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WEIGHTED SUBSPACES OF HARDY SPACES

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1. Introduction. A function f in H^p on the unit disc U of the complex plane has the uniform growth

$$f(z) = O(1 - |z|)^{-1/p}.$$

We consider in this paper a subspace H^p_γ of H^p with better uniform growth

$$f(z) = O(1 - |z|)^{-\gamma}, \quad 0 < \gamma \leqq 1/p.$$

For the previous results on H^p_γ see [**5**, **6**, **7**]. We start with proving an inequality on H^p which is related to the Hardy-Stein identity (Theorem 2.1) in Section 2. This is applied in the subsequent section to prove some space imbedding theorems related to H^p_γ (Theorems 3.1 and 3.5). These theorems have some known theorems as their corollaries. Finally we prove some coefficient relations on H^p_γ in the last section.

The authors wish to thank Professor Patrick Ahern for the helpful conversations during his visit to Korea. Actually he suggested to the first author the possibility of Theorem 2.1 some years ago. They also express the sincere thanks to the referee for the suggestion of rewriting the paper in the context of H^p_γ.

1.1. *H^p and H^p_γ.* For $0 < p < \infty$, the Hardy space H^p is the class of those functions f holomorphic in U for which

$$\|f\|_p = \sup_{0 \leqq r < 1} M_p(r, f) < \infty,$$

where

$$M_p(r, f) = \left(\int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi}\right)^{1/p}.$$

See [**2**] for the theory of H^p.

For $0 < p < \infty$ and $0 < \gamma \leqq 1/p$, H^p_γ is defined as the class of those $f \in H^p$ for which

$$f(z) = O(1 - |z|)^{-\gamma}.$$

We note that $H^p_{1/p} = H^p$. For $f \in H^p_\gamma$, we define

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$$\|f\|_{p,\gamma} = \max(\|f\|_p, \sup_z (1 - |z|)^\gamma |f(z)|).$$

It is routine to check that H^p_γ ($p \geqq 1$) is a Banach space and H^p_γ ($0 < p < 1$) a Frechet space. See [**5**, **6**, **7**] for more on H^p_γ. A different notation is used in [**6**, **7**].

1.2. $A^{p,\alpha}$ *and* $A^{p,\alpha}_\gamma$. For $0 < p < \infty$ and $\alpha > -1$, the weighted Bergman space $A^{p,\alpha}$ is the class of functions f holomorphic in U for which

$$\int_0^1 (1 - r)^\alpha M_p(r, f)^p dr < \infty.$$

The space $A^{p,\alpha}$ has been extensively studied. See [**1**, **2**, **3**, **6**, **7**, **8**, **9**] for example. A function $f \in A^{p,\alpha}$ has the uniform growth

$$f(z) = O(1 - |z|)^{-(\alpha+2)/p}.$$

For $0 < \gamma < (\alpha + 2)/p$, we define

$$A^{p,\alpha}_\gamma = \{f \in A^{p,\alpha}: \sup_z (1 - |z|)^\gamma |f(z)| < \infty\}.$$

We do not use any linear space theory of $A^{p,\alpha}$ or $A^{p,\alpha}_\gamma$ in this paper.

1.3. *Fractional integrals.* If $f(z) = \sum f_k z^k$ is holomorphic in U, we define the fractional integral $I^\beta f(z)$ of order $\beta > 0$ as

$$I^\beta f(z) = \frac{1}{\Gamma(\beta)} \int_0^1 \left(\log \frac{1}{\rho}\right)^{\beta - 1} f(\rho z) d\rho.$$

See [**3**, **5**].

1.4. For f holomorphic in U and $0 < r \leqq 1$, we define f_r as $f_r(z) = f(rz)$, $z \in U$.

1.5. *Constants.* Throughout this paper $C(\ldots)$ will denote a positive constant depending only on the arguments $(\ldots)$. The magnitude of $C(\ldots)$ may vary from occurrence to occurrence even in the proof of the same theorem.

2. An inequality related to the Hardy-Stein identity. Throughout this section we assume that f is holomorphic in U with $f(0) = 0$. The Hardy-Stein identity says that, for $0 < p < \infty$,

$$(1) \quad M_p(r, f)^p = \frac{p^2}{2\pi} \int_0^r \int_0^{2\pi} |f(\rho e^{i\theta})|^{p-2} |f'(\rho e^{i\theta})|^2 \log \frac{r}{\rho} \rho d\rho d\theta.$$

See [**4**, **11**]. For $0 < p < \infty$ and $0 < \alpha < \infty$, we set

$$J(p, \alpha; f) = \int_0^1 \int_0^{2\pi} |f(\rho e^{i\theta})|^{p-\alpha} |f'(\rho e^{i\theta})|^\alpha \left(\log \frac{1}{\rho}\right)^{\alpha - 1} \rho d\rho d\theta.$$

By letting $r \to 1$ in (1), we have

(I) $$\|f\|_p^p = \frac{p^2}{2\pi} J(p, 2; f).$$

We prove an inequality related to (I) in the following theorem.

2.1. THEOREM. *Let* $0 < \alpha \leqq 2$ *and* $\alpha \leqq p < \infty$. *Then there is a positive constant* $C = C(p, \alpha)$ *such that*

$$\|f\|_p^p \leqq CJ(p, \alpha; f).$$

For the proof of Theorem 2.1 we note the following facts whose easy proofs we omit.

(II) If $0 \leqq \alpha \leqq p$ and f is holomorphic in U then $|f|^{p-\alpha}|f'|^\alpha$ is subharmonic in U.

(III) If $0 < \alpha \leqq p$ then $J(p, \alpha; f_r) \leqq J(p, \alpha; f)$, $0 < r \leqq 1$ (by II).

(IV) For a fixed p, $\log J(p, \alpha; f)$ is a convex function of α $(0 < \alpha < \infty)$. That is, if $0 < \alpha \leqq \beta \leqq \gamma < \infty$ then

$$J(p, \beta; f) \leqq J(p, \alpha; f)^t J(p, \gamma; f)^{1-t},$$

where $t = (\gamma - \beta)/(\gamma - \alpha)$ (by the Hölder's inequality).

We also recall the following results of Littlewood and Paley [**9**, **12**].

(V) If $0 < p \leqq 2$, then there is a positive constant $C(p)$ such that

$$M_p(r, f)^p \leqq C(p)J(p, p; f_r).$$

(VI) If $2 \leqq p < \infty$, then there is a positive constant $C(p)$ such that

$$J(p, p; f_r) \leqq C(p)M_p(r, f)^p.$$

2.2. *Proof of Theorem* 2.1. Case 1. $\alpha \leqq 2 \leqq p$. Set $t = (p - 2)/(p - \alpha)$.

$$\begin{aligned} M_p(r, f)^p &\leqq C(p)J(p, 2; f_r) \quad \text{(by I)} \\ &\leqq C(p)J(p, \alpha; f_r)^t J(p, p; f_r)^{1-t} \quad \text{(by IV)} \\ &\leqq C(p)J(p, \alpha; f_r)^t M_p(r, f)^{p(1-t)} \quad \text{(by VI)}. \end{aligned}$$

Therefore

$$\begin{aligned} M_p(r, f)^p &\leqq C(p, \alpha)J(p, \alpha; f_r) \\ &\leqq C(p, \alpha)J(p, \alpha; f) \quad \text{(by III)} \end{aligned}$$

so that

$$\|f\|_p^p \leqq C(p, \alpha)J(p, \alpha; f).$$

Case 2. $\alpha \leqq p \leqq 2$. Set $t = (2 - p)/(2 - \alpha)$.

$$\begin{aligned} M_p(r, f)^p &\leqq C(p)J(p, p; f_r) \quad \text{(by V)} \\ &\leqq C(p)J(p, \alpha; f_r)^t J(p, 2; f_r)^{1-t} \quad \text{(by IV)} \\ &\leqq C(p)J(p, \alpha; f_r)^t M_p(r, f)^{p(1-t)} \quad \text{(by I).} \end{aligned}$$

Therefore

$$\begin{aligned} M_p(r, f)^p &\leqq C(p, \alpha)J(p, \alpha; f_r) \\ &\leqq C(p, \alpha)J(p, \alpha; f) \quad \text{(by III)} \end{aligned}$$

so that

$$\|f\|_p^p \leqq C(p, \alpha)J(p, \alpha; f).$$

This completes the proof.

3. Imbedding theorems. For f holomorphic in U, we use the following notations:

$$M_\gamma(\theta) = M_\gamma(f; \theta) = \sup_{0 \leqq r < 1} (1 - r)^\gamma |f(re^{i\theta})|,$$

and

$$M(\theta) = M(f; \theta) = \sup_{0 \leqq r < 1} |f(re^{i\theta})|.$$

We use a technique of Ahern **[5]** to prove

3.1. THEOREM. *Let* $0 < p, q, s < \infty$, $q > \alpha > 0$, $\gamma > 0$, $p \geqq (q - \alpha)s$ *and let*

$$u = p\alpha s/(p - (q - \alpha)s).$$

Then there exists a positive constant $C = C(p, q, s, \alpha, \gamma)$ *such that if* $f \in H^p$ *with* $M_\gamma \in L^u(\partial U)$ *then*

$$\text{(1)} \quad \int_0^{2\pi} \left(\int_0^1 |f(re^{i\theta})|^q (1 - r)^{\alpha\gamma - 1} dr \right)^s d\theta \leqq C\|f\|_p^{(q-\alpha)s} \|M_\gamma\|_{L^u}^{\alpha s}.$$

Proof. Assume $f \not\equiv 0$. If $f \in H^p$ then $M(\theta) < \infty$ for almost every θ by the complex maximal theorem. For such a θ, we have

$$0 < M_\gamma(\theta)/M(\theta) \leqq 1.$$

Setting $\rho = 1 - (M_\gamma(\theta)/M(\theta))^{1/\gamma}$ we have, for almost every θ,

$$\begin{aligned} &\int_0^1 |f(re^{i\theta})|^q (1 - r)^{\alpha\gamma - 1} dr \\ &\leqq M_\gamma(\theta)^q \int_0^\rho (1 - r)^{\gamma(\alpha - q) - 1} dr + M(\theta)^q \int_\rho^1 (1 - r)^{\alpha\gamma - 1} dr \\ &\leqq C(q, \alpha, \gamma) M(\theta)^{q - \alpha} M_\gamma(\theta)^\alpha. \end{aligned}$$

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Now, if we apply Hölder's inequality and the complex maximal theorem, we have

$$(1) \quad \leqq C(q, \alpha, \gamma)\left(\int_0^{2\pi} M(\theta)^p d\theta\right)^{(q-\alpha)s/p}\left(\int_0^{2\pi} M_\gamma(\theta)^u d\theta\right)^{\alpha s/u}$$
$$\leqq C(p, q, s, \alpha, \gamma)\|f\|_p^{(q-\alpha)s}\|M_\gamma\|_{L^u}^{\alpha s}.$$

This completes the proof.

3.2. Corollary. *Let* $0 < p < \infty$, $q > \alpha > 0$ *and let* $0 < \gamma \leqq 1/p$. *Then there is a positive constant* $C = C(p, q, \alpha, \gamma)$ *such that if* $f \in H_\gamma^p$ *then*

$$\int_0^{2\pi}\left(\int_0^1 |f(re^{i\theta})|^q(1 - r)^{\alpha\gamma - 1}dr\right)^{p/(q-\alpha)} d\theta \leqq C\|f\|_{p,\gamma}^{pq/(q-\alpha)}.$$

Proof. We note that

$$\|M_\gamma\|_{L^\infty} = \sup_{z\in U}(1 - |z|)^\gamma|f(z)|.$$

The corollary is now a special case of Theorem 3.1 where $s = p/(q - \alpha)$.

3.3. Corollary [**5**, Theorem B]. *If* $0 < p < q < \infty$ *and* $0 < \gamma \leqq 1/p$ *then*

$$H_\gamma^p \subset A_\gamma^{q,\gamma(q-p)-1}.$$

Proof. Set $\alpha = q - p$ in Corollary 3.2.

3.4. Corollary [**5**, Theorem 2.1]. *If* $f \in H_\gamma^p$ *then*

$$I^\beta f \in H_{\gamma-\beta}^{\gamma p/(\gamma-\beta)}$$

where $0 < \beta < \gamma \leqq 1/p$.

Proof. If we set $\alpha = \beta/\gamma$ and $q = 1$ in Corollary 3.2, then we get

$$\int_0^{2\pi}\left(\int_0^1 |f(re^{i\theta})|(1 - r)^{\beta-1}dr\right)^{\gamma p/(\gamma-\beta)} d\theta \leqq C(p, \beta, \gamma)\|f\|_{p,\gamma}^{\gamma p/(\gamma-\beta)}.$$

If $0 < \beta \leqq 1$, then

$$\left(\log\frac{1}{r}\right)^{\beta-1} \leqq (1 - r)^{\beta-1};$$

so that

$$|I^\beta f(re^{i\theta})| \leqq \frac{1}{\Gamma(\beta)}\int_0^1 (1 - \rho)^{\beta-1}|f(\rho re^{i\theta})|d\rho.$$

Therefore

$$I^\beta f \in H_{\gamma-\beta}^{\gamma p/(\gamma-\beta)}.$$

If $\beta > 1$, we can write $\beta = \beta_1 + \beta_2 + \ldots + \beta_n$ with $0 < \beta_1, \beta_2, \ldots, \beta_n < 1$. The successive applications of the above argument with $\beta_1, \beta_2, \ldots, \beta_n$ proves the corollary.

Theorems 2.1 and 3.1 are applied in the proof of the following theorem.

3.5. THEOREM. *Let* $0 < s < p < \infty$, $0 < \gamma < s/(p - s)$ *and let* $\gamma \leqq 1/p$. *If*

$$f' \in A_{\gamma+1}^{s,s-\gamma(p-s)-1}$$

then $f \in H_\gamma^p$.

Proof. We may assume $f(0) = 0$. We set $\delta = \gamma(p - s)$ and choose $0 < \epsilon < \min(1, s)$. By Hölder's inequality with the conjugate indices $s/(s - \epsilon)$ and s/ϵ, we have

$$(2)\quad \int_0^1 (1 - \rho)^{\epsilon-1}|f_r(\rho e^{i\theta})|^{p-\epsilon}|f_r'(\rho e^{i\theta})|^{\epsilon}\rho d\rho$$
$$\leqq \left(\int_0^1 (1 - \rho)^{\delta\epsilon/(s-\epsilon)}|f_r(\rho e^{i\theta})|^{(p-\epsilon)s/(s-\epsilon)}(1 - \rho)^{-1}\rho d\rho\right)^{(s-\epsilon)/s}$$
$$\times \left(\int_0^1 (1 - \rho)^{s-\delta}|f_r'(\rho e^{i\theta})|^{s}(1 - \rho)^{-1}\rho d\rho\right)^{\epsilon/s}.$$

If we apply Hölder's inequality again to (2), we get

$$(3)\quad \int_0^{2\pi}\int_0^1 (1 - \rho)^{\epsilon-1}|f_r(\rho e^{i\theta})|^{p-\epsilon}|f_r'(\rho e^{i\theta})|^{\epsilon}\rho d\rho d\theta$$
$$\leqq \left(\int_0^{2\pi}\int_0^1 (1 - \rho)^{\delta\epsilon/(s-\epsilon)-1}|f_r(\rho e^{i\theta})|^{(p-\epsilon)s/(s-\epsilon)}\rho d\rho d\theta\right)^{(s-\epsilon)/s}$$
$$\times \left(\int_0^{2\pi}\int_0^1 (1 - \rho)^{s-\delta-1}|f_r'(\rho e^{i\theta})|^{s}\rho d\rho d\theta\right)^{\epsilon/s}$$
$$= (A)^{(s-\epsilon)/s}(B)^{\epsilon/s}.$$

If we set $\alpha = (p - s)\epsilon/(s - \epsilon) > 0$ and $q = (p - \epsilon)s/(s - \epsilon)$ then $p/(q - \alpha) = 1$; so by Theorem 3.1

$$(A) \leqq \int_0^{2\pi}\int_0^1 (1 - \rho)^{\alpha\gamma-1}|f_r(\rho e^{i\theta})|^{q}d\rho d\theta$$
$$\leqq C(p, q, \alpha, \gamma)M_p(r, f)^p\|M_\gamma\|_{L^\infty}^{\alpha}.$$

Since

$$\left(\log\frac{1}{\rho}\right)^{\epsilon-1} \leqq (1 - \rho)^{\epsilon-1},$$

we have, by (3) and Theorem 2.1,

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$$M_p(r, f)^p \leqq C(p, q, \alpha, \gamma)^{(s-\epsilon)/s} M_p(r, f)^{p(s-\epsilon)/s} \|M_\gamma\|_{L^\infty}^{\alpha(s-\epsilon)/s} B^{\epsilon/s}.$$

Therefore

$$\begin{aligned} &M_p(r, f)^p \\ &\leqq C(p, q, \alpha, \gamma)^{(s-\epsilon)/\epsilon} \|M_\gamma\|_{L^\infty}^{\alpha(s-\epsilon)/\epsilon} B \\ &\leqq C(p, s, \gamma) \|M_\gamma\|_{L^\infty}^{\alpha(s-\epsilon)/\epsilon} \int_0^1 (1-\rho)^{s-\gamma(p-s)-1} M_s(\rho, f')^s d\rho. \end{aligned}$$

Thus

$$\|f\|_p^p \leqq C(p, s, \gamma) \|M_\gamma\|_{L^\infty}^{\alpha(s-\epsilon)/\epsilon} \int_0^1 (1-\rho)^{s-\gamma(p-s)-1} M_s(\rho, f')^s d\rho.$$

Note that

$$f'(z) = O(1 - |z|)^{-(\gamma+1)}$$

implies

$$f(z) = O(1 - |z|)^{-\gamma}.$$

This completes the proof.

The special case $\gamma = 1/p$ gives an interesting corollary, which can also be derived from a result of Flett [**3**, Theorem 2 (i)].

3.6. Corollary *Let* $0 < p < \infty$ *and let* $p/(1 + p) < s < p$. *If*

$$f' \in A^{s,s(p+1)/p-2},$$

then $f \in H^p$.

The following corollary extends a result of Kim [**5**] for all $p > 0$.

3.7. Corollary. *Let* $0 < p < \infty$, $\alpha > -1$ *and let* $(\alpha + 1)/p < \beta < \gamma \leqq (\alpha + 2)/p$. *If* $f \in A_\gamma^{p,\alpha}$, *then*

$$I^\beta f \in H^q_{\gamma-\beta}$$

where $q = (\gamma p - \alpha - 1)/(\gamma - \beta)$.

Proof. The case $0 < p \leqq 2$ is proved in [**5**, Corollary 2.3]. Assume $p > 2$. By a result of Hardy and Littlewood (see [**5**] for example),

$$(I^\beta f)' \in A^{p,\alpha-(\beta-1)p}_{\gamma-\beta+1},$$

since we can check that $\alpha - (\beta - 1)p > -1$. By Theorem 3.5, we have $I^\beta f \in H^q$ where

$$p - (\gamma - \beta)(q - p) - 1 = \alpha - (\beta - 1)p,$$

i. e., $q = (\gamma p - \alpha - 1)/(\gamma - \beta)$. This completes the proof.

4. Taylor coefficients. We will prove two theorems on the Taylor coefficients of H^p_γ functions. We use the following theorem of M. Mateljević and M. Pavlović [**1**, **10**]. The same technique was used in [**8**].

THEOREM A. *If* $s, \alpha > 0$ *then there are positive constants* $A(s, \alpha)$ *and* $B(s, \alpha)$ *such that if* $a_k \geqq 0$, $k = 1, 2, 3, \dots$

$$A(s, \alpha) \sum_0^\infty 2^{-n\alpha}\Big(\sum_{k \in I_n} a_k\Big)^s \leqq \int_0^1 (1 - r)^{\alpha - 1}\Big(\sum_{k=1}^\infty a_k r^k\Big)^s dr$$

$$\leqq B(s, \alpha) \sum_0^\infty 2^{-n\alpha}\Big(\sum_{k \in I_n} a_k\Big)^s$$

where $I_n = \{k : 2^n \leqq k < 2^{n+1}\}$.

4.1. THEOREM. *Let* $2 \leqq s < p < \infty$, $1/s + 1/t = 1$ *and let* $0 < \gamma \leqq 1/p$. *Then there is a positive constant* $C = C(p, s, \gamma)$ *such that if*

$$f(z) = \sum_1^\infty f_k z^k = O(1 - |z|)^{-\gamma}$$

then

$$(1) \quad \|f\|_{p,\gamma} \leqq C \sum_0^\infty \Big(\sum_{I_n} (k^{\gamma p(1/s - 1/p)}|f_k|)^t\Big)^{s/t}.$$

Proof. Since

$$f'(z) = \sum_1^\infty k f_k z^{k-1},$$

we have, by the Hausdorff-Young inequality [**2**, Theorem 6.1],

$$M_s(r, f') \leqq \sum_1^\infty (|k f_k r^{k-1}|^t)^{1/t}.$$

By Theorem A, we get

$$\int_0^1 (1 - r)^{\alpha - 1} M_s(r, f')^s dr \leqq \int_0^1 (1 - r)^{\alpha - 1}(\sum |k f_k r^{k-1}|^t)^{s/t}$$

$$\leqq B(s, \alpha) \sum_0^\infty 2^{-n\alpha}\Big(\sum_{I_n} |k f_k|^t\Big)^{s/t}$$

$$\leqq B(s, \alpha) \sum_0^\infty \Big(\sum_{I_n} |k^{1-\alpha/s} f_k|^t\Big)^{s/t}.$$

If we set $\alpha = s - \gamma(p - s)$, we have (1) for some positive constant $C(p, s, \gamma)$ by Theorem 3.5.

4.2. THEOREM. *Let* $0 < p < q$, $1 \leqq q < 2$, $1/q + 1/s = 1$, *and let* $0 < \gamma \leqq 1/p$. *Then there is a positive constant* $C = C(p, q, \gamma)$ *such that if*

$$f(z) = \sum_{1}^{\infty} f_k z^k \in H_\gamma^p$$

then

$$\sum_{0}^{\infty} \left(\sum_{I_n} |k^{\gamma p(1/p - 1/q)} f_k|^s \right)^{q/s} \leqq C \|f\|_{p,\gamma}^p.$$

Proof. By Corollary 3.3, we have

$$\int_0^1 (1 - r)^{\gamma(q-p)-1} M_q(r, f)^q dr \leqq C(p, q, \gamma) \|f\|_{p,\gamma}^p.$$

By the Hausdorff-Young inequality [**2**, Theorem 6.1],

$$\left(\sum_{1}^{\infty} |f_k r^k|^s \right)^{1/s} \leqq M_q(r, f).$$

By Theorem A again, we have

$$\int_0^1 (1 - r)^{\gamma(q-p)-1} M_q(r, f)^q dr$$

$$\geqq \int_0^1 (1 - r)^{\gamma(q-p)-1} \left(\sum_{1}^{\infty} |f_k r^k|^s \right)^{q/s}$$

$$\geqq C(p, q, \gamma) \sum_{0}^{\infty} 2^{-n\gamma(q-p)} \left(\sum_{I_n} |f_k|^s \right)^{q/s}$$

$$\geqq C(p, q, \gamma) \sum_{0}^{\infty} \left(\sum_{I_n} |k^{\gamma p(1/p - 1/q)} f_k|^s \right)^{q/s}.$$

The theorem follows.

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