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PARTS FORMULAS INVOLVING CONDITIONAL FEYNMAN INTEGRALS

SEUNG JUN CHANG AND DAVID SKOUG

In this paper we first obtain a basic formula for the conditional analytic Feynman integral of the first variation of a functional on Wiener space. We then apply this basic result to obtain several integration by parts formulas for conditional analytic Feynman integrals and conditional Fourier-Feynman transforms.

1. INTRODUCTION

Let $C_0[0,T]$ denote the one-parameter Wiener space, that is the space of $\mathbb{R}$-valued continuous functions $x(t)$ on $[0,T]$ with $x(0)=0$. Let $\mathcal{M}$ denote the class of all Wiener measurable subsets of $C_0[0,T]$ and let m denote Wiener measure. $\bigl(C_0[0,T],\ \mathcal{M},m\bigr)$ is a complete measure space and we denote the Wiener integral of a Wiener integrable functional F by

$$\int_{C_0[0,T]} F(x)m(dx).$$

A subset E of $C_0[0,T]$ is said to be scale-invariant measurable ([**7, 13**]) provided $\rho E \in \mathcal{M}$ for all $\rho > 0$, and a scale-invariant measurable set N is said to be scale-invariant null provided $m(\rho N) = 0$ for each $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere. If two functionals F and G are equal scale-invariant almost everywhere , we write $F \approx G$. For a rather detailed discussion of scale-invariant measurable and its relation with other topics see [**13**]. It was also pointed out in [**13**] that the concept of scale-invariant measurable, rather than Borel measurability or Wiener measurability, is precisely correct for the analytic Fourier-Feynman transform theory and the analytic Feynman integration theory. Thus throughout this paper we shall assume that each functional F (or G or H) we consider satisfies the conditions:

$$F : C_0[0,T] \to \mathbb{C}, \tag{1.1}$$

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is defined scale-invariant almost everywhere and is scale-invariant measurable.

(1.2) $$\int_{C_0[0,T]} |F(\rho x)| m(dx) < \infty \text{ for each } \rho > 0.$$

Let $\mathbb{C}_+$ and $\widetilde{\mathbb{C}}_+$ denote the complex numbers with positive real part and the nonzero complex numbers with nonnegative real part respectively. Let F satisfy conditions (1.1) and (1.2) above, and for $\lambda > 0$, let

$$J(\lambda) = \int_{C_0[0,T]} F(\lambda^{-(1/2)}x) m(dx).$$

If there exists a function $J^*(\lambda)$ analytic in $\mathbb{C}_+$ such that $J^*(\lambda) = J(\lambda)$ for all $\lambda > 0$, then $J^*(\lambda)$ is defined to be the analytic Wiener integral of F over $C_0[0,T]$ with parameter λ, and for λ in $\mathbb{C}_+$ we write

(1.3) $$\int_{C_0[0,T]}^{\text{anw}_\lambda} F(x) m(dx) = J^*(\lambda).$$

Let $q \neq 0$ be a real parameter and let F be a functional whose analytic Wiener integral exists for all $\lambda \in \mathbb{C}_+$. If the following limit exists, we call it the analytic Feynman integral of F with parameter q and we write

(1.4) $$\int_{C_0[0,T]}^{\text{anf}_q} F(x) m(dx) = \lim_{\lambda \to -iq} \int_{C_0[0,T]}^{\text{anw}_\lambda} F(x) m(dx)$$

where $\lambda \to -iq$ through values in $\mathbb{C}_+$.

The concept of an L_1 analytic Fourier-Feynman transform was introduced by Brue in [**1**], while in [**4**], Cameron and Storvick introduced an L_2 analytic Fourier-Feynman transform. In [**12**], Johnson and Skoug developed an L_p analytic Fourier-Feynman transform which extended the results in [**1, 4**] and gave various relationships between the L_1 and L_2 theories.

Next we state the definition of the L_p analytic Fourier-Feynman transform ([**12**]) using (1.3) and (1.4) above. First for $\lambda \in \mathbb{C}_+$ and $y \in C_0[0,T]$, let

(1.5) $$T_\lambda(F)(y) = \int_{C_0[0,T]}^{\text{anw}_\lambda} F(x+y) m(dx).$$

In the standard Fourier theory the integrals involved are often interpreted in the mean; a similar concept is useful in the Fourier-Feynman transform theory ([**12**, p. 104]). Let $p \in (1,2]$ and let p and p' be related by $1/p + 1/p' = 1$. Let $\{H_n\}$ and H be scale-invariant measurable functionals such that for each $\rho > 0$,

$$\lim_{n\to\infty} \int_{C_0[0,T]} \left|H_n(\rho y) - H(\rho y)\right|^{p'} m(dy) = 0.$$

Then we write

$$H \approx \underset{n\to\infty}{\text{l. i. m.}}\ H_n$$

and we call H the scale-invariant limit in the mean of order p'. A similar definition is understood when n is replaced by the continuously varying parameter λ. Let real $q \neq 0$ be given. For $1 < p \leqslant 2$ we define the L_p analytic Fourier-Feynman transform, $T_q^{(p)}(F)$ of F, by the formula ($\lambda \in \mathbb{C}_+$),

$$T_q^{(p)}(F)(y) = \underset{\lambda\to -iq}{\text{l. i. m.}}\ T_\lambda(F)(y) \tag{1.6}$$

if it exists. We define the L_1 analytic Fourier-Feynman transform, $T_q^{(1)}$ of F, by the formula ($\lambda \in \mathbb{C}_+$),

$$T_q^{(1)}(F)(y) = \lim_{\lambda\to -iq} T_\lambda(F)(y) \tag{1.7}$$

if it exists. We note that for $1 \leqslant p \leqslant 2$, $T_q^{(p)}(F)$ is defined only scale-invariant almost everywhere . We also note that if $T_q^{(p)}(F_1)$ exists and if $F_1 \approx F_2$, then $T_q^{(p)}(F_2)$ exists and $T_q^{(p)}(F_1) \approx T_q^{(p)}(F_2)$.

Next we give the definition of the first variation δF of a functional F [**2, 6**].

DEFINITION: Let F be a Wiener measurable functional on $C_0[0,T]$, and let $w \in C_0[0,T]$. Then

$$\delta F(x \mid w) = \frac{\partial}{\partial k} F(x + kw)\Big|_{k=0} \tag{1.8}$$

(if it exists) is called the first variation of $F(x)$. However throughout this paper we shall always choose w to be an element of A where

$$A = \{w \in C_0[0,T] : w \text{ is absolutely continuous on } [0,T] \text{ with } w' \in L_2[0,T]\}. \tag{1.9}$$

See [**10, 14, 19**] for some relationships which exist between the Fourier-Feynman transform and the first variation for various classes of functionals.

Throughout this paper, for u and v in $L_2[0,T]$ and $x \in C_0[0,T]$, we let $\langle u, x\rangle$ denote the Paley-Wiener-Zygmund integral $\int_0^T u(s)dx(s)$ and $(u,v) = \int_0^T u(s)v(s)ds$.

We finish this section by stating the following well-known translation theorem using the above notation ([**3**]).

TRANSLATION THEOREM. *For $\lambda > 0$ and $w \in A$,*

$$\int_{C_0[0,T]}^{\text{anw}_\lambda} F(x+w)m(dx) \doteq \exp\Big\{-\frac{\lambda}{2}\|w'\|^2\Big\} \int_{C_0[0,T]}^{\text{anw}_\lambda} F(x)\exp\{\lambda\langle w', x\rangle\}m(dx) \tag{1.10}$$

where $\doteq$ means that if either side of equation (1.10) *exists, both side exist and equality holds.*

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2. Conditional Feynman integrals and transforms

For the definitions and related work involving conditional Feynman integrals and transforms see [**8, 9, 11, 15, 16, 17, 20**]. Throughout this paper we shall always condition by

$$X(x) = x(T). \tag{2.1}$$

For $\lambda > 0$ and $\eta \in \mathbb{R}$ let

$$J_\lambda(\eta) = E\Big(F\big(\lambda^{-(1/2)}x\big) \parallel \lambda^{-(1/2)}x(T) = \eta\Big) \tag{2.2}$$

denote the conditional Wiener integral of $F\big(\lambda^{-(1/2)}x\big)$ given $\lambda^{-(1/2)}x(T)$. If for almost all $\eta \in \mathbb{R}$, there exists a function $J_\lambda^*(\eta)$, analytic in λ on $\mathbb{C}_+$ such that $J_\lambda^*(\eta) = J_\lambda(\eta)$ for $\lambda > 0$, then $J_\lambda^*(\eta)$ is defined to be the conditional analytic Wiener integral of $F(x)$ given $x(T)$ with parameter λ and for $\lambda \in \mathbb{C}+$ we write

$$J_\lambda^*(\eta) = E^{\mathrm{anw}_\lambda}\big(F(x) \parallel x(T) = \eta\big). \tag{2.3}$$

If for fixed real $q \neq 0$, $\lim_{\lambda \to -iq} J_\lambda^*(\eta)$ exists for almost all $\eta \in \mathbb{R}$, we denote the value of this limit by

$$E^{\mathrm{anf}_q}\big(F(x) \parallel x(T) = \eta\big) \tag{2.4}$$

and we call it the conditional analytic Feynman integral of F given X with parameter q.

Remark 1. In [**16**], Park and Skoug give a formula for expressing conditional Wiener integrals in terms of ordinary Wiener integrals; namely that for $\lambda > 0$,

$$\begin{aligned} E\Big(F\big(\lambda^{-(1/2)}x\big) \parallel \lambda^{-(1/2)}x(T) = \eta\Big) \\ = \int_{C_0[0,T]} F\Big(\lambda^{-(1/2)}x(\cdot) - \frac{\cdot}{T}\lambda^{-(1/2)}x(T) + \frac{\cdot}{T}\eta\Big)\, m(dx). \end{aligned} \tag{2.5}$$

Thus we have that

$$E^{\mathrm{anw}_\lambda}\big(F(x) \parallel x(T) = \eta\big) = \int_{C_0[0,T]}^{\mathrm{anw}_\lambda} F\Big(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta\Big)\, m(dx) \tag{2.6}$$

and

$$E^{\mathrm{anf}_q}\big(F(x) \parallel x(T) = \eta\big) = \int_{C_0[0,T]}^{\mathrm{anf}_q} F\Big(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta\Big)\, m(dx) \tag{2.7}$$

where in (2.6) and (2.7) the existence of either side implies the existence of the other side and their equality.

Next we state the definition of the conditional Fourier-Feynman transform using (1.6), (1.7), (2.6) and (2.7) above. For $\lambda \in \mathbb{C}_+$, $\eta \in \mathbb{R}$ and $y \in C_0[0,T]$, let $T_\lambda(F\|X)(y,\eta)$ denote the conditional analytic Wiener integral of $F(x+y)$ given $X(x) = x(T)$; that is to say

$$
\begin{aligned}
T_\lambda(F\|X)(y,\eta) &= E^{\text{anw}_\lambda}\Big(F(y+x) \parallel x(T) = \eta\Big) \\
&= \int_{C_0[0,T]}^{\text{anw}_\lambda} F\Big(y(\cdot) + x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta\Big) m(dx).
\end{aligned} \tag{2.8}
$$

Then for $p \in [1,2]$ we define the conditional Fourier-Feynman transform, $T_q^{(p)}(F\|X)$ of F, by the formula ($\lambda \in \mathbb{C}_+$),

$$
T_q^{(p)}(F\|X)(y,\eta) = \begin{cases} \operatorname*{l.i.m.}_{\lambda\to -iq} T_\lambda(F\|X)(y,\eta), & 1 < p \leqslant 2 \\ \lim_{\lambda\to -iq} T_\lambda(F\|X)(y,\eta), & p = 1 \end{cases} \tag{2.9}
$$

if it exists. Note that for $p = 1$,

$$
T_q^{(1)}(F\|X)(y,\eta) = \int_{C_0[0,T]}^{\text{anf}_q} F\Big(y(\cdot) + x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta\Big) m(dx).
$$

3. Main results

Our first result is a fundamental theorem in which we express the conditional analytic Feynman integral of the first variation of a functional in terms of an ordinary (that is, non-conditional) analytic Feynman integral. In [**2**], Cameron (see [**6**, Theorem A, p. 145]) expressed the Wiener integral of the first variation of a functional F in terms of the Wiener integral of the product of F by a linear functional, and in [**6**, Theorem 1], Cameron and Storvick obtained a similar result for analytic Feynman integrals.

REMARK 2. Throughout this paper the main conditions we impose upon the functionals F, G and H, in addition to conditions (1.1) and (1.2) above, are the conditions (3.1), (3.8), (3.9), et cetera, below. These conditions ensure the existence of various integrals (or conditional integrals), and they justify the various interchanges of differentiation and integration used in the proofs.

THEOREM 1. *Let $w_1 \in A$ and $\eta \in \mathbb{R}$ be given. For each $\rho > 0$, assume that $F(\rho x)$ has a first variation $\delta F(\rho x \mid \rho w_1)$ for all $x \in C_0[0,T]$ such that for some positive function $\gamma(\rho)$,*

$$
\sup_{|k|\leqslant\gamma(\rho)} \Big|\delta F\Big(\rho x(\cdot) - \frac{\cdot}{T}\rho x(T) + \frac{\cdot}{T}\rho\eta + \rho k w_1 \mid \rho w_1\Big)\Big| \tag{3.1}
$$

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is a Wiener integrable function of x *over* $C_0[0,T]$. *Then for all* $q \in \mathbb{R}$, $q \neq 0$,

$$E^{\text{anf}_q}\Big(\delta F(x \mid w_1) \,\|\, x(T) = \eta\Big) \doteq -iq\int_{C_0[0,T]}^{\text{anf}_q} F\Big(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta\Big)\langle w_1', x\rangle m(dx) \tag{3.2}$$

where $\doteq$ *means that if either side exists, both sides exist and equality holds. Furthermore, if* (3.1) *also holds with* $w_1(t)$ *replaced with* $w_0(t) = t/T$ *on* $[0,T]$, *and if either side of* (3.2) *exists, then*

$$\begin{aligned} E^{\text{anf}_q}\Big(\delta F(x \mid w_1) \,\|\, x(T) = \eta\Big) &= -iqE^{\text{anf}_q}\Big(F(x)\langle w_1', x\rangle \,\|\, x(T) = \eta\Big) \\ &\quad + T(w_0', w_1')E^{\text{anf}_q}\Big(\delta F(x \mid w_0) \,\|\, x(T) = \eta\Big) \\ &\quad + iq\eta(w_0', w_1')E^{\text{anf}_q}\Big(F(x) \,\|\, x(T) = \eta\Big). \end{aligned} \tag{3.3}$$

PROOF: First proceeding formally with $\lambda > 0$, and then using equation (1.10), we see that

$$\begin{aligned} &E^{\text{anw}_\lambda}\Big(\delta F(x \mid w_1) \,\|\, x(T) = \eta\Big) \\ &= E^{\text{anw}_\lambda}\Big(\frac{\partial}{\partial k}F(x + kw_1)\Big|_{k=0} \,\|\, x(T) = \eta\Big) \\ &= \int_{C_0[0,T]}^{\text{anw}_\lambda} \frac{\partial}{\partial k}\Big[F\Big(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta + kw_1(\cdot)\Big)\Big]\Big|_{k=0} m(dx) \\ &= \frac{\partial}{\partial k}\Big[\int_{C_0[0,T]}^{\text{anw}_\lambda} F\Big(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta + kw_1(\cdot)\Big)m(dx)\Big]\Big|_{k=0} \\ &= \frac{\partial}{\partial k}\Big[\exp\Big\{-\frac{\lambda k^2}{2}\|w_1'\|^2\Big\}\int_{C_0[0,T]}^{\text{anw}_\lambda} F\Big(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta\Big) \\ &\qquad \cdot \exp\big\{\lambda k\langle w_1', x\rangle\big\}m(dx)\Big]\Big|_{k=0} \\ &= \lambda\int_{C_0[0,T]}^{\text{anw}_\lambda} F\Big(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta\Big)\langle w_1', x\rangle m(dx). \end{aligned} \tag{3.4}$$

But condition (3.1) justifies the above interchanges of differentiation and integration, and so the existence of either side of equation (3.2) implies the existence of all the expressions in equation (3.4) and their equality. Hence for all $\lambda > 0$,

$$E^{\text{anw}_\lambda}\Big(\delta F(x \mid w_1) \,\|\, x(T) = \eta\Big) \doteq \lambda\int_{C_0[0,T]}^{\text{anw}_\lambda} F\Big(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta\Big)\langle w_1', x\rangle m(dx). \tag{3.5}$$

Next we note that if either side of (3.2) exists for all real $q \neq 0$, then equation (3.5) holds for all $\lambda \in \mathbb{C}_+$. Finally, letting $\lambda \to -iq$ through values in $\mathbb{C}_+$, we obtain equation (3.2).

To establish equation (3.3), note that for all $\lambda > 0$,

$$
\begin{aligned}
&\lambda \int_{C_0[0,T]}^{\mathrm{anw}_\lambda} F\Big(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta\Big)\langle w_1', x\rangle m(dx) \\
&\quad = \lambda \int_{C_0[0,T]}^{\mathrm{anw}_\lambda} F\Big(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta\Big)\Big\langle w_1', x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta\Big\rangle m(dx) \\
&\qquad + \lambda \int_{C_0[0,T]}^{\mathrm{anw}_\lambda} F\Big(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta\Big)\Big\langle w_1', \frac{\cdot}{T}x(T)\Big\rangle m(dx) \\
&\qquad - \lambda \int_{C_0[0,T]}^{\mathrm{anw}_\lambda} F\Big(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta\Big)\Big\langle w_1', \frac{\cdot}{T}\eta\Big\rangle m(dx) \\
&\quad = \lambda E^{\mathrm{anw}_\lambda}\Big(F(x)\langle w_1', x\rangle \,\|\, x(T) = \eta\Big) \\
&\qquad + T(w_0', w_1')E^{\mathrm{anw}_\lambda}\Big(\delta F(x \mid w_0) \,\|\, x(T) = \eta\Big) \\
&\qquad - \lambda\eta(w_0', w_1')E^{\mathrm{anw}_\lambda}\Big(F(x) \,\|\, x(T) = \eta\Big).
\end{aligned} \tag{3.6}
$$

Then, if the right hand of (3.2) holds for all real $q \neq 0$, equation (3.6) holds for all $\lambda \in \mathbb{C}_+$. Finally letting $\lambda \to -iq$ through values in $\mathbb{C}_+$, we obtain equation (3.3) as desired. ∎

Our first corollary of Theorem 1 follows from a careful examination of equation (3.3) and yields a formula for the conditional analytic Feynman integral of F multiplied by the linear factor $\langle w_1', x\rangle$.

COROLLARY 1. *Let w_1, w_0, η, and F be as in Theorem 1 above and assume that $E^{\mathrm{anf}_q}(\delta F(x \mid w_1)\|x(T) = \eta)$ exists. Then*

$$
\begin{aligned}
E^{\mathrm{anf}_q}\Big(F(x)\langle w_1', x\rangle \,\|\, x(T) = \eta\Big) &= \frac{i}{q}E^{\mathrm{anf}_q}\Big(\delta F(x \mid w_1) \,\|\, x(T) = \eta\Big) \\
&\quad - \frac{iT}{q}(w_0', w_1')E^{\mathrm{anf}_q}\Big(\delta F(x \mid w_0) \,\|\, x(T) = \eta\Big) \\
&\quad + \eta(w_0', w_1')E^{\mathrm{anf}_q}\Big(F(x) \,\|\, x(T) = \eta\Big).
\end{aligned} \tag{3.7}
$$

In our next theorem we obtain an integration by parts formula for conditional analytic Feynman integrals.

THEOREM 2. *Let $w_1 \in A$ and $\eta \in \mathbb{R}$ be given. For each $\rho > 0$ assume that $G(\rho x)$ and $H(\rho x)$ have first variations $\delta G(\rho x \mid \rho w_1)$ and $\delta H(\rho x \mid \rho w_1)$ for all $x \in$*

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$C_0[0,T]$ such that for some positive function $\gamma(\rho)$,

$$\sup_{|k|\leqslant\gamma(\rho)}\Big|G\Big(\rho x(\cdot)-\frac{\cdot}{T}\rho x(T)+\frac{\cdot}{T}\rho\eta+k\rho w_1\Big) \cdot \delta H\Big(\rho x(\cdot)-\frac{\cdot}{T}\rho x(T)+\frac{\cdot}{T}\rho\eta+k\rho w_1 \mid \rho w_1\Big)\Big| \tag{3.8}$$

and

$$\sup_{|k|\leqslant\gamma(\rho)}\Big|H\Big(\rho x(\cdot)-\frac{\cdot}{T}\rho x(T)+\frac{\cdot}{T}\rho\eta+k\rho w_1\Big) \cdot \delta G\Big(\rho x(\cdot)-\frac{\cdot}{T}\rho x(T)+\frac{\cdot}{T}\rho\eta+k\rho w_1 \mid \rho w_1\Big)\Big| \tag{3.9}$$

are Wiener integrable functions of x over $C_0[0,T]$. Then for all $q\in\mathbb{R}$, $q\neq 0$,

$$\begin{aligned} &E^{\mathrm{anf}_q}\Big(G(x)\delta H(x\mid w_1)+\delta G(x\mid w_1)H(x)\,\Big\|\,x(T)=\eta\Big) \\ &\doteq -iq\int_{C_0[0,T]}^{\mathrm{anf}_q} G\Big(x(\cdot)-\frac{\cdot}{T}x(T)+\frac{\cdot}{T}\eta\Big) \cdot H\Big(x(\cdot)-\frac{\cdot}{T}x(T)+\frac{\cdot}{T}\eta\Big)\langle w_1',x\rangle m(dx). \end{aligned} \tag{3.10}$$

Furthermore, if either side of equation (3.10) exists, then

$$\begin{aligned} &E^{\mathrm{anf}_q}\Big(G(x)\delta H(x\mid w_1)+\delta G(x\mid w_1)H(x)\,\Big\|\,x(T)=\eta\Big) \\ &= -iqE^{\mathrm{anf}_q}\Big(G(x)H(x)\langle w_1',x\rangle\,\Big\|\,x(T)=\eta\Big) \\ &\quad + T(w_0',w_1')E^{\mathrm{anf}_q}\Big(G(x)\delta H(x\mid w_0)+\delta G(x\mid w_0)H(x)\,\Big\|\,x(T)=\eta\Big) \\ &\quad + iq\eta(w_0',w_1')E^{\mathrm{anf}_q}\Big(G(x)H(x)\,\Big\|\,x(T)=\eta\Big) \end{aligned} \tag{3.11}$$

where $w_0(t)=t/T$ on $[0,T]$.

Proof: Let $F(x)=G(x)H(x)$. Then, since

$$\delta F(\rho x\mid\rho w_1)=G(\rho x)\delta H(\rho x\mid\rho w_1)+\delta G(\rho x\mid\rho w_1)H(\rho x)$$

for all $\rho>0$, Theorem 2 follows immediately from Theorem 1. ∎

We obtain our next corollary by letting $H(x)=G(x)$ in Theorem 2 above.

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Corollary 2. *Let $w_1 \in A$ and $\eta \in \mathbb{R}$ be given. For each $\rho > 0$ assume that $G(\rho x)$ has a first variation $\delta G(\rho x \mid \rho w_1)$ for all $x \in C_0[0,T]$ such that for some positive function $\gamma(\rho)$,*

$$\sup_{|k|\leqslant\gamma(\rho)} \left|G\Big(\rho x(\cdot) - \frac{\cdot}{T}\rho x(T) + \frac{\cdot}{T}\rho\eta + k\rho w_1\Big)\cdot \delta G\Big(\rho x(\cdot) - \frac{\cdot}{T}\rho x(T) + \frac{\cdot}{T}\rho\eta + k\rho w_1 \mid \rho w_1\Big)\right|$$

is a Wiener integrable function of x over $C_0[0,T]$. Then for all $q \in \mathbb{R}$, $q \neq 0$,

$$\text{(3.12)}\quad E^{\text{anf}_q}\Big(G(x)\delta G(x \mid w_1) \parallel x(T) = \eta\Big) \doteq -\frac{iq}{2}\int^{\text{anf}_q}_{C_0[0,T]}\Big[G\Big(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta\Big)\Big]^2 \langle w_1', x\rangle m(dx).$$

Furthermore, if either side of (3.12) exists, then

$$\text{(3.13)}\quad \begin{aligned} &E^{\text{anf}_q}\Big(G(x)\delta G(x \mid w_1) \parallel x(T) = \eta\Big) \\ &\quad = -\frac{iq}{2}E^{\text{anf}_q}\Big(\big[G(x)\big]^2\langle w_1', x\rangle \parallel x(T) = \eta\Big) \\ &\qquad + T(w_0', w_1')E^{\text{anf}_q}\Big(G(x)\delta G(x \mid w_0) \parallel x(T) = \eta\Big) \\ &\qquad + \frac{iq\eta}{2}(w_0', w_1')E^{\text{anf}_q}\Big(\big[G(x)\big]^2 \parallel x(T) = \eta\Big) \end{aligned}$$

where $w_0(t) = t/T$ on $[0,T]$.

In our next corollary we obtain a formula for the conditional analytic Feynman integral of a functional G multiplied by the two linear factors $\langle w_1', x\rangle$ and $\langle w_2', x\rangle$.

Corollary 3. *Let w_1 and w_2 be elements of A, let $\eta \in \mathbb{R}$, and let $F(x) = G(x)\langle w_2', x\rangle$. For each $\rho > 0$, assume that $F(\rho x)$ has a first variation $\delta F(\rho x \mid \rho w_1)$ for all $x \in C_0[0,T]$ such that for some positive function $\gamma(\rho)$, the expression given by (3.1) is a Wiener integrable function of x over $C_0[0,T]$. Also assume that $E^{\text{anf}_q}\big(\delta F(x \mid w_1) \parallel x(T) = \eta\big)$ exists. Then*

$$\text{(3.14)}\quad \begin{aligned} &E^{\text{anf}_q}\Big(G(x)\langle w_2', x\rangle\langle w_1', x\rangle \parallel x(T) = \eta\Big) \\ &\quad = \frac{i}{q}E^{\text{anf}_q}\Big((w_1', w_2')G(x) + \langle w_2', x\rangle\delta G(x \mid w_1) \parallel x(T) = \eta\Big) \\ &\qquad - \frac{iT}{q}(w_0', w_1')E^{\text{anf}_q}\Big((w_0', w_2')G(x) + \delta G(x \mid w_0)\langle w_2', x\rangle \parallel x(T) = \eta\Big) \\ &\qquad + \eta(w_0', w_1')E^{\text{anf}_q}\Big(G(x)\langle w_2', x\rangle \parallel x(T) = \eta\Big) \end{aligned}$$

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where $w_0(t) = t/T$ *on* $[0, T]$.

PROOF: Let $H(x) = \langle w_2', x\rangle$. Then a direct calculation shows that $\delta H(x \mid w_1) = (w_1', w_2')$. Now equation (3.14) follows directly from equation (3.7) with $F(x) = G(x)\langle w_2', x\rangle$ or from equation (3.11) with $H(x) = \langle w_2', x\rangle$. ∎

Our next corollary involves the Fourier-Feynman Transforms, $T_q^{(p)}(G)$ of G, for fixed $p \in [1,2]$ and $q \in \mathbb{R}$, $q \neq 0$.

COROLLARY 4. *Let $w_1 \in A$ and $\eta \in \mathbb{R}$ be given. For each $\rho > 0$, assume that $T_q^{(p)}(G)(\rho x)$ has a first variation $\delta T_q^{(p)}(G)(\rho x \mid \rho w_1)$ for all $x \in C_0[0,T]$ such that for some positive function $\gamma(\rho)$,*

$$(3.15)\qquad \sup_{|k|\leqslant\gamma(\rho)} \left|\delta T_q^{(p)}(G)\Big(\rho x(\cdot) - \frac{\cdot}{T}\rho x(T) + \frac{\cdot}{T}\rho\eta + k\rho w_1 \;\Big|\; \rho w_1\Big)\right|$$

is a Wiener integrable function of x over $C_0[0,T]$. Then

$$(3.16)\qquad E^{\mathrm{anf}_q}\Big(\delta T_q^{(p)}(G)(x \mid w_1) \;\big\|\; x(T) = \eta\Big) \doteq -iq\int_{C_0[0,T]}^{\mathrm{anf}_q} T_q^{(p)}(G)\Big(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta\Big)\langle w_1', x\rangle m(dx).$$

Furthermore, if the expression in (3.15) is also a Wiener integrable function of x when $w_1(t)$ is replaced with $w_0(t) = t/T$, and if either side of equation (3.16) exists, then

$$(3.17)\qquad \begin{aligned} E^{\mathrm{anf}_q}\Big(\delta T_q^{(p)}(G)(w \mid w_1) \;\big\|\; x(T) = \eta\Big) &= -iqE^{\mathrm{anf}_q}\Big(T_q^{(p)}(G)(x)\langle w_1', x\rangle \;\big\|\; x(T) = \eta\Big) \\ &\quad + T(w_0', w_1')E^{\mathrm{anf}_q}\Big(\delta T_q^{(p)}(G)(x \mid w_0) \;\big\|\; x(T) = \eta\Big) \\ &\quad + iq\eta(w_0', w_1')E^{\mathrm{anf}_q}\Big(T_q^{(p)}(G)(x) \;\big\|\; x(T) = \eta\Big). \end{aligned}$$

PROOF: Simply apply Theorem 1 with $F(x) = T_q^{(p)}(G)(x)$. ∎

Our next integration by parts formula involves the Fourier-Feynman Transforms, $T_q^{(p)}(G)$ and $T_q^{(p)}(H)$.

THEOREM 3. *Let $w_1 \in A$ and $\eta \in \mathbb{R}$ be given. For each $\rho > 0$ assume that $T_q^{(p)}(G)(\rho x)$ and $T_q^{(p)}(H)(\rho x)$ have first variations $\delta T_q^{(p)}(G)(\rho x \mid \rho w_1)$ and $\delta T_q^{(p)}(H)(\rho x \mid \rho w_1)$ for all $x \in C_0[0,T]$ such that for some positive function $\gamma(\rho)$,*

$$(3.18)\qquad \sup_{|k|\leqslant\gamma(\rho)} \Big|T_q^{(p)}(G)\Big(\rho x(\cdot) - \frac{\cdot}{T}\rho x(T) + \frac{\cdot}{T}\rho\eta + k\rho w_1\Big) \cdot \delta T_q^{(p)}(H)\Big(\rho x(\cdot) - \frac{\cdot}{T}\rho x(T) + \frac{\cdot}{T}\rho\eta + k\rho w_1 \;\Big|\; \rho w_1\Big)\Big|$$

and

$$\sup_{|k|\leqslant\gamma(\rho)}\Big|T_q^{(p)}(H)\Big(\rho x(\cdot)-\frac{\cdot}{T}\rho x(T)+\frac{\cdot}{T}\rho\eta+k\rho w_1\Big)\cdot\delta T_q^{(p)}(G)\Big(\rho x(\cdot)-\frac{\cdot}{T}\rho x(T)+\frac{\cdot}{T}\rho\eta+k\rho w_1\ \Big|\ \rho w_1\Big)\Big| \tag{3.19}$$

are Wiener integrable functions of x *over* $C_0[0,T]$. *Then,*

$$\begin{aligned}E^{\text{anf}_q}&\Big(T_q^{(p)}(G)(x)\delta T_q^{(p)}(H)(x\mid w_1)+\delta T_q^{(p)}(G)(x\mid w_1)T_q^{(p)}(H)(x)\ \Big\|\ x(T)=\eta\Big)\\ &\doteq -iq\int_{C_0[0,T]}^{\text{anf}_q}T_q^{(p)}(G)\Big(x(\cdot)-\frac{\cdot}{T}x(T)+\frac{\cdot}{T}\eta\Big)\cdot T_q^{(p)}(H)\Big(x(\cdot)-\frac{\cdot}{T}x(T)+\frac{\cdot}{T}\eta\Big)\langle w_1',x\rangle m(dx).\end{aligned}\tag{3.20}$$

Furthermore, if either side of (3.20) *exists, then*

$$\begin{aligned}&E^{\text{anf}_q}\Big(T_q^{(p)}(G)(x)\delta T_q^{(p)}(H)(x\mid w_1)+\delta T_q^{(p)}(G)(x\mid w_1)T_q^{(p)}(H)(x)\ \Big\|\ x(T)=\eta\Big)\\ &\quad=-iqE^{\text{anf}_q}\Big(T_q^{(p)}(G)(x)T_q^{(p)}(H)(x)\langle w_1',x\rangle\ \Big\|\ x(T)=\eta\Big)\\ &\qquad+T(w_0',w_1')E^{\text{anf}_q}\Big(T_q^{(p)}(G)(x)\delta T_q^{(p)}(H)(x\mid w_0)\\ &\qquad\qquad+\delta T_q^{(p)}(G)(x\mid w_0)T_q^{(p)}(H)(x)\ \Big\|\ x(T)=\eta\Big)\\ &\qquad+iq\eta(w_0',w_1')E^{\text{anf}_q}\Big(T_q^{(p)}(G)(x)T_q^{(p)}(H)(x)\ \Big\|\ x(T)=\eta\Big)\end{aligned}\tag{3.21}$$

where $w_0(t)=t/T$ *on* $[0,T]$.

PROOF: Simply apply Theorem 1 with $F(x)=T_q^{(p)}(G)(x)T_q^{(p)}(H)(x)$. ∎

The following corollary follows by choosing $H(x)=G(x)$ in Theorem 3 above.

COROLLARY 5. *Let* $w_1\in A$ *and* $\eta\in\mathbb{R}$ *be given. For each* $\rho>0$, *assume that* $T_q^{(p)}(G)(\rho x)$ *has a first variation* $\delta T_q^{(p)}(G)(\rho x\mid\rho w_1)$ *for all* $x\in C_0[0,T]$ *such that for some positive function* $\gamma(\rho)$,

$$\sup_{|k|\leqslant\gamma(\rho)}\Big|T_q^{(p)}(G)\Big(\rho x(\cdot)-\frac{\cdot}{T}\rho x(T)+\frac{\cdot}{T}\rho\eta+k\rho w_1\Big)\cdot\delta T_q^{(p)}(G)\Big(\rho x(\cdot)-\frac{\cdot}{T}\rho x(T)+\frac{\cdot}{T}\rho\eta+k\rho w_1\ \Big|\ \rho w_1\Big)\Big| \tag{3.22}$$

is a Wiener integrable function of x *over* $C_0[0,T]$. *Then,*

$$\begin{aligned}&E^{\text{anf}_q}\Big(T_q^{(p)}(G)(x)\delta T_q^{(p)}(G)(x\mid w_1)\ \Big\|\ x(T)=\eta\Big)\\ &\qquad\doteq-\frac{iq}{2}\int_{C_0[0,T]}^{\text{anf}_q}\Big[T_q^{(p)}(G)\Big(x(\cdot)-\frac{\cdot}{T}x(T)+\frac{\cdot}{T}\eta\Big)\Big]^2\langle w_1',x\rangle m(dx).\end{aligned}\tag{3.23}$$

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Furthermore, if either side of (3.23) *exists, then* (3.21) *also holds with each* H *replaced with* G.

COROLLARY 6. *Let* $w_1 \in A$ *and* $\eta \in \mathbb{R}$ *be given. For each* $\rho > 0$, *assume that* $G(\rho x)$ *and* $T_q^{(p)}(H)(\rho x)$ *have first variations* $\delta G(\rho x \mid \rho w_1)$ *and* $\delta T_q^{(p)}(H)(\rho x \mid \rho w_1)$ *for all* $x \in C_0[0,T]$ *such that for some positive function* $\gamma(\rho)$,

$$\tag{3.24} \sup_{|k| \leqslant \gamma(\rho)} \Big| G\Big(\rho x(\cdot) - \frac{\cdot}{T}\rho x(T) + \frac{\cdot}{T}\rho\eta + k\rho w_1\Big) \cdot \delta T_q^{(p)}(H)\Big(\rho x(\cdot) - \frac{\cdot}{T}\rho x(T) + \frac{\cdot}{T}\rho\eta + k\rho w_1 \mid \rho w_1\Big)\Big|$$

and

$$\tag{3.25} \sup_{|k| \leqslant \gamma(\rho)} \Big| \delta G\Big(\rho x(\cdot) - \frac{\cdot}{T}\rho x(T) + \frac{\cdot}{T}\rho\eta + k\rho w_1 \mid \rho w_1\Big) \cdot T_q^{(p)}(H)\Big(\rho x(\cdot) - \frac{\cdot}{T}\rho x(T) + \frac{\cdot}{T}\rho\eta + k\rho w_1\Big)\Big|$$

are Wiener integrable functions of x *over* $C_0[0,T]$. *Then,*

$$\tag{3.26} \begin{aligned} &E^{\mathrm{anf}_q}\big(G(x)\delta T_q^{(p)}(H)(x \mid w_1) + \delta G(x \mid w_1) T_q^{(p)}(H)(x) \,\|\, x(T) = \eta\big) \\ &\doteq -iq \int_{C_0[0,T]}^{\mathrm{anf}_q} G\Big(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta\Big) \cdot T_q^{(p)}(H)\Big(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta\Big)\langle w_1', x\rangle m(dx). \end{aligned}$$

Furthermore, if either side of equation (3.26) *exists, then*

$$\tag{3.27} \begin{aligned} &E^{\mathrm{anf}_q}\big(G(x)\delta T_q^{(p)}(H)(x \mid w_1) + \delta G(x \mid w_1) T_q^{(p)}(H)(x) \,\|\, x(T) = \eta\big) \\ &= -iqE^{\mathrm{anf}_q}\big(G(x)T_q^{(p)}(H)(x)\langle w_1', x\rangle \,\|\, x(T) = \eta\big) \\ &\quad + T(w_0', w_1')E^{\mathrm{anf}_q}\big(G(x)\delta T_q^{(p)}(H)(x \mid w_0) + \delta G(x \mid w_0)T_q^{(p)}(H)(x) \,\|\, x(T) = \eta\big) \\ &\quad + iq\eta(w_0', w_1')E^{\mathrm{anf}_q}\big(G(x)T_q^{(p)}(H)(x) \,\|\, x(T) = \eta\big). \end{aligned}$$

PROOF: Simply apply Theorem 1 with $F(x) = G(x)T_q^{(p)}(H)(x)$. ∎

4. ADDITIONAL RESULTS

In our first result below we obtain an interesting integration by parts formula involving the conditional Fourier-Feynman transform's, $T_q^{(p)}(G\|X)$ and $T_q^{(p)}(H\|X)$; see equation (2.9) above.

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THEOREM 4. *Let $w_1 \in A$ and $\eta_1, \eta_2, \eta_3 \in \mathbb{R}$ be given. For each $\rho > 0$ assume that $T_q^{(p)}(G\|X)(\rho x, \eta_1)$ and $T_q^{(p)}(H\|X)(\rho x, \eta_2)$ have first variations $\delta T_q^{(p)}(G\|X)(\rho x \mid \rho w_1, \eta_1)$ and $\delta T_q^{(p)}(H\|X)(\rho x \mid \rho w_1, \eta_2)$ for all $x \in C_0[0,T]$ such that for some positive function $\gamma(\rho)$,*

$$\tag{4.1} \sup_{|k|\leqslant\gamma(\rho)} \Big|\delta T_q^{(p)}(G\|X)\Big(\rho x(\cdot) - \frac{\cdot}{T}\rho x(T) + \frac{\cdot}{T}\rho\eta_3 + k\rho w_1 \Big| \rho w_1, \eta_1\Big) \cdot T_q^{(p)}(H\|X)\Big(\rho x(\cdot) - \frac{\cdot}{T}\rho x(T) + \frac{\cdot}{T}\rho\eta_3 + k\rho w_1, \eta_2\Big)\Big|$$

and

$$\tag{4.2} \sup_{|k|\leqslant\gamma(\rho)} \Big|T_q^{(p)}(G\|X)\Big(\rho x(\cdot) - \frac{\cdot}{T}\rho x(T) + \frac{\cdot}{T}\rho\eta_3 + k\rho w_1, \eta_1\Big) \cdot \delta T_q^{(p)}(H\|X)\Big(\rho x(\cdot) - \frac{\cdot}{T}\rho x(T) + \frac{\cdot}{T}\rho\eta_3 + k\rho w_1 \Big| \rho w_1, \eta_2\Big)\Big|$$

are Wiener integrable functions of x over $C_0[0,T]$. Then

$$\tag{4.3} \begin{aligned} &E^{\mathrm{anf}_q}\Big(T_q^{(p)}(G\|X)(x,\eta_1)\delta T_q^{(p)}(H\|X)(x \mid w_1, \eta_2) \\ &\qquad + \delta T_q^{(p)}(G\|X)(x \mid w_1, \eta_1) T_q^{(p)}(H\|X)(x, \eta_2) \;\|\; x(T) = \eta_3\Big) \\ &\doteq -iq \int_{C_0[0,T]}^{\mathrm{anf}_q} T_q^{(p)}(G\|X)\Big(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta_3, \eta_1\Big) \\ &\qquad \cdot T_q^{(p)}(H\|X)\Big(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta_3, \eta_2\Big)\langle w_1', x\rangle m(dx). \end{aligned}$$

Furthermore, if either side of (4.3) exists, then

$$\tag{4.4} \begin{aligned} &E^{\mathrm{anf}_q}\Big(T_q^{(p)}(G\|X)(x,\eta_1)\delta T_q^{(p)}(H\|X)(x \mid w_1, \eta_2) \\ &\qquad + \delta T_q^{(p)}(G\|X)(x \mid w_1, \eta_1) T_q^{(p)}(H\|X)(x, \eta_2) \;\|\; x(T) = \eta_3\Big) \\ &= -iqE^{\mathrm{anf}_q}\Big(T_q^{(p)}(G\|X)(x,\eta_1)T_q^{(p)}(H\|X)(x,\eta_2)\langle w_1', x\rangle \;\|\; x(T) = \eta_3\Big) \\ &\quad + T(w_0', w_1')E^{\mathrm{anf}_q}\Big(T_q^{(p)}(G\|X)(x,\eta_1)\delta T_q^{(p)}(H\|X)(x \mid w_0, \eta_2) \\ &\quad + \delta T_q^{(p)}(G\|X)(x \mid w_0, \eta_1) T_q^{(p)}(H\|X)(x, \eta_2) \;\|\; x(T) = \eta_3\Big) \\ &\quad + iq\eta_3(w_0', w_1')E^{\mathrm{anf}_q}\Big(T_q^{(p)}(G\|X)(x,\eta_1)T_q^{(p)}(H\|X)(x,\eta_2) \;\|\; x(T) = \eta_3\Big) \end{aligned}$$

where, as usual, $w_0(t) = t/T$ on $[0,T]$.

PROOF: Simply apply Theorem 1 with $F(x) = T_q^{(p)}(G\|X)(x,\eta_1)T_q^{(p)}(H\|X)(x,\eta_2)$. ∎

Choosing $H(x)$ to be identically equal to one on $C_0[0,T]$ yields our next corollary; to obtain Corollary 8 we simply choose $H(x) = G(x)$.

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COROLLARY 7. *Let $w_1 \in A$ and $\eta_1, \eta_3 \in \mathbb{R}$ be given. For each $\rho > 0$ assume that $T_q^{(p)}(G\|X)(\rho x, \eta_1)$ has a first variation $\delta T_q^{(p)}(G\|X)(\rho x \mid \rho w_1, \eta_1)$ for all $x \in C_0[0,T]$ such that for some positive function $\gamma(\rho)$,*

$$\sup_{|k|\leqslant\gamma(\rho)} \left|\delta T_q^{(p)}(G\|X)\Big(\rho x(\cdot) - \dot{\overline{T}}\rho x(T) + \dot{\overline{T}}\rho\eta_3 + k\rho w_1 \mid \rho w_1, \eta_1\Big)\right| \tag{4.5}$$

is an Wiener integrable function of x over $C_0[0,T]$. Then

$$E^{\mathrm{anf}_q}\Big(\delta T_q^{(p)}(G\|X)(x \mid w_1, \eta_1) \;\|\; x(T) = \eta_3\Big) \doteq -iq\int_{C_0[0,T]}^{\mathrm{anf}_q} T_q^{(p)}(G\|X)\Big(x(\cdot) - \dot{\overline{T}}x(T) + \dot{\overline{T}}\eta_3, \eta_1\Big)\langle w_1', x\rangle m(dx). \tag{4.6}$$

Furthermore, if either side of equation (4.6) exists, then

$$\begin{aligned} &E^{\mathrm{anf}_q}\Big(\delta T_q^{(p)}(G\|X)(x \mid w_1, \eta_1) \;\|\; x(T) = \eta_3\Big) \\ &\quad = -iqE^{\mathrm{anf}_q}\Big(T_q^{(p)}(G\|X)(x, \eta_1)\langle w_1', x\rangle \;\|\; x(T) = \eta_3\Big) \\ &\qquad + T(w_0', w_1')E^{\mathrm{anf}_q}\Big(\delta T_q^{(p)}(G\|X)(x \mid w_0, \eta_1) \;\|\; x(T) = \eta_3\Big) \\ &\qquad + iq\eta_3(w_0', w_1')E^{\mathrm{anf}_q}\Big(T_q^{(p)}(G\|X)(x, \eta_1) \;\|\; x(T) = \eta_3\Big). \end{aligned} \tag{4.7}$$

COROLLARY 8. *Assume that condition (4.1) holds with $H(x) = G(x)$. Then*

$$E^{\mathrm{anf}_q}\Big(T_q^{(p)}(G\|X)(x, \eta_1)\delta T_q^{(p)}(G\|X)(x \mid w_1, \eta_1) \;\|\; x(T) = \eta_3\Big) \doteq -\frac{iq}{2}\int_{C_0[0,T]}^{\mathrm{anf}_q}\Big[T_q^{(p)}(G\|X)\Big(x(\cdot) - \dot{\overline{T}}x(T) + \dot{\overline{T}}\eta_3, \eta_1\Big)\Big]^2\langle w_1', x\rangle m(dx). \tag{4.8}$$

Furthermore, if either side of (4.8) exists, then

$$\begin{aligned} &E^{\mathrm{anf}_q}\Big(T_q^{(p)}(G\|X)(x, \eta_1)\delta T_q^{(p)}(G\|X)(x \mid w_1, \eta_1) \;\|\; x(T) = \eta_3\Big) \\ &\quad = -\frac{iq}{2}E^{\mathrm{anf}_q}\Big(\big[T_q^{(p)}(G\|X)(x, \eta_1)\big]^2\langle w_1', x\rangle \;\|\; x(T) = \eta_3\Big) \\ &\qquad + T(w_0', w_1')E^{\mathrm{anf}_q}\Big(T_q^{(p)}(G\|X)(x, \eta_1)\delta T_q^{(p)}(G\|X)(x \mid w_0, \eta_1) \;\|\; x(T) = \eta_3\Big) \\ &\qquad + \frac{iq}{2}\eta_3(w_0', w_1')E^{\mathrm{anf}_q}\Big(\big[T_q^{(p)}(G\|X)(x, \eta_1)\big]^2 \;\|\; x(T) = \eta_3\Big). \end{aligned} \tag{4.9}$$

We finish this paper by mentioning that the hypotheses (and hence the conclusions) of Theorems 1-4 and Corollaries 1-8 above are indeed satisfied by several large classes of functionals; we shall very briefly discuss two such classes.

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1. The Banach algebra $\mathcal{S}$, introduced by Cameron and Storvick in [5], consists of functionals expressible in the form

$$F(x) = \int_{L_2[0,T]} \exp\Big\{i\langle u, x\rangle\Big\} df(u) \tag{4.10}$$

for scale-invariant almost everywhere $x \in C_0[0,T]$, where the associated measure f is an element of $M\big(L_2[0,T]\big)$, the space of $\mathbb{C}$-valued countably additive Borel measures on $L_2[0,T]$. Now let

$$\mathcal{K} = \Big\{F \in \mathcal{S} : \int_{L_2[0,T]} \|u\|_2 |df(u)| < \infty\Big\}. \tag{4.11}$$

Then, all of the above theorems and corollaries are valid provided that all of the functionals F, G and H involved are elements of $\mathcal{K}$. For example for $G \in \mathcal{K}$, a direct calculation shows that

$$T_q^{(p)}(G\|X)(x, \eta_1) = \int_{L_2[0,T]} \exp\Big\{i\langle u, x\rangle + i\eta_1 b - \frac{i}{2q}\int_0^T \big[u(s) - b\big]^2 ds\Big\} dg(u),$$

and hence,

$$\begin{aligned}\delta T_q^{(p)}&(G\|X)(x \mid w_1, \eta_1)\\ &= \int_{L_2[0,T]} i(u, w_1') \exp\Big\{i\langle u, x\rangle + i\eta_1 b - \frac{i}{2q}\int_0^T \big[u(s) - b\big]^2 ds\Big\} dg(u)\end{aligned}$$

for scale-invariant almost everywhere $x \in C_0[0,T]$ where $b = 1/T \int_0^T u(s)ds$.

Thus for scale-invariant almost everywhere $x \in C_0[0,T]$, we easily obtain that

$$\big|T_q^{(p)}(G\|X)(x, \eta_1)\big| \leqslant \int_{L_2[0,T]} |dg(u)| < \infty$$

and

$$\big|\delta T_q^{(p)}(G\|X)(x \mid w_1, \eta_1)\big| \leqslant \|w_1'\|_2 \int_{L_2[0,T]} \|u\|_2 |df(u)| < \infty.$$

Hence, by carrying out the same calculations for $H \in \mathcal{K}$, we see that the expressions in (4.1) and (4.2) are certainly integrable functions of x over $C_0[0,T]$ since $m(C_0[0,T]) = 1$. Thus Theorem 4 and Corollaries 7 and 8 hold for all G and H in $\mathcal{K}$. The results in Section 3 for F, G and H in $\mathcal{K}$ follow by similar calculations.

2. In [18], Park and Skoug obtained various integration by parts formulas involving analytic Feynman integrals for functionals of the form

$$F(x) = f\big(\langle\alpha_1, x\rangle, \ldots, \langle\alpha_n, x\rangle\big) \tag{4.12}$$

for scale-invariant almost everywhere $x \in C_0[0,T]$ where $\{\alpha_1, \dots, \alpha_n\}$ is an orthonormal set of functions in $L_2[0,T]$. Proceeding formally we see that

$$\delta F(x \mid w_1) = \sum_{j=1}^{n} (\alpha_j, w_1') f_j(\langle \alpha_1, x\rangle, \dots, \langle \alpha_n, x\rangle),$$

and that

$$\delta T_q^{(p)}(F)(x \mid w_1) = \Big(-\frac{iq}{2\pi}\Big)^{n/2} \int_{\mathbb{R}^n} \Big(\sum_{j=1}^{n} (\alpha_j, w_1') f_j(v_1, \dots, v_n) \cdot \exp\Big\{ \frac{iq}{2} \Big[\big(v_1 - \langle \alpha_1, x\rangle\big)^2 + \cdots + \big(v_n - \langle \alpha_n, x\rangle\big)^2 \Big] \Big\} \Big) dv_1 \dots dv_n.$$

Thus, putting appropriate continuity and integrability conditions on $f : \mathbb{R}^n \to \mathbb{R}$ and its partial derivatives $f_1 : \mathbb{R}^n \to \mathbb{R}, \dots, f_n : \mathbb{R}^n \to \mathbb{R}$, one can show that the four theorems and eight corollaries established above hold for various functionals of the form (4.12).

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Department of Mathematics
Dankook University
Cheonan 330-714
Korea
e-mail: sejchang@dankook.ac.kr

Department of Mathematics and Statistics
University of Nebraska-Lincoln
Lincoln, NE 68588-0323
United States of America
e-mail: dskoug@math.unl.edu