv) What are the singularities of the moduli space of cscK metrics around cscK foldable toric varieties?

Conflicts of Interest

None.

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